SAS/OR® 9.1.3 User's Guide: Mathematical Programming 3.2

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SAS/OR® 9.1.3 User's Guide: Mathematical Programming 3.2, Volumes 1-4

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Contents

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The procedures in SAS/OR software were implemented by the Operations Research and Development Department. Substantial support was given to the project by other members of the Analytical Solutions Division. Core Development Division, Display Products Division, Graphics Division, and the Host Systems Division also contributed to this product.

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John Stone	North Carolina State University
Keith R. Weiss	ICI Americas Inc.

The final responsibility for the SAS System lies with SAS Institute alone. We hope that you will always let us know your opinions about the SAS System and its documentation. It is through your participation that SAS software is continuously improved.

What's New in SAS/OR 9.1.3, Release 3.1 and Release 3.2

Overview

SAS/OR version 9.1.3, release 3.1 and release 3.2 contain several new and enhanced features designed to expand the scale and scope of problems that SAS/OR can address. These enhancements also make it easier for you to use the capabilities of SAS/OR. Brief descriptions of the new features are presented in the following sections. For more information, refer to the SAS/OR documentation, available in the following six volumes:

- SAS/OR User's Guide: Bills of Material Processing
- SAS/OR User's Guide: Constraint Programming
- SAS/OR User's Guide: Local Search Optimization
- SAS/OR User's Guide: Mathematical Programming
- SAS/OR User's Guide: Project Management
- SAS/OR User's Guide: The QSIM Application

Online help can also be found under the corresponding classification.

The NETFLOW Procedure

The NETFLOW procedure for network flow optimization contains a new feature that enables you to specify and solve generalized network problems. In generalized networks the amount of flow entering an arc might not equal the amount of flow leaving the arc, signifying a loss or a gain as flow traverses the arc. A new PROC NETFLOW option, GENNET, indicates that the network is generalized. Generalized networks have a broad range of practical applications, including the following:

- transportation of perishable goods (weight loss due to drying)
- financial investment account balances (interest rates)
- manufacturing (yield ratios)
- electrical power generation (loss during transmission along lines)

Another new option, EXCESS=, enables you to use PROC NETFLOW to solve an even wider variety of network flow optimization problems for both standard and generalized networks. As a result, PROC NETFLOW is equipped to deal with many frequently encountered challenges to successful network flow optimization, such as the following:

- networks with excess supply or demand
- networks containing nodes with unknown supply and demand values
- networks with nodes having range constraints on supply and demand

The OPTLP Procedure

The OPTLP procedure provides linear programming solvers and enables you to choose from three linear programming solvers: primal simplex, dual simplex, and interior point (experimental). The simplex solvers implement a two-phase simplex method, and the interior point solver implements a primal-dual predictor-corrector algorithm. All three solvers are newly rewritten and are designed for excellent performance and scalability. Presolvers, which work aggressively to reduce the effective size of problems before the solvers are invoked, are also provided.

New in SAS/OR 9.1.3 release 3.2, the TIMETYPE= option enables you to specify the type of time (real time or CPU time) that can be limited via the MAXTIME= option and reported via the _OROPTLP_ macro variable.

PROC OPTLP accepts linear programming problems that are submitted in an MPS-format SAS data set. The MPS-format SAS data set corresponds closely to the MPS-format text file (commonly used in the optimization community) and was first introduced in SAS/OR version 9.1.3, release 2.1.

The OPTMILP Procedure

The new OPTMILP procedure solves mixed-integer linear programming problems with an LP-based branch-and-bound algorithm that has been completely rewritten for this release. The algorithm also implements advanced techniques including presolvers, cutting planes, and primal heuristics. The resulting improvements in efficiency enable you to use PROC OPTMILP to solve larger and more complex optimization problems than you could solve with previous releases of SAS/OR.

PROC OPTMILP accepts mixed-integer linear programming problems that are submitted in an MPS-format SAS data set.

The OPTMODEL Procedure

The OPTMODEL procedure provides a modeling environment tailored to building, solving, and maintaining optimization models. This makes the process of translating the symbolic formulation of an optimization model into PROC OPTMODEL virtually transparent, since the modeling language mimics the symbolic algebra of the formulation as closely as possible. PROC OPTMODEL also streamlines and simplifies the critical process of populating optimization models with data from SAS data sets. All of this transparency produces models that are more easily inspected for completeness and correctness, more easily corrected, and more easily modified, whether through structural changes or through the substitution of new data for old.

The OPTMODEL procedure comprises the powerful OPTMODEL modeling language and state-of-the-art solvers for several classes of mathematical programming problems. Seven solvers are available to OPTMODEL:

Problem	Solver
Linear Programming	**LP**
Mixed Integer Programming	**MILP**
Quadratic Programming (experimental)	**QP**
Nonlinear Programming, Unconstrained	**NLPU**
General Nonlinear Programming	**NLPC**
General Nonlinear Programming	**SQP**
General Nonlinear Programming (experimental)	**IPNLP**

The OPTQP Procedure

The OPTQP procedure solves quadratic programming problems with a new infeasible primal-dual predictor-corrector interior point algorithm. Performance is excellent for both sparse and dense quadratic programming problems, and PROC OPTQP excels at solving large problems efficiently.

PROC OPTQP accepts quadratic programming problems that are submitted in a QPS-format SAS data set. The QPS-format SAS data set corresponds closely to the format of the QPS text file (a widely accepted extension of the MPS format) and was introduced in SAS/OR 9.1.3, release 2.1.

Earned Value Management Macros (Experimental)

The set of experimental Earned Value Management macros complement the current SAS/OR procedures for project and resource scheduling (PROC CPM and PROC PM) by providing diagnostic information about the execution of scheduled projects. Earned Value Management (EVM) is growing in prominence and acceptance in the project management community due to its ability to turn information about partially completed projects into valid, early projections of overall project performance. EVM measures current project execution against the project execution plan on a cost and schedule basis.

SAS/OR provides two sets of EVM macros: a set of four analytical macros to compute EVM metrics, and a set of six macros to create graphical reports based on these metrics. A wide variety of EVM metrics and performance projections, for both task-by-task and project-wide evaluations, are supported.

Microsoft Project Conversion Macro (Experimental)

The SAS macros %MDBTOPM and %MP2KTOPM have been used in previous releases of SAS/OR to convert files saved by Microsoft Project 98 and Microsoft Project 2000 (and later), respectively, into SAS data sets that can be used as input for project scheduling with SAS/OR. Now these two macros are combined in the SAS macro %MSPTOSAS, which converts Microsoft Project 98 (and later) data. This experimental macro generates the necessary SAS data sets, determines the values of the relevant options, and invokes PROC PM in SAS/OR with the converted project data. The %MSPTOSAS macro enables you to use Microsoft Project for the input of project data and still take advantage of the excellent project and resource scheduling capabilities of SAS/OR.

Execution of the %MSPTOSAS macro requires SAS/ACCESS software.

Using This Book

Purpose

SAS/OR User's Guide: Mathematical Programming provides a complete reference for the mathematical programming procedures in SAS/OR software. This book serves as the primary documentation for the INTPOINT, LP, NETFLOW, and NLP procedures, in addition to the new OPTMODEL, OPTLP, OPTQP, and OPTMILP procedures, the various solvers called by the OPTMODEL procedure, and the MPS-format SAS data set specification.

"Using This Book" describes the organization of this book and the conventions used in the text and example code. To gain full benefit from using this book, you should familiarize yourself with the information presented in this section and refer to it when needed. "Additional Documentation" at the end of this section provides references to other books that contain information on related topics.

Organization

Chapter 1 contains a brief overview of the mathematical programming procedures in SAS/OR software and provides an introduction to optimization and the use of the optimization tools in the SAS System. The first chapter also describes the flow of data between the procedures and how the components of the SAS System fit together.

After the introductory chapter, the next five chapters describe the INTPOINT, LP, NETFLOW, NLP, and OPTMODEL procedures. The next seven chapters describe the linear programming, mixed integer linear programming, quadratic programming, nonlinear optimization, sequential quadratic programming, interior point nonlinear programming, and unconstrained nonlinear programming solvers, which are used by the OPTMODEL procedure. The next chapter is the specification of the newly introduced MPS-format SAS data set. The last three chapters describe the new OPTLP, OPTQP, and OPTMILP procedures for solving linear programming, quadratic programming, and mixed integer linear programming problems, respectively. Each procedure description is self-contained; you need to be familiar with only the basic features of the SAS System and SAS terminology to use most procedures. The statements and syntax necessary to run each procedure are presented in a uniform format throughout this book.

The following list summarizes the types of information provided for each procedure:

Overview provides a general description of what the procedure does. It outlines major capabilities of the procedure and lists all input and output data sets that are used with it.

Getting Started illustrates simple uses of the procedure using a few short examples. It provides introductory *hands-on* information for the procedure.

Syntax constitutes the major reference section for the syntax of the procedure. First, the statement syntax is summarized. Next, a functional summary table lists all the statements and options in the procedure, classified by function. In addition, the online version includes a Dictionary of Options, which provides an alphabetical list of all options. Following these tables, the PROC statement is described, and then all other statements are described in alphabetical order.

Details describes the features of the procedure, including algorithmic details and computational methods. It also explains how the various options interact with each other. This section describes input and output data sets in greater detail, with definitions of the output variables, and explains the format of printed output, if any.

Examples consists of examples designed to illustrate the use of the procedure. Each example includes a description of the problem and lists the options highlighted by the example. The example shows the data and the SAS statements needed, and includes the output produced. You can duplicate the examples by copying the statements and data and running the SAS program. The SAS Sample Library contains the code used to run the examples shown in this book; consult your SAS Software representative for specific information about the Sample Library.

References lists references that are relevant to the chapter.

Typographical Conventions

The printed version of *SAS/OR User's Guide: Mathematical Programming* uses various type styles, as explained by the following list:

roman	is the standard type style used for most text.

UPPERCASE ROMAN	is used for SAS statements, options, and other SAS language elements when they appear in the text. However, you can enter these elements in your own SAS code in lowercase, uppercase, or a mixture of the two. This style is also used for identifying arguments and values (in the Syntax specifications) that are literals (for example, to denote valid keywords for a specific option).
UPPERCASE BOLD	is used in the "Syntax" section to identify SAS keywords, such as the names of procedures, statements, and options.
bold	is used in the "Syntax" section to identify options.
helvetica	is used for the names of SAS variables and data sets when they appear in the text.
oblique	is used for user-supplied values for options (for example, VARSELECT= *rule*).
italic	is used for terms that are defined in the text, for emphasis, and for references to publications.
`monospace`	is used to show examples of SAS statements. In most cases, this book uses lowercase type for SAS code. You can enter your own SAS code in lowercase, uppercase, or a mixture of the two. This style is also used for values of character variables when they appear in the text.

Conventions for Examples

Most of the output shown in this book is produced with the following SAS System options:

```
options linesize=80 pagesize=60 nonumber nodate;
```

Accessing the SAS/OR Sample Library

The SAS/OR sample library includes many examples that illustrate the use of SAS/OR software, including the examples used in this documentation. To access these sample programs, select **Learning to Use SAS->Sample SAS Programs** from the **SAS Help and Documentation** window, and then select **SAS/OR** from the list of available topics.

Online Help System and Updates

You can access online help information about SAS/OR software in two ways, depending on whether you are using the SAS windowing environment in the command line mode or the pull-down menu mode.

If you are using a command line, you can access the SAS/OR help menus by typing **help or** on the command line. If you are using the pull-down menus, you can select **SAS Help and Documentation->SAS Products** from the **Help** pull-down menu, and then select **SAS/OR** from the list of available topics.

Additional Documentation for SAS/OR Software

In addition to *SAS/OR User's Guide: Mathematical Programming,* you may find these other documents helpful when using SAS/OR software:

SAS/OR User's Guide: Bills of Material Processing
provides documentation for the BOM procedure and all bill-of-material post-processing SAS macros. The BOM procedure and SAS macros provide the ability to generate different reports and to perform several transactions to maintain and update bills of material.

SAS/OR User's Guide: Constraint Programming
provides documentation for the constraint programming procedure in SAS/OR software. This book serves as the primary documentation for the CLP procedure, an experimental procedure new to SAS/OR software.

SAS/OR User's Guide: Local Search Optimization
provides documentation for the local search optimization procedure in SAS/OR software. This book serves as the primary documentation for the GA procedure, an experimental procedure that uses genetic algorithms to solve optimization problems.

SAS/OR User's Guide: Project Management
provides documentation for the project management procedures in SAS/OR software. This book serves as the primary documentation for the CPM, DTREE, GANTT, NETDRAW, and PM procedures, as well as the PROJMAN Application, a graphical user interface for project management.

SAS/OR User's Guide: The QSIM Application
provides documentation for the QSIM Application, which is used to build and analyze models of queueing systems using discrete event simulation. This book shows you how to build models using the simple point-and-click graphical user interface, how to run the models, and how to collect and analyze the sample data to give you insight into the behavior of the system.

SAS/OR Software: Project Management Examples, Version 6
contains a series of examples that illustrate how to use SAS/OR software to manage projects. Each chapter contains a complete project management scenario and describes how to use PROC GANTT, PROC CPM, and PROC NETDRAW, in addi-

tion to other reporting and graphing procedures in the SAS System, to perform the necessary project management tasks.

SAS/IRP User's Guide: Inventory Replenishment Planning
provides documentation for SAS/IRP software. This book serves as the primary documentation for the IRP procedure for determining replenishment policies, as well as the %IRPSIM SAS programming macro for simulating replenishment policies.

Chapter 1
Introduction to Optimization

Chapter Contents

Chapter 1
Introduction to Optimization

Overview

Operations Research tools are directed toward the solution of resource management and planning problems. Models in Operations Research are representations of the structure of a physical object or a conceptual or business process. Using the tools of Operations Research involves the following:

- defining a structural model of the system under investigation
- collecting the data for the model
- analyzing the model

SAS/OR software is a set of procedures for exploring models of distribution networks, production systems, resource allocation problems, and scheduling problems using the tools of Operations Research.

The following list suggests some of the application areas where optimization-based decision support systems have been used. In practice, models often contain elements of several applications listed here.

- **Product-mix problems** find the mix of products that generates the largest return when there are several products competing for limited resources.
- **Blending problems** find the mix of ingredients to be used in a product so that it meets minimum standards at minimum cost.
- **Time-staged problems** are models whose structure repeats as a function of time. Production and inventory models are classic examples of time-staged problems. In each period, production plus inventory minus current demand equals inventory carried to the next period.
- **Scheduling problems** assign people to times, places, or tasks so as to optimize people's preferences or performance while satisfying the demands of the schedule.
- **Multiple objective problems** have multiple, possibly conflicting, objectives. Typically, the objectives are prioritized and the problems are solved sequentially in a priority order.
- **Capital budgeting and project selection problems** ask for the project or set of projects that will yield the greatest return.
- **Location problems** seek the set of locations that meets the distribution needs at minimum cost.
- **Cutting stock problems** find the partition of raw material that minimizes waste and fulfills demand.

A problem is formalized with the construction of a model to represent it. These models, called mathematical programs, are represented in SAS data sets and then solved using SAS/OR procedures. The solution of mathematical programs is called mathematical programming. Since mathematical programs are represented in SAS data sets, they can be saved, easily changed, and re-solved. The SAS/OR procedures also output SAS data sets containing the solutions. These can then be used to produce customized reports. In addition, this structure enables you to build decision support systems using the tools of Operations Research and other tools in the SAS System as building blocks.

The basic optimization problem is that of minimizing or maximizing an objective function subject to constraints imposed on the variables of that function. The objective function and constraints can be linear or nonlinear; the constraints can be bound constraints, equality or inequality constraints, or integer constraints. Traditionally, optimization problems are divided into linear programming (LP; all functions and constraints are linear) and nonlinear programming (NLP).

The data describing the model are supplied to an optimizer (such as one of the procedures described in this book), an optimizing algorithm is used to determine the optimal values for the decision variables so the objective is either maximized or minimized, the optimal values assigned to decision variables are on or between allowable bounds, and the constraints are obeyed. Determining the optimal values is the process called *optimization*.

This chapter describes how to use SAS/OR software to solve a wide variety of optimization problems. We describe various types of optimization problems, indicate which SAS/OR procedure you can use, and show how you provide data, run the procedure, and obtain optimal solutions.

In the next section we broadly classify the SAS/OR procedures based on the types of mathematical programming problems they can solve.

Linear Programming Problems

PROC OPTLP

PROC OPTLP solves linear programming problems that are submitted either in an MPS-format file or in an MPS-format SAS data set.

The MPS file format is a format commonly used for describing linear programming (LP) and integer programming (IP) problems (Murtagh 1981; IBM 1988). MPS-format files are in text format and have specific conventions for the order in which the different pieces of the mathematical model are specified. The MPS-format SAS data set corresponds closely to the MPS file format and is used to describe linear programming problems for PROC OPTLP. For more details, refer to Chapter 14, "The MPS-Format SAS Data Set."

PROC OPTLP provides three solvers to solve the LP: primal simplex, dual simplex, and interior point. The simplex solvers implement a two-phase simplex method, and

the interior point solver implements a primal-dual predictor-corrector algorithm. For more details refer to Chapter 15, "The OPTLP Procedure."

PROC OPTMODEL

PROC OPTMODEL provides a language for concisely modeling linear programming problems. The language allows a model to be expressed in a form that matches the mathematical formulation. Within OPTMODEL you can declare a model, pass it directly to various solvers, and review the solver result. You can also save an instance of a linear model in data set form for use by PROC OPTLP. For more details, refer to Chapter 6, "The OPTMODEL Procedure."

PROC LP

The LP procedure solves linear and mixed integer programs with a primal simplex solver. It can perform several types of post-optimality analysis, including range analysis, sensitivity analysis, and parametric programming. The procedure can also be used interactively.

PROC LP requires a problem data set that contains the model. In addition, a primal and active data set can be used for warm starting a problem that has been partially solved previously.

The problem data describing the model can be in one of two formats: dense or sparse. The dense format represents the model as a rectangular coefficient matrix. For details see the section "Dense Format" on page 11. The sparse format, on the other hand, represents only the nonzero elements of a rectangular coefficient matrix. See the section "Sparse Format" on page 12 for more details.

For more details on PROC LP refer to Chapter 3, "The LP Procedure."

PROC INTPOINT

The INTPOINT procedure solves linear programming problems using the interior point algorithm.

The constraint data can be specified in either the sparse or dense input format. This is the same format that is used by PROC LP; therefore, any model-building techniques that apply to models for PROC LP also apply to PROC INTPOINT.

For more details on PROC INTPOINT refer to Chapter 2, "The INTPOINT Procedure."

Network Problems

PROC NETFLOW

The NETFLOW procedure solves network flow problems with linear side constraints using either a network simplex algorithm or an interior point algorithm. In addition, it can solve linear programming (LP) problems using the interior point algorithm.

Networks and the Network Simplex Algorithm

PROC NETFLOW's network simplex algorithm solves pure network flow problems and network flow problems with linear side constraints. The procedure accepts the network specification in formats that are particularly suited to networks. Although network problems could be solved by PROC LP, the NETFLOW procedure generally solves network flow problems more efficiently than PROC LP.

Network flow problems, such as finding the minimum cost flow in a network, require model representation in a format that is specialized for network structures. The network is represented in two data sets: a node data set that names the nodes in the network and gives supply and demand information at them, and an arc data set that defines the arcs in the network using the node names and gives arc costs and capacities. In addition, a side-constraint data set is included that gives any side constraints that apply to the flow through the network. Examples of these are found later in this chapter.

The constraint data can be specified in either the sparse or dense input format. This is the same format that is used by PROC LP; therefore, any model-building techniques that apply to models for PROC LP also apply to network flow models having side constraints.

Linear and Network Programs Solved by the Interior Point Algorithm

The data required by PROC NETFLOW for a linear program resemble the data for nonarc variables and constraints for constrained network problems. They are similar to the data required by PROC LP.

The LP representation requires a data set that defines the variables in the LP using variable names, and gives objective function coefficients and upper and lower bounds. In addition, a constraint data set can be included that specifies any constraints.

When solving a constrained network problem, you can specify the INTPOINT option to indicate that the interior point algorithm is to be used. The input data are the same whether the simplex or interior point method is used. The interior point method is often faster when problems have many side constraints.

The constraint data can be specified in either the sparse or dense input format. This is the same format that is used by PROC LP; therefore, any model-building techniques that apply to models for PROC LP also apply to LP models solved by PROC NETFLOW.

PROC INTPOINT

The INTPOINT procedure solves the Network Program with Side Constraints (NPSC) problem using the interior point algorithm.

The data required by PROC INTPOINT are similar to the data required by PROC NETFLOW when solving network flow models using the interior point algorithm.

The constraint data can be specified in either the sparse or dense input format. This is the same format that is used by PROC LP and PROC NETFLOW; therefore, any model-building techniques that apply to models for PROC LP or PROC NETFLOW also apply to PROC INTPOINT.

For more details on PROC INTPOINT refer to Chapter 2, "The INTPOINT Procedure."

Mixed Integer Linear Problems

PROC OPTMILP

The OPTMILP procedure solves general mixed integer linear programs (MILPs) —linear programs in which a subset of the decision variables are constrained to be integers. The OPTMILP procedure solves MILPs with an LP-based branch-and-bound algorithm augmented by advanced techniques such as cutting planes and primal heuristics.

The OPTMILP procedure requires a MILP to be specified using a SAS data set that adheres to the MPS format. See Chapter 14, "The MPS-Format SAS Data Set," for details about the MPS-format data set.

PROC OPTMODEL

PROC OPTMODEL provides a language for concisely modeling mixed integer linear programming problems. The language allows a model to be expressed in a form that matches the mathematical formulation. Within OPTMODEL you can declare a model, pass it directly to various solvers, and review the solver result. You can also save an instance of a mixed integer linear model in data set form for use by PROC OPTMILP. For more details, refer to Chapter 6, "The OPTMODEL Procedure."

PROC LP

The LP procedure solves MILPs with a primal simplex solver. To solve a MILP you need to identify the integer variables. You can do this with a row in the input data set that has the keyword INTEGER for the type variable. It is important to note that integer variables must have upper bounds explicitly defined.

As with linear programs, you can specify MIP problem data using sparse or dense format. For more details see Chapter 3, "The LP Procedure."

Quadratic Programming Problems

PROC OPTQP

The OPTQP procedure solves quadratic programs—problems with quadratic objective function and a collection of linear constraints, including general linear constraints along with lower and/or upper bounds on the decision variables.

You can specify the problem input data in one of two formats: QPS-format flat file or QPS-format SAS data set. For details on the QPS-format data specification, refer to Chapter 14, "The MPS-Format SAS Data Set." For more details on the OPTQP procedure, refer to Chapter 17, "The OPTQP Procedure."

PROC OPTMODEL

PROC OPTMODEL provides a language for concisely modeling quadratic programming problems. The language allows a model to be expressed in a form that matches the mathematical formulation. Within OPTMODEL you can declare a model, pass it directly to various solvers, and review the solver result. You can also save an instance of a quadratic model in data set form for use by PROC OPTQP. For more details, refer to Chapter 6, "The OPTMODEL Procedure."

Nonlinear Problems

PROC OPTMODEL

PROC OPTMODEL provides a language for concisely modeling nonlinear programming (NLP) problems. The language allows a model to be expressed in a form that matches the mathematical formulation. Within OPTMODEL you can declare a model, pass it directly to various solvers, and review the solver result. For more details, refer to Chapter 6, "The OPTMODEL Procedure."

You can solve the following types of nonlinear programming problems using PROC OPTMODEL:

- **Nonlinear objective function, linear constraints**: Invoke the constrained nonlinear programming (NLPC) solver. For more details about the NLPC solver, refer to Chapter 10, "The NLPC Nonlinear Optimization Solver."
- **Nonlinear objective function, nonlinear constraints**: Invoke the sequential programming (SQP) or interior point nonlinear programming (IPNLP) solver. For more details about the SQP solver, refer to Chapter 13, "The Sequential Quadratic Programming Solver." For more details about the IPNLP solver, refer to Chapter 7, "The Interior Point Nonlinear Programming Solver."
- **Nonlinear objective function, no constraints**: Invoke the unconstrained nonlinear programming (NLPU) solver. For more details about the NLPU solver, refer to Chapter 11, "The Unconstrained Nonlinear Programming Solver."

PROC NLP

The NLP procedure (**N**on**L**inear **P**rogramming) offers a set of optimization techniques for minimizing or maximizing a continuous nonlinear function subject to linear and nonlinear, equality and inequality, and lower and upper bound constraints. Problems of this type are found in many settings ranging from optimal control to maximum likelihood estimation.

Nonlinear programs can be input into the procedure in various ways. The objective, constraint, and derivative functions are specified using the programming statements of PROC NLP. In addition, information in SAS data sets can be used to define the structure of objectives and constraints, and to specify constants used in objectives, constraints, and derivatives.

PROC NLP uses the following data sets to input various pieces of information:

- The DATA= data set enables you to specify data shared by all functions involved in a least-squares problem.
- The INQUAD= data set contains the arrays appearing in a quadratic programming problem.
- The INEST= data set specifies initial values for the decision variables, the values of constants that are referred to in the program statements, and simple boundary and general linear constraints.
- The MODEL= data set specifies a model (functions, constraints, derivatives) saved at a previous execution of the NLP procedure.

As an alternative to supplying data in SAS data sets, some or all data for the model can be specified using SAS programming statements. These are similar to those used in the SAS DATA step.

For more details on PROC NLP refer to Chapter 4, “The NLP Procedure.”

Model Building

Model generation and maintenance are often difficult and expensive aspects of applying mathematical programming techniques. The flexible input formats for the optimization procedures in SAS/OR software simplify this task.

PROC LP

A small product-mix problem serves as a starting point for a discussion of different types of model formats supported in SAS/OR software.

A candy manufacturer makes two products: chocolates and toffee. What combination of chocolates and toffee should be produced in a day in order to maximize the company's profit? Chocolates contribute \$0.25 per pound to profit, and toffee contributes \$0.75 per pound. The decision variables are *chocolates* and *toffee*.

Four processes are used to manufacture the candy:

1. Process 1 combines and cooks the basic ingredients for both chocolates and toffee.
2. Process 2 adds colors and flavors to the toffee, then cools and shapes the confection.
3. Process 3 chops and mixes nuts and raisins, adds them to the chocolates, and then cools and cuts the bars.
4. Process 4 is packaging: chocolates are placed in individual paper shells; toffee is wrapped in cellophane packages.

During the day, there are 7.5 hours (27,000 seconds) available for each process.

Firm time standards have been established for each process. For Process 1, mixing and cooking take 15 seconds for each pound of chocolate, and 40 seconds for each pound of toffee. Process 2 takes 56.25 seconds per pound of toffee. For Process 3, each pound of chocolate requires 18.75 seconds of processing. In packaging, a pound of chocolates can be wrapped in 12 seconds, whereas a pound of toffee requires 50 seconds. These data are summarized as follows:

	Available Time	Required per Pound	
Process	(sec)	chocolates (sec)	toffee (sec)
1 Cooking	27,000	15	40
2 Color/Flavor	27,000		56.25
3 Condiments	27,000	18.75	
4 Packaging	27,000	12	50

The objective is to

Maximize: $0.25(chocolates) + 0.75(toffee)$

which is the company's total profit.

The production of the candy is limited by the time available for each process. The limits placed on production by Process 1 are expressed by the following inequality:

Process 1: $15(chocolates) + 40(toffee) \leq 27{,}000$

Process 1 can handle any combination of chocolates and toffee that satisfies this inequality.

The limits on production by other processes generate constraints described by the following inequalities:

Process 2: $56.25(toffee) \leq 27{,}000$

Process 3: $18.75(chocolates) \leq 27{,}000$

Process 4: $12(chocolates) + 50(toffee) \leq 27{,}000$

This linear program illustrates the type of problem known as a product mix example. The mix of products that maximizes the objective without violating the constraints is the solution. Two formats — dense or sparse — can be used to represent this model.

Dense Format

The following DATA step creates a SAS data set for this product mix problem. Notice that the values of CHOCO and TOFFEE in the data set are the coefficients of those variables in the equations corresponding to the objective function and constraints. The variable _id_ contains a character string that names the rows in the data set. The variable _type_ is a character variable that contains keywords that describe the type of each row in the problem data set. The variable _rhs_ contains the right-hand-side values.

```
data factory;
   input _id_ $ CHOCO TOFFEE _type_ $ _rhs_;
   datalines;
object      0.25    0.75   MAX      .
process1   15.00   40.00   LE   27000
process2    0.00   56.25   LE   27000
process3   18.75    0.00   LE   27000
process4   12.00   50.00   LE   27000
;
```

To solve this problem by using PROC LP, specify the following:

```
proc lp data = factory;
run;
```

You can also solve this problem by using PROC OPTLP. PROC OPTLP requires a linear program to be specified using a SAS data set that adheres to the MPS format, a widely accepted format in the optimization community. You can use the SAS macro %LP2MPSD to convert typical PROC LP format data sets into MPS-format SAS data sets. The macro is available online at the SAS Customer Support Center.

Sparse Format

Typically, mathematical programming models are sparse. That is, few of the coefficients in the constraint matrix are nonzero. The dense problem format shown in the previous section is an inefficient way to represent sparse models. The LP procedure also accepts data in a sparse input format. Only the nonzero coefficients must be specified. It is consistent with the standard MPS sparse format, and much more flexible; models using the MPS format can be easily converted to the LP format.

Although the factory example in the last section is not sparse, an example of the sparse input format for that problem is illustrated here. The sparse data set has four variables: a row type identifying variable (_type_), a row name variable (_row_), a column name variable (_col_), and a coefficient variable (_coef_).

```
data sp_factory;
   format _type_ $8. _row_ $10. _col_ $10.;
   input _type_ $_row_ $ _col_ $ _coef_ ;
   datalines;
max       object     .            .
.         object     chocolate    .25
.         object     toffee       .75
le        process1   .            .
.         process1   chocolate    15
.         process1   toffee       40
.         process1   _RHS_        27000
le        process2   .            .
.         process2   toffee       56.25
.         process2   _RHS_        27000
le        process3   .            .
.         process3   chocolate    18.75
.         process3   _RHS_        27000
le        process4   .            .
.         process4   chocolate    12
.         process4   toffee       50
.         process4   _RHS_        27000
;
```

To solve this problem using PROC LP specify the following:

```
proc lp data = sp_factory
   sparsecondata;
run;
```

The Solution Summary (shown in Figure 1.1) gives information about the solution that was found, including whether the optimizer terminated successfully after finding the optimum.

When PROC LP solves a problem, it uses an iterative process. First, the procedure finds a feasible solution that satisfies the constraints. Second, it finds the optimal solution from the set of feasible solutions. The Solution Summary lists the number of iterations in each of these phases, the number of variables in the initial feasible

solution, the time the procedure required to solve the problem, and the number of matrix inversions necessary.

```
                        The LP Procedure

                        Solution Summary

                     Terminated Successfully

Objective Value                                  475

Phase 1 Iterations                                 0
Phase 2 Iterations                                 3
Phase 3 Iterations                                 0
Integer Iterations                                 0
Integer Solutions                                  0
Initial Basic Feasible Variables                   6
Time Used (seconds)                                0
Number of Inversions                               3

Epsilon                                         1E-8
Infinity                                1.797693E308
Maximum Phase 1 Iterations                       100
Maximum Phase 2 Iterations                       100
Maximum Phase 3 Iterations                  99999999
Maximum Integer Iterations                       100
Time Limit (seconds)                             120
```

Figure 1.1. Solution Summary

After performing three Phase 2 iterations, the procedure terminated successfully with an optimal objective value of 475.

Separating the Data from the Model Structure

It is often desirable to keep the data separate from the structure of the model. This is useful for large models with numerous identifiable components. The data are best organized in rectangular tables that can be easily examined and modified. Then, before the problem is solved, the model is built using the stored data. This process of model building is known as *matrix generation*. In conjunction with the sparse format, the SAS DATA step provides a good matrix generation language.

For example, consider the candy manufacturing example introduced previously. Suppose that, for the user interface, it is more convenient to organize the data so that each record describes the information related to each product (namely, the contribution to the objective function and the unit amount needed for each process). A DATA step for saving the data might look like this:

```
data manfg;
   format product $12.;
   input product $ object process1 - process4 ;
   datalines;
chocolate         .25    15  0.00 18.75    12
toffee            .75    40 56.25  0.00    50
licorice         1.00    29 30.00 20.00    20
jelly_beans       .85    10  0.00 30.00    10
```

```
_RHS_              .    27000 27000 27000 27000
;
```

Notice that there is a special record at the end having product _RHS_. This record gives the amounts of time available for each of the processes. This information could have been stored in another data set. The next example illustrates a model where the data are stored in separate data sets.

Building the model involves adding the data to the structure. There are as many ways to do this as there are programmers and problems. The following DATA step shows one way to use the candy data to build a sparse format model to solve the product mix problem.

```
data model;
   array process object process1-process4;
   format _type_ $8. _row_ $12. _col_ $12. ;
   keep _type_ _row_ _col_ _coef_;

   set manfg;          /* read the manufacturing data */

   /* build the object function */

   if _n_=1 then do;
      _type_='max'; _row_='object'; _col_=' '; _coef_=.;
      output;
   end;

   /* build the constraints */

   do over process;
      if _i_>1 then do;
         _type_='le'; _row_='process'||put(_i_-1,1.);
      end;
      else             _row_='object';
      _col_=product; _coef_=process;
      output;
   end;
run;
```

The sparse format data set is shown in Figure 1.2.

Obs	_type_	_row_	_col_	_coef_
1	max	object		.
2	max	object	chocolate	0.25
3	le	process1	chocolate	15.00
4	le	process2	chocolate	0.00
5	le	process3	chocolate	18.75
6	le	process4	chocolate	12.00
7		object	toffee	0.75
8	le	process1	toffee	40.00
9	le	process2	toffee	56.25
10	le	process3	toffee	0.00
11	le	process4	toffee	50.00
12		object	licorice	1.00
13	le	process1	licorice	29.00
14	le	process2	licorice	30.00
15	le	process3	licorice	20.00
16	le	process4	licorice	20.00
17		object	jelly_beans	0.85
18	le	process1	jelly_beans	10.00
19	le	process2	jelly_beans	0.00
20	le	process3	jelly_beans	30.00
21	le	process4	jelly_beans	10.00
22		object	_RHS_	.
23	le	process1	_RHS_	27000.00
24	le	process2	_RHS_	27000.00
25	le	process3	_RHS_	27000.00
26	le	process4	_RHS_	27000.00

Figure 1.2. Sparse Data Format

The model data set looks a little different from the sparse representation of the candy model shown earlier. It not only includes additional products (licorice and jelly beans), but it also defines the model in a different order. Since the sparse format is robust, the model can be generated in ways that are convenient for the DATA step program.

If the problem had more products, you could increase the size of the manfg data set to include the new product data. Also, if the problem had more than four processes, you could add the new process variables to the manfg data set and increase the size of the process array in the model data set. With these two simple changes and additional data, a product mix problem having hundreds of processes and products can be solved.

PROC NETFLOW

Network flow problems can be described by specifying the nodes in the network and their supplies and demands, and the arcs in the network and their costs, capacities, and lower flow bounds. Consider the simple transshipment problem in Figure 1.3 as an illustration.

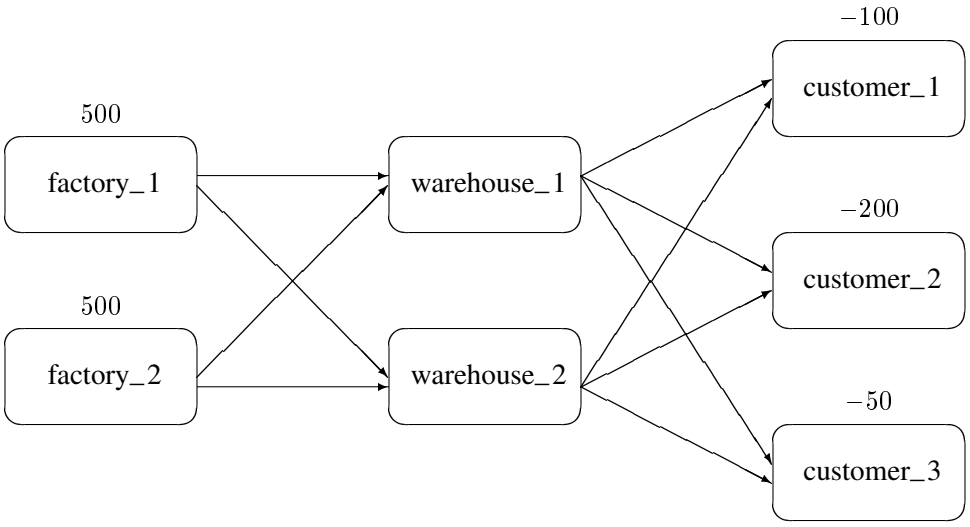


Figure 1.3. Transshipment Problem

Suppose the candy manufacturing company has two factories, two warehouses, and three customers for chocolate. The two factories each have a production capacity of 500 pounds per day. The three customers have demands of 100, 200, and 50 pounds per day, respectively.

The following data set describes the supplies (positive values for the supdem variable) and the demands (negative values for the supdem variable) for each of the customers and factories.

```
data nodes;
   format node $10. ;
   input node $ supdem;
   datalines;
customer_1   -100
customer_2   -200
customer_3    -50
factory_1     500
factory_2     500
;
```

Suppose that there are two warehouses that are used to store the chocolate before shipment to the customers, and that there are different costs for shipping between each factory, warehouse, and customer. What is the minimum cost routing for supplying the customers?

Arcs are described in another data set. Each observation defines a new arc in the network and gives data about the arc. For example, there is an arc between the node factory_1 and the node warehouse_1. Each unit of flow on that arc costs 10.

Although this example does not include it, lower and upper bounds on the flow across that arc can be listed here.

```
data network;
   format from $12. to $12.;
   input from $ to $ cost ;
   datalines;
factory_1     warehouse_1  10
factory_2     warehouse_1   5
factory_1     warehouse_2   7
factory_2     warehouse_2   9
warehouse_1   customer_1    3
warehouse_1   customer_2    4
warehouse_1   customer_3    4
warehouse_2   customer_1    5
warehouse_2   customer_2    5
warehouse_2   customer_3    6
;
```

You can use PROC NETFLOW to find the minimum cost routing. This procedure takes the model as defined in the network and nodes data sets and finds the minimum cost flow.

```
proc netflow arcout=arc_sav
             arcdata=network nodedata=nodes;
   node node;      /* node data set information */
   supdem supdem;
   tail from;      /* arc data set information */
   head to;
   cost cost;
   run;

proc print;
   var from to cost _capac_ _lo_ _supply_ _demand_
       _flow_ _fcost_ _rcost_;
   sum _fcost_;
   run;
```

PROC NETFLOW produces the following messages in the SAS log:

```
NOTE: Number of nodes= 7 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 3 .
NOTE: Total supply= 1000 , total demand= 350 .
NOTE: Number of arcs= 10 .
NOTE: Number of iterations performed (neglecting
      any constraints)= 7 .
NOTE: Of these, 2 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 3050 .
NOTE: The data set WORK.ARC_SAV has 10 observations
      and 13 variables.
```

The solution (Figure 1.4) saved in the arc_sav data set shows the optimal amount of chocolate to send across each arc (the amount to ship from each factory to each warehouse and from each warehouse to each customer) in the network per day.

Obs	from	to	cost	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_COST_	_RCOST_
1	warehouse_1	customer_1	3	99999999	0	.	100	100	300	.
2	warehouse_2	customer_1	5	99999999	0	.	100	0	0	4
3	warehouse_1	customer_2	4	99999999	0	.	200	200	800	.
4	warehouse_2	customer_2	5	99999999	0	.	200	0	0	3
5	warehouse_1	customer_3	4	99999999	0	.	50	50	200	.
6	warehouse_2	customer_3	6	99999999	0	.	50	0	0	4
7	factory_1	warehouse_1	10	99999999	0	500	.	0	0	5
8	factory_2	warehouse_1	5	99999999	0	500	.	350	1750	.
9	factory_1	warehouse_2	7	99999999	0	500	.	0	0	.
10	factory_2	warehouse_2	9	99999999	0	500	.	0	0	2
									====	
									3050	

Figure 1.4. ARCOUT Data Set

Notice which arcs have positive flow (_FLOW_ is greater than 0). These arcs indicate the amount of chocolate that should be sent from factory_2 to warehouse_1 and from there to the three customers. The model indicates no production at factory_1 and no use of warehouse_2.

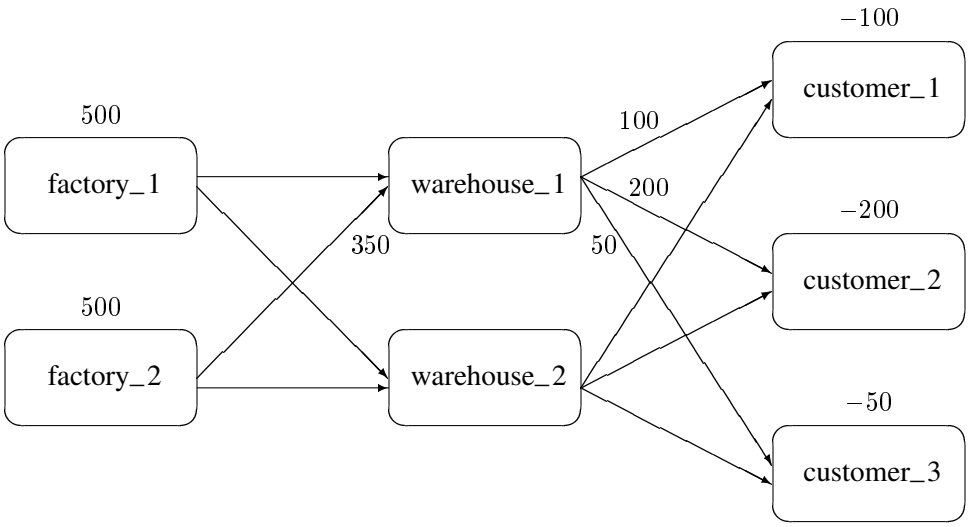


Figure 1.5. Optimal Solution for the Transshipment Problem

PROC OPTMODEL

Modeling a Linear Programming Problem

Consider the candy manufacturer's problem described in the section "PROC LP" on page 10. You can formulate the problem using PROC OPTMODEL and solve it using the primal simplex solver as follows:

```
proc optmodel;

  /* declare variables */
  var choco, toffee;

  /* maximize objective function (profit) */
  maximize profit = 0.25*choco + 0.75*toffee;

  /* subject to constraints */
  con process1: 15*choco + 40*toffee    <= 27000;
  con process2:            56.25*toffee <= 27000;
  con process3: 18.75*choco             <= 27000;
  con process4: 12*choco + 50*toffee    <= 27000;

  /* solve LP using primal simplex solver */
  solve with lp / solver = primal_spx;

  /* display solution */
  print choco toffee;
quit;
```

The optimal objective value and the optimal solution are displayed in the following summary output:

```
                Proc OPTMODEL

               Solver Results

        Technique          Primal Simplex
        Status                    Optimal
        Iterations                      2
        Objective                  profit
        Objective Value               475


                 choco    toffee

                  1000       300
```

You can observe from the preceding example that PROC OPTMODEL provides an easy and intuitive way of modeling and solving mathematical programming models.

Modeling a Nonlinear Programming Problem

The following optimization problem illustrates how you can use some features of PROC OPTMODEL to formulate and solve nonlinear programming problems. The objective of the problem is to find coefficients for an approximation function that matches the values of a given function, $f(x)$, at a set of points P. The approximation is a rational function with degree d in the numerator and denominator:

$$r(x) = \frac{\alpha_0 + \sum_{i=1}^{d} \alpha_i x^i}{\beta_0 + \sum_{i=1}^{d} \beta_i x^i}$$

The problem can be formulated by minimizing the sum of squared errors at each point in P:

$$\min \quad \sum_{x \in P} [r(x) - f(x)]^2$$

The following code implements this model. The function $f(x) = 2^x$ is approximated over a set of points P in the range 0 to 1. The function values are saved in a data set that is used by PROC OPTMODEL to set model parameters:

```
data points;
   /* generate data points */
   keep f x;
   do i = 0 to 100;
      x = i/100;
      f = 2**x;
      output;
   end;

proc optmodel;
   /* declare, read, and save our data points */
   set points;
   number f{points};
   read data points into points = [x] f;

   /* declare variables and model parameters */
   number d=1; /* linear polynomial */
   var a{0..d};
   var b{0..d} init 1;
   constraint fixb0: b[0] = 1;

   /* minimize sum of squared errors */
   min z=sum{x in points}
               ((a[0] + sum{i in 1..d} a[i]*x**i) /
                (b[0] + sum{i in 1..d} b[i]*x**i) - f[x])**2;

   /* solve and show coefficients */
   solve;
   print a b;
   quit;
```

The expression for the objective z is defined using operators that parallel the mathematical form. In this case the polynomials in the rational function are linear, so d is equal to 1.

The constraint fixb0 forces the constant term of the rational function denominator, b[0], to equal 1. This causes the resulting coefficients to be normalized. The OPTMODEL presolver preprocesses the problem to remove the constraint. An unconstrained solver is used after substituting for b[0].

The SOLVE statement selects a solver, calls it, and displays the status. The PRINT command then prints the values of coefficient arrays a and b:

```
          Proc OPTMODEL

         Solver Results

Technique                  L-BFGS
Status                     Normal
Iterations                     10
Objective                       z
Objective Value        0.0000591


 [1]              a              b

  0         0.99817        1.00000
  1         0.42064       -0.29129
```

The approximation for $f(x) = 2^x$ between 0 and 1 is therefore

$$f_{\text{approx}}(x) = \frac{0.99817 + 0.42064x}{1 - 0.29129x}$$

Matrix Generation

It is desirable to keep data in separate tables, and then to automate model building and reporting. This example illustrates a problem that has elements of both a product mix problem and a blending problem. Suppose four kinds of ties are made: all silk, all polyester, a 50-50 polyester-cotton blend, and a 70-30 cotton-polyester blend.

The data include cost and supplies of raw material, selling price, minimum contract sales, maximum demand of the finished products, and the proportions of raw materials that go into each product. The objective is to find the product mix that maximizes profit.

The data are saved in three SAS data sets. The program that follows demonstrates one way for these data to be saved.

```
data material;
   format descpt $20.;
   input descpt $ cost supply;
   datalines;
silk_material             .21   25.8
polyester_material        .6    22.0
cotton_material           .9    13.6
;

data tie;
   format descpt $20.;
   input descpt $ price contract demand;
   datalines;
all_silk                 6.70    6.0    7.00
all_polyester            3.55   10.0   14.00
poly_cotton_blend        4.31   13.0   16.00
cotton_poly_blend        4.81    6.0    8.50
;

data manfg;
   format descpt $20.;
   input descpt $ silk poly cotton;
   datalines;
all_silk                  100     0      0
all_polyester               0   100      0
poly_cotton_blend           0    50     50
cotton_poly_blend           0    30     70
;
```

The following program takes the raw data from the three data sets and builds a linear program model in the data set called model. Although it is designed for the three-resource, four-product problem described here, it can easily be extended to include more resources and products. The model-building DATA step remains essentially the same; all that changes are the dimensions of loops and arrays. Of course, the data tables must expand to accommodate the new data.

```
data model;
   array  raw_mat {3} $ 20 ;
   array  raw_comp {3} silk poly cotton;
   length _type_ $ 8  _col_ $ 20  _row_ $ 20 _coef_ 8 ;
   keep   _type_      _col_       _row_      _coef_ ;

   /* define the objective, lower, and upper bound rows */

   _row_='profit'; _type_='max';     output;
   _row_='lower';  _type_='lowerbd'; output;
   _row_='upper';  _type_='upperbd'; output;
   _type_=' ';

   /* the object and upper rows for the raw materials */

   do i=1 to 3;
      set material;
      raw_mat[i]=descpt;  _col_=descpt;
      _row_='profit';    _coef_=-cost;  output;
      _row_='upper';     _coef_=supply; output;
   end;

   /* the object, upper, and lower rows for the products */

   do i=1 to 4;
      set tie;
      _col_=descpt;
      _row_='profit'; _coef_=price;    output;
      _row_='lower';  _coef_=contract; output;
      _row_='upper';  _coef_=demand;   output;
   end;

   /* the coefficient matrix for manufacturing */

   _type_='eq';
   do i=1 to 4;          /* loop for each raw material */
      set manfg;
      do j=1 to 3;       /* loop for each product      */

         _col_=descpt;   /* % of material in product   */
         _row_  = raw_mat[j];
         _coef_ = raw_comp[j]/100;
         output;

         _col_  = raw_mat[j];  _coef_ = -1;
         output;

         /* the right-hand side */

         if i=1 then do;
            _col_='_RHS_';
            _coef_=0;
            output;
         end;
```

```
      end;
      _type_='   ';
   end;
   stop;
run;
```

The model is solved using PROC LP, which saves the solution in the PRIMALOUT data set named solution. PROC PRINT displays the solution, shown in Figure 1.6.

```
proc lp sparsedata primalout=solution;

proc print ;
   id _var_;
   var _lbound_--_r_cost_;
run;
```

VAR	_LBOUND_	_VALUE_	_UBOUND_	_PRICE_	_R_COST_
all_polyester	10	11.800	14.0	3.55	0.000
all_silk	6	7.000	7.0	6.70	6.490
cotton_material	0	13.600	13.6	-0.90	4.170
cotton_poly_blend	6	8.500	8.5	4.81	0.196
polyester_material	0	22.000	22.0	-0.60	2.950
poly_cotton_blend	13	15.300	16.0	4.31	0.000
silk_material	0	7.000	25.8	-0.21	0.000
PHASE_1_OBJECTIVE	0	0.000	0.0	0.00	0.000
profit	0	168.708	1.7977E308	0.00	0.000

Figure 1.6. Solution Data Set

The solution shows that 11.8 units of polyester ties, 7 units of silk ties, 8.5 units of the cotton-polyester blend, and 15.3 units of the polyester-cotton blend should be produced. It also shows the amounts of raw materials that go into this product mix to generate a total profit of 168.708.

Exploiting Model Structure

Another example helps to illustrate how the model can be simplified by exploiting the structure in the model when using the NETFLOW procedure.

Recall the chocolate transshipment problem discussed previously. The solution required no production at factory_1 and no storage at warehouse_2. Suppose this solution, although optimal, is unacceptable. An additional constraint requiring the production at the two factories to be balanced is needed. Now, the production at the two factories can differ by, at most, 100 units. Such a constraint might look like this:

```
-100 <= (factory_1_warehouse_1 + factory_1_warehouse_2 -
         factory_2_warehouse_1 - factory_2_warehouse_2) <= 100
```

The network and supply and demand information are saved in the following two data sets:

```
data network;
   format from $12. to $12.;
   input  from $ to $ cost ;
   datalines;
factory_1    warehouse_1  10
factory_2    warehouse_1   5
factory_1    warehouse_2   7
factory_2    warehouse_2   9
warehouse_1  customer_1   3
warehouse_1  customer_2   4
warehouse_1  customer_3   4
warehouse_2  customer_1   5
warehouse_2  customer_2   5
warehouse_2  customer_3   6
;

data nodes;
   format node $12. ;
   input node $  supdem;
   datalines;
customer_1   -100
customer_2   -200
customer_3    -50
factory_1     500
factory_2     500
;
```

The factory-balancing constraint is not a part of the network. It is represented in the sparse format in a data set for side constraints.

```
data side_con;
   format _type_ $8. _row_ $8. _col_ $21. ;
   input  _type_  _row_  _col_  _coef_ ;
   datalines;
eq       balance   .                        .
.        balance  factory_1_warehouse_1     1
.        balance  factory_1_warehouse_2     1
.        balance  factory_2_warehouse_1    -1
.        balance  factory_2_warehouse_2    -1
.        balance  diff                     -1
lo       lowerbd  diff                   -100
up       upperbd  diff                    100
;
```

This data set contains an equality constraint that sets the value of DIFF to be the amount that factory 1 production exceeds factory 2 production. It also contains implicit bounds on the DIFF variable. Note that the DIFF variable is a nonarc variable.

You can use the following call to PROC NETFLOW to solve the problem:

```
proc netflow
   conout=con_sav
```

```
      arcdata=network nodedata=nodes condata=side_con
      sparsecondata ;
      node node;
      supdem supdem;
      tail from;
      head to;
      cost cost;
      run;

   proc print;
      var from to _name_ cost _capac_ _lo_ _supply_ _demand_
          _flow_ _fcost_ _rcost_;
      sum _fcost_;
      run;
```

The solution is saved in the con_sav data set, as displayed in Figure 1.7.

Obs	from	to	_NAME_	cost	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_	_RCOST_
1	warehouse_1	customer_1		3	99999999	0	.	100	100	300	.
2	warehouse_2	customer_1		5	99999999	0	.	100	0	0	1.0
3	warehouse_1	customer_2		4	99999999	0	.	200	75	300	.
4	warehouse_2	customer_2		5	99999999	0	.	200	125	625	.
5	warehouse_1	customer_3		4	99999999	0	.	50	50	200	.
6	warehouse_2	customer_3		6	99999999	0	.	50	0	0	1.0
7	factory_1	warehouse_1		10	99999999	0	500	.	0	0	2.0
8	factory_2	warehouse_1		5	99999999	0	500	.	225	1125	.
9	factory_1	warehouse_2		7	99999999	0	500	.	125	875	.
10	factory_2	warehouse_2		9	99999999	0	500	.	0	0	5.0
11			diff	0	100	-100	.	.	-100	0	1.5
										3425	

Figure 1.7. CON_SAV Data Set

Notice that the solution now has production balanced across the factories; the production at factory 2 exceeds that at factory 1 by 100 units.

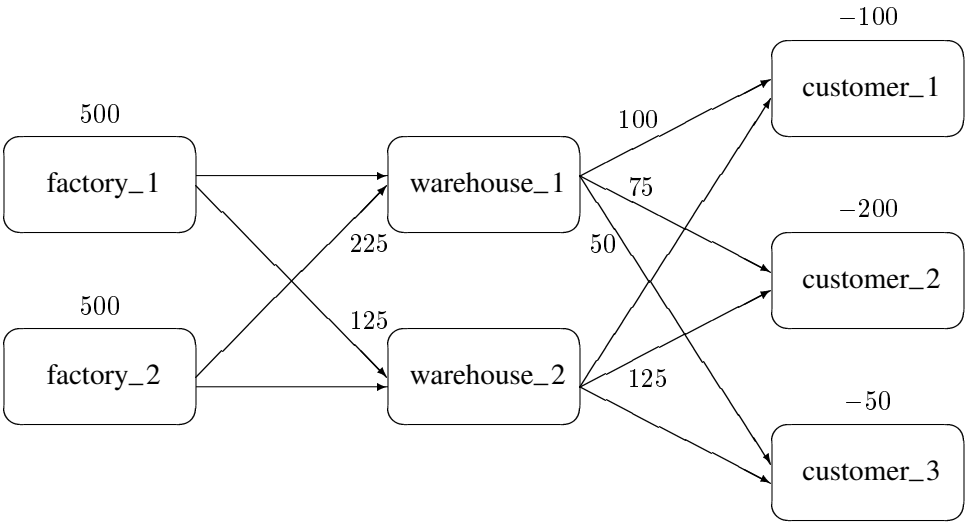


Figure 1.8. Constrained Optimum for the Transshipment Problem

Report Writing

The reporting of the solution is also an important aspect of modeling. Since the optimization procedures save the solution in one or more SAS data sets, reports can be written using any of the tools in the SAS language.

The DATA Step

Use of the DATA step and PROC PRINT is the most common way to produce reports. For example, from the data set solution shown in Figure 1.6, a table showing the revenue of the optimal production plan and a table of the cost of material can be produced with the following program.

```
data product(keep= _var_ _value_ _price_ revenue)
     material(keep=_var_ _value_ _price_ cost);
   set solution;
   if _price_>0 then do;
      revenue=_price_*_value_; output product;
   end;
   else if _price_<0 then do;
      _price_=-_price_;
      cost = _price_*_value_; output material;
   end;
run;


/* display the product report */

proc print data=product;
   id _var_;
   var _value_ _price_ revenue ;
   sum revenue;
   title 'Revenue Generated from Tie Sales';
run;

/* display the materials report */

proc print data=material;
   id _var_;
   var _value_ _price_ cost;
   sum cost;
   title 'Cost of Raw Materials';
run;
```

This DATA step reads the solution data set saved by PROC LP and segregates the records based on whether they correspond to materials or products—namely whether the contribution to profit is positive or negative. Each of these is then displayed to produce Figure 1.9.

Revenue Generated from Tie Sales

VAR	_VALUE_	_PRICE_	revenue
all_polyester	11.8	3.55	41.890
all_silk	7.0	6.70	46.900
cotton_poly_blend	8.5	4.81	40.885
poly_cotton_blend	15.3	4.31	65.943
			=======
			195.618

Cost of Raw Materials

VAR	_VALUE_	_PRICE_	cost
cotton_material	13.6	0.90	12.24
polyester_material	22.0	0.60	13.20
silk_material	7.0	0.21	1.47
			=====
			26.91

Figure 1.9. Tie Problem: Revenues and Costs

Other Reporting Procedures

The GCHART procedure can be a useful tool for displaying the solution to mathematical programming models. The con_solv data set that contains the solution to the balanced transshipment problem can be effectively displayed using PROC GCHART. In Figure 1.10, the amount that is shipped from each factory and warehouse can be seen by submitting the following SAS code:

```
title;
proc gchart data=con_sav;
   hbar from / sumvar=_flow_;
run;
```

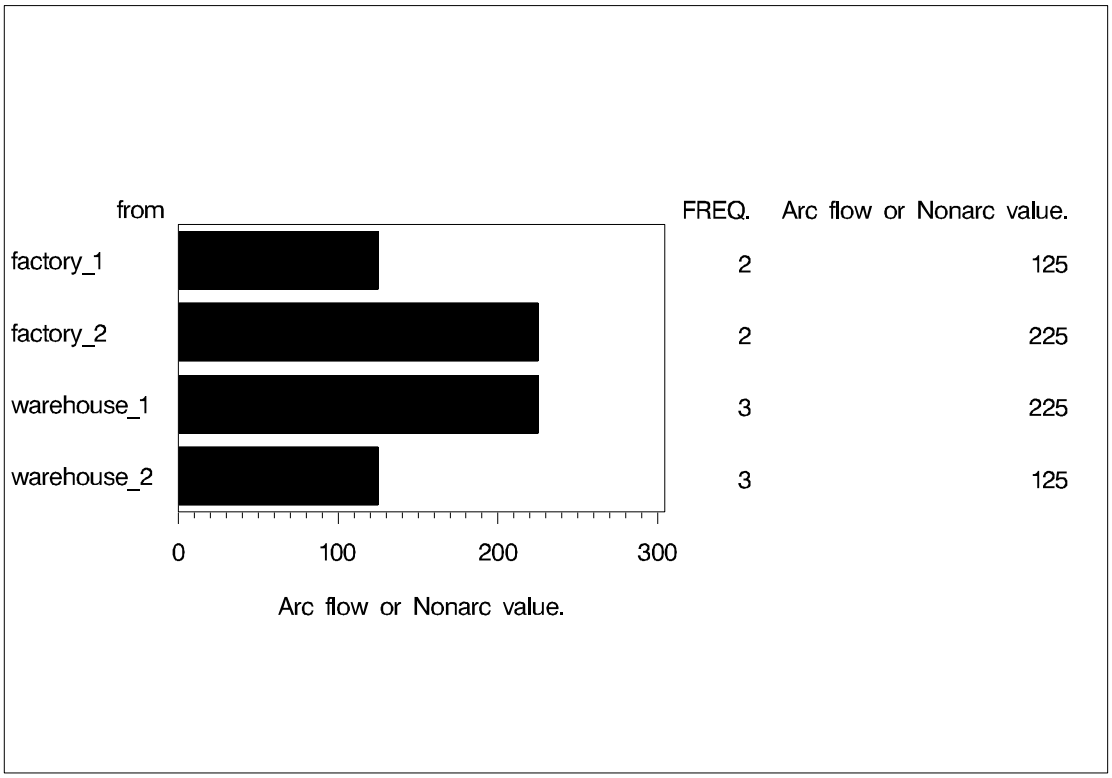


Figure 1.10. Tie Problem: Throughputs

The horizontal bar chart is just one way of displaying the solution to a mathematical program. The solution to the Tie Product Mix problem that was solved using PROC LP can also be illustrated using PROC GCHART. Here, a pie chart shows the relative contribution of each product to total revenues.

```
proc gchart data=product;
   pie _var_ / sumvar=revenue;
title 'Projected Tie Sales Revenue';
run;
```

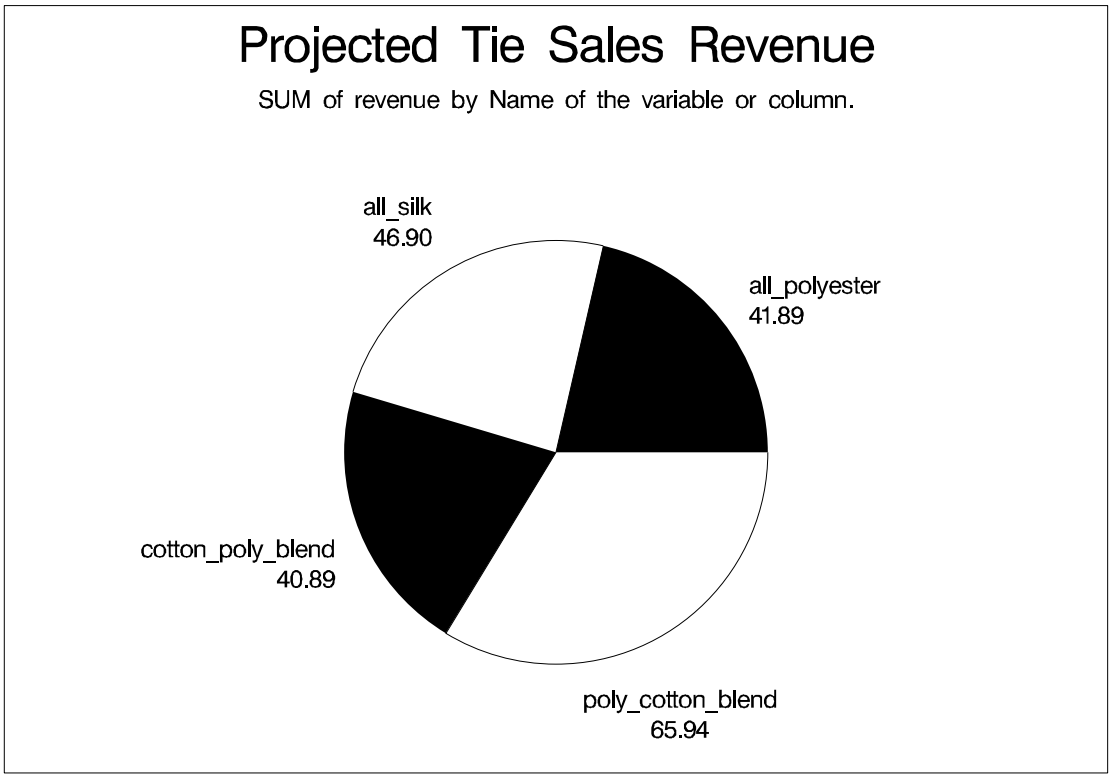


Figure 1.11. Tie Problem: Projected Tie Sales Revenue

The TABULATE procedure is another procedure that can help automate solution reporting. Several examples in Chapter 3, "The LP Procedure," illustrate its use.

References

IBM (1988), *Mathematical Programming System Extended/370 (MPSX/370) Version 2 Program Reference Manual*, volume SH19-6553-0, IBM.

Murtagh, B. A. (1981), *Advanced Linear Programming, Computation and Practice*, New York: McGraw-Hill.

Rosenbrock, H. H. (1960), "An Automatic Method for Finding the Greatest or Least Value of a Function," *Computer Journal*, 3, 175–184.

Chapter 2
The INTPOINT Procedure

Chapter Contents

Chapter 2
The INTPOINT Procedure

Overview: INTPOINT Procedure

The INTPOINT procedure solves the Network Program with Side Constraints (NPSC) problem (defined in the section "Mathematical Description of NPSC" on page 36) and the more general Linear Programming (LP) problem (defined in the section "Mathematical Description of LP" on page 38). NPSC and LP models can be used to describe a wide variety of real-world applications ranging from production, inventory, and distribution problems to financial applications.

Whether your problem is NPSC or LP, PROC INTPOINT uses the same optimization algorithm, the interior point algorithm. This algorithm is outlined in the section "The Interior Point Algorithm" on page 38.

While many of your problems may best be formulated as LP problems, there may be other instances when your problems are better formulated as NPSC problems. The section "Network Models" on page 46 describes typical models that have a network component and suggests reasons why NPSC may be preferable to LP. The section "Getting Started: NPSC Problems" on page 54 outlines how you supply data of any NPSC problem to PROC INTPOINT and call the procedure. After it reads the NPSC data, PROC INTPOINT converts the problem into an equivalent LP problem, performs interior point optimization, then converts the solution it finds back into a form you can use as the optimum to the original NPSC model.

If your model is an LP problem, the way you supply the data to PROC INTPOINT and run the procedure is described in the section "Getting Started: LP Problems" on page 61.

You can also solve LP problems by using PROC OPTLP. PROC OPTLP requires a linear program to be specified using a SAS data set that adheres to the MPS format, a widely accepted format in the optimization community. You can use the SAS macro %LP2MPSD to convert typical PROC LP format data sets into MPS-format SAS data sets. The macro is available online at the SAS Customer Support Center.

The remainder of this chapter is organized as follows:

- The section "Typical PROC INTPOINT Run" on page 69 describes how to use this procedure.
- The section "Syntax: INTPOINT Procedure" on page 70 describes all the statements and options of PROC INTPOINT.
- The section "Functional Summary" on page 71 lists the statements and options that can be used to control PROC INTPOINT.
- The section "Details: INTPOINT Procedure" on page 100 contains detailed explanations, descriptions, and advice on the use and behavior of the procedure.

- PROC INTPOINT is demonstrated by solving several examples in the section "Examples: INTPOINT Procedure" on page 132.

Mathematical Description of NPSC

A network consists of a collection of nodes joined by a collection of arcs. The arcs connect nodes and convey flow of one or more commodities that are supplied at supply nodes and demanded at demand nodes in the network. Each arc has a cost per unit of flow, a flow capacity, and a lower flow bound associated with it. An important concept in network modeling is *conservation of flow*. Conservation of flow means that the total flow in arcs directed toward a node, plus the supply at the node, minus the demand at the node, equals the total flow in arcs directed away from the node.

Often all the details of a problem cannot be specified in a network model alone. In many of these cases, these details can be represented by the addition of side constraints to the model. Side constraints are linear functions of arc variables (variables containing flow through an arc) and nonarc variables (variables that are not part of the network). The data for a side constraint consist of coefficients of arcs and coefficients of nonarc variables, a constraint type (that is, $\leq$, $=$, or $\geq$) and a right-hand-side value (rhs). A nonarc variable has a name, an objective function coefficient analogous to an arc cost, an upper bound analogous to an arc capacity, and a lower bound analogous to an arc lower flow bound.

If a network component of NPSC is removed by merging arcs and nonarc variables into a single set of variables, and if the flow conservation constraints and side constraints are merged into a single set of constraints, the result is an LP problem. PROC INTPOINT will automatically transform an NPSC problem into an equivalent LP problem, perform the optimization, then transform the problem back into its original form. By doing this, PROC INTPOINT finds the flow through the network and the values of any nonarc variables that minimize the total cost of the solution. Flow conservation is met, flow through each arc is on or between the arc's lower flow bound and capacity, the value of each nonarc variable is on or between the nonarc's lower and upper bounds, and the side constraints are satisfied.

Note that, since many LPs have large embedded networks, PROC INTPOINT is an attractive alternative to the LP procedure in many cases. Rather than formulating all problems as LPs, network models remain conceptually easy since they are based on network diagrams that represent the problem pictorially. PROC INTPOINT accepts the network specification in a format that is particularly suited to networks. This not only simplifies problem description but also aids in the interpretation of the solution. The conversion to and from the equivalent LP is done "behind the scenes" by the procedure.

If a network programming problem with side constraints has n nodes, a arcs, g nonarc variables, and k side constraints, then the formal statement of the problem solved by PROC INTPOINT is

$$\begin{array}{ll} \text{minimize} & c^T x + d^T z \\ \text{subject to} & Fx = b \\ & Hx + Qz \{\geq, =, \leq\} r \\ & l \leq x \leq u \\ & m \leq z \leq v \end{array}$$

where

- c is the $a \times 1$ arc variable objective function coefficient vector (the cost vector)
- x is the $a \times 1$ arc variable value vector (the flow vector)
- d is the $g \times 1$ nonarc variable objective function coefficient vector
- z is the $g \times 1$ nonarc variable value vector
- F is the $n \times a$ node-arc incidence matrix of the network, where

$$F_{i,j} = \begin{cases} -1, & \text{if arc } j \text{ is directed from node } i \\ 1, & \text{if arc } j \text{ is directed toward node } i \\ 0, & \text{otherwise} \end{cases}$$

- b is the $n \times 1$ node supply/demand vector, where

$$b_i = \begin{cases} s, & \text{if node } i \text{ has supply capability of } s \text{ units of flow} \\ -d, & \text{if node } i \text{ has demand of } d \text{ units of flow} \\ 0, & \text{if node } i \text{ is a transshipment node} \end{cases}$$

- H is the $k \times a$ side constraint coefficient matrix for arc variables, where $H_{i,j}$ is the coefficient of arc j in the ith side constraint
- Q is the $k \times g$ side constraint coefficient matrix for nonarc variables, where $Q_{i,j}$ is the coefficient of nonarc j in the ith side constraint
- r is the $k \times 1$ side constraint right-hand-side vector
- l is the $a \times 1$ arc lower flow bound vector
- u is the $a \times 1$ arc capacity vector
- m is the $g \times 1$ nonarc variable lower bound vector
- v is the $g \times 1$ nonarc variable upper bound vector

The INTPOINT procedure can also be used to solve an unconstrained network problem, that is, one in which H, Q, d, r, and z do not exist. It can also be used to solve a network problem with side constraints but no nonarc variables, in which case Q, d, and z do not exist.

Mathematical Description of LP

A linear programming (LP) problem has a linear objective function and a collection of linear constraints. PROC INTPOINT finds the values of variables that minimize the total cost of the solution. The value of each variable is on or between the variable's lower and upper bounds, and the constraints are satisfied.

If an LP has g variables and k constraints, then the formal statement of the problem solved by PROC INTPOINT is

$$\begin{array}{ll} \text{minimize} & d^T z \\ \text{subject to} & Qz \{\geq, =, \leq\} r \\ & m \leq z \leq v \end{array}$$

where

- d is the $g \times 1$ variable objective function coefficient vector
- z is the $g \times 1$ variable value vector
- Q is the $k \times g$ constraint coefficient matrix for the variables, where $Q_{i,j}$ is the coefficient of variable j in the ith constraint
- r is the $k \times 1$ side constraint right-hand-side vector
- m is the $g \times 1$ variable lower bound vector
- v is the $g \times 1$ variable upper bound vector

The Interior Point Algorithm

The simplex algorithm, developed shortly after World War II, was for many years the main method used to solve linear programming problems. Over the last fifteen years, however, the interior point algorithm has been developed. This algorithm also solves linear programming problems. From the start it showed great theoretical promise, and considerable research in the area resulted in practical implementations that performed competitively with the simplex algorithm. More recently, interior point algorithms have evolved to become superior to the simplex algorithm, in general, especially when the problems are large.

There are many variations of interior point algorithms. PROC INTPOINT uses the Primal-Dual with Predictor-Corrector algorithm. More information on this particular algorithm and related theory can be found in the texts by Roos, Terlaky, and Vial (1997), Wright (1996), and Ye (1996).

Interior Point Algorithmic Details

After preprocessing, the linear program to be solved is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

This is the *primal* problem. The matrices d, z, and Q of NPSC have been renamed c, x, and A, respectively, as these symbols are by convention used more, the problem to be solved is different from the original because of preprocessing, and there has been a change of primal variable to transform the LP into one whose variables have zero lower bounds. To simplify the algebra here, assume that variables have infinite upper bounds, and constraints are equalities. (Interior point algorithms do efficiently handle finite upper bounds, and it is easy to introduce primal slack variables to change inequalities into equalities.) The problem has n variables; i is a variable number; k is an iteration number, and if used as a subscript or superscript it denotes "of iteration k".

There exists an equivalent problem, the *dual* problem, stated as

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c \\ & s \geq 0 \end{array}$$

where y are dual variables, and s are dual constraint slacks.

The interior point algorithm solves the system of equations to satisfy the Karush-Kuhn-Tucker (KKT) conditions for optimality:

$$\begin{aligned} & Ax = b \\ & A^T y + s = c \\ & XSe = 0 \\ & x \geq 0 \\ & s \geq 0 \end{aligned}$$

where

$$\begin{aligned} & S = \text{diag}(s) \text{ (that is, } S_{i,j} = s_i \text{ if } i = j, S_{i,j} = 0 \text{ otherwise)} \\ & X = \text{diag}(x) \\ & e_i = 1 \, \forall i \end{aligned}$$

These are the conditions for feasibility, with the *complementarity* condition $XSe = 0$ added. Complementarity forces the optimal objectives of the primal and dual to be equal, $c^T x_{opt} = b^T y_{opt}$, as

$$0 = x_{opt}^T s_{opt} = s_{opt}^T x_{opt} = (c - A^T y_{opt})^T x_{opt} =$$
$$c^T x_{opt} - y_{opt}^T (A x_{opt}) = c^T x_{opt} - b^T y_{opt}$$

Before the optimum is reached, a solution (x, y, s) may not satisfy the KKT conditions:

- Primal constraints may be violated, $infeas_c = b - Ax \neq 0$.
- Dual constraints may be violated, $infeas_d = c - A^T y - s \neq 0$.
- Complementarity may not be satisfied, $x^T s = c^T x - b^T y \neq 0$. This is called the *duality gap*.

The interior point algorithm works by using Newton's method to find a direction to move $(\Delta x^k, \Delta y^k, \Delta s^k)$ from the current solution (x^k, y^k, s^k) toward a better solution:

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where α is the *step length* and is assigned a value as large as possible but not so large that an x_i^{k+1} or s_i^{k+1} is "too close" to zero. The direction in which to move is found using

$$\begin{aligned} A\Delta x^k &= infeas_c \\ A^T\Delta y^k + \Delta s^k &= infeas_d \\ S^k\Delta x^k + X^k\Delta s^k &= -X^k S^k e \end{aligned}$$

To greatly improve performance, the third equation is changed to

$$S^k\Delta x^k + X^k\Delta s^k = -X^k S^k e + \sigma_k \mu_k e$$

where $\mu_k = (x^k)^T s^k / n$, the average complementarity, and $0 \leq \sigma_k \leq 1$.

The effect now is to find a direction in which to move to reduce infeasibilities and to reduce the complementarity toward zero, but if any $x_i^k s_i^k$ is too close to zero, it is "nudged out" to μ, and any $x_i^k s_i^k$ that is larger than μ is "nudged into" μ. A σ_k close to or equal to 0.0 biases a direction toward the optimum, and a value of σ_k close to or equal to 1.0 "centers" the direction toward a point where all pairwise products $x_i^k s_i^k = \mu$. Such points make up the *central path* in the interior. Although centering directions make little, if any, progress in reducing μ and moving the solution closer to the optimum, substantial progress toward the optimum can usually be made in the next iteration.

The central path is crucial to why the interior point algorithm is so efficient. As μ is decreased, this path "guides" the algorithm to the optimum through the interior of feasible space. Without centering, the algorithm would find a series of solutions near each other close to the boundary of feasible space. Step lengths along the direction would be small and many more iterations would probably be required to reach the optimum.

That in a nutshell is the primal-dual interior point algorithm. Varieties of the algorithm differ in the way α and σ_k are chosen and the direction adjusted during each

iteration. A wealth of information can be found in the texts by Roos, Terlaky, and Vial (1997), Wright (1996), and Ye (1996).

The calculation of the direction is the most time-consuming step of the interior point algorithm. Assume the kth iteration is being performed, so the subscript and superscript k can be dropped from the algebra:

$$\begin{aligned} A\Delta x &= infeas_c \\ A^T\Delta y + \Delta s &= infeas_d \\ S\Delta x + X\Delta s &= -XSe + \sigma\mu e \end{aligned}$$

Rearranging the second equation,

$$\Delta s = infeas_d - A^T\Delta y$$

Rearranging the third equation,

$$\begin{aligned} \Delta s &= X^{-1}(-S\Delta x - XSe + \sigma\mu e) \\ \Delta s &= -\Theta\Delta x - Se + X^{-1}\sigma\mu e \end{aligned}$$

where $\Theta = SX^{-1}$.

Equating these two expressions for Δs and rearranging,

$$\begin{aligned} -\Theta\Delta x - Se + X^{-1}\sigma\mu e &= infeas_d - A^T\Delta y \\ -\Theta\Delta x &= Se - X^{-1}\sigma\mu e + infeas_d - A^T\Delta y \\ \Delta x &= \Theta^{-1}(-Se + X^{-1}\sigma\mu e - infeas_d + A^T\Delta y) \\ \Delta x &= \rho + \Theta^{-1}A^T\Delta y \end{aligned}$$

where $\rho = \Theta^{-1}(-Se + X^{-1}\sigma\mu e - infeas_d)$.

Substituting into the first direction equation,

$$\begin{aligned} A\Delta x &= infeas_c \\ A(\rho + \Theta^{-1}A^T\Delta y) &= infeas_c \\ A\Theta^{-1}A^T\Delta y &= infeas_c - A\rho \\ \Delta y &= (A\Theta^{-1}A^T)^{-1}(infeas_c - A\rho) \end{aligned}$$

Θ, ρ, Δy, Δx, and Δs are calculated in that order. The hardest term is the factorization of the $(A\Theta^{-1}A^T)$ matrix to determine Δy. Fortunately, although the *values* of $(A\Theta^{-1}A^T)$ are different for each iteration, the *locations* of the nonzeros in this matrix remain fixed; the nonzero locations are the same as those in the matrix (AA^T).

This is because $\Theta^{-1} = XS^{-1}$ is a diagonal matrix that has the effect of merely scaling the columns of (AA^T).

The fact that the nonzeros in $A\Theta^{-1}A^T$ have a constant pattern is exploited by all interior point algorithms and is a major reason for their excellent performance. Before iterations begin, AA^T is examined and its rows and columns are symmetrically permuted so that during Cholesky factorization, the number of *fill-ins* created is smaller. A list of arithmetic operations to perform the factorization is saved in concise computer data structures (working with memory locations rather than actual numerical values). This is called *symbolic factorization*. During iterations, when memory has been initialized with numerical values, the operations list is performed sequentially. Determining how the factorization should be performed again and again is unnecessary.

The Primal-Dual Predictor-Corrector Interior Point Algorithm

The variant of the interior point algorithm implemented in PROC INTPOINT is a Primal-Dual Predictor-Corrector interior point algorithm. At first, Newton's method is used to find a direction $(\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff})$ to move, but calculated as if μ is zero, that is, as a step with no centering, known as an *affine* step:

$$
\begin{aligned}
&A\Delta x^k_{aff} = infeas_c \\
&A^T\Delta y^k_{aff} + \Delta s^k_{aff} = infeas_d \\
&S^k\Delta x^k_{aff} + X^k\Delta s^k_{aff} = -X^kS^ke
\end{aligned}
$$

$$
(x^k_{aff}, y^k_{aff}, s^k_{aff}) = (x^k, y^k, s^k) + \alpha(\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff})
$$

where α is the *step length* as before.

Complementarity x^Ts is calculated at $(x^k_{aff}, y^k_{aff}, s^k_{aff})$ and compared with the complementarity at the starting point (x^k, y^k, s^k), and the success of the affine step is gauged. If the affine step was successful in reducing the complementarity by a substantial amount, the need for centering is not great, and σ_k in the following linear system is assigned a value close to zero. If, however, the affine step was unsuccessful, centering would be beneficial, and σ_k in the following linear system is assigned a value closer to 1.0. The value of σ_k is therefore adaptively altered depending on the progress made toward the optimum.

A second linear system is solved to determine a centering vector $(\Delta x^k_c, \Delta y^k_c, \Delta s^k_c)$ from $(x^k_{aff}, y^k_{aff}, s^k_{aff})$:

$$
\begin{aligned}
&A\Delta x^k_c = 0 \\
&A^T\Delta y^k_c + \Delta s^k_c = 0 \\
&S^k\Delta x^k_c + X^k\Delta s^k_c = -X^k_{aff}S^k_{aff}e + \sigma_k\mu_k e
\end{aligned}
$$

Then

$$(\Delta x^k, \Delta y^k, \Delta s^k) = (\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff}) + (\Delta x^k_c, \Delta y^k_c, \Delta s^k_c)$$

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where, as before, α is the *step length* assigned a value as large as possible but not so large that an x_i^{k+1} or s_i^{k+1} is "too close" to zero.

Although the Predictor-Corrector variant entails solving two linear systems instead of one, fewer iterations are usually required to reach the optimum. The additional overhead of calculating the second linear system is small, as the factorization of the $(A\Theta^{-1}A^T)$ matrix has already been performed to solve the first linear system.

Interior Point: Upper Bounds

If the LP had upper bounds ($0 \leq x \leq u$ where u is the upper bound vector), then the primal and dual problems, the duality gap, and the KKT conditions would have to be expanded.

The primal linear program to be solved is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

where $0 \leq x \leq u$ is split into $x \geq 0$ and $x \leq u$. Let z be primal slack so that $x + z = u$, and associate dual variables w with these constraints. The interior point algorithm solves the system of equations to satisfy the Karush-Kuhn-Tucker (KKT) conditions for optimality:

$$\begin{aligned} Ax &= b \\ x + z &= u \\ A^T y + s - w &= c \\ XSe &= 0 \\ ZWe &= 0 \\ x, s, z, w &\geq 0 \end{aligned}$$

These are the conditions for feasibility, with the *complementarity* conditions $XSe = 0$ and $ZWe = 0$ added. Complementarity forces the optimal objectives of the primal and dual to be equal, $c^T x_{opt} = b^T y_{opt} - u^T w_{opt}$, as

$$0 = z_{opt}^T w_{opt} = (u - x_{opt})^T w_{opt} = u^T w_{opt} - x_{opt}^T w_{opt}$$

$$0 = x_{opt}^T s_{opt} = s_{opt}^T x_{opt} = (c - A^T y_{opt} + w_{opt})^T x_{opt} =$$
$$c^T x_{opt} - y_{opt}^T (A x_{opt}) + w_{opt}^T x_{opt} = c^T x_{opt} - b^T y_{opt} + u^T w_{opt}$$

Before the optimum is reached, a solution (x, y, s, z, w) might not satisfy the KKT conditions:

- Primal bound constraints may be violated, $infeas_b = u - x - z \neq 0$.
- Primal constraints may be violated, $infeas_c = b - Ax \neq 0$.
- Dual constraints may be violated, $infeas_d = c - A^T y - s + w \neq 0$.
- Complementarity conditions may not be satisfied, $x^T s \neq 0$ or $z^T w \neq 0$.

The calculations of the interior point algorithm can easily be derived in a fashion similar to calculations for when an LP has no upper bounds. See the paper by Lustig, Marsten, and Shanno (1992).

In some iteration k, the *affine* step system that must be solved is

$$
\begin{aligned}
&\Delta x_{aff} + \Delta z_{aff} = infeas_b \\
&A\Delta x_{aff} = infeas_c \\
&A^T \Delta y_{aff} + \Delta s_{aff} - \Delta w_{aff} = infeas_d \\
&S\Delta x_{aff} + X\Delta s_{aff} = -XSe \\
&Z\Delta w_{aff} + W\Delta z_{aff} = -ZWe
\end{aligned}
$$

Therefore, the computations involved in solving the affine step are

$$
\begin{aligned}
&\Theta = SX^{-1} + WZ^{-1} \\
&\rho = \Theta^{-1}(infeas_d + (S - W)e - Z^{-1}W\ infeas_b) \\
&\Delta y_{aff} = (A\Theta^{-1}A^T)^{-1}(infeas_c + A\rho) \\
&\Delta x_{aff} = \Theta^{-1}A^T\Delta y_{aff} - \rho \\
&\Delta z_{aff} = infeas_b - \Delta x_{aff} \\
&\Delta w_{aff} = -We - Z^{-1}W\Delta z_{aff} \\
&\Delta s_{aff} = -Se - X^{-1}S\Delta x_{aff}
\end{aligned}
$$

$$
\begin{aligned}
&(x_{aff}, y_{aff}, s_{aff}, z_{aff}, w_{aff}) = (x, y, s, z, w) + \\
&\alpha(\Delta x_{aff}, \Delta y_{aff}, \Delta s_{aff}, \Delta z_{aff}, \Delta w_{aff})
\end{aligned}
$$

and α is the *step length* as before.

A second linear system is solved to determine a centering vector $(\Delta x_c, \Delta y_c, \Delta s_c, \Delta z_c, \Delta w_c)$ from $(x_{aff}, y_{aff}, s_{aff}, z_{aff}, w_{aff})$:

$$
\begin{aligned}
&\Delta x_c + \Delta z_c = 0 \\
&A\Delta x_c = 0 \\
&A^T\Delta y_c + \Delta s_c - \Delta w_c = 0 \\
&S\Delta x_c + X\Delta s_c = -X_{aff}S_{aff}e + \sigma\mu e
\end{aligned}
$$

$$Z\Delta w_c + W\Delta z_c = -Z_{aff}W_{aff}e + \sigma\mu e$$

where

$\zeta_{start} = x^T s + z^T w$, complementarity at the start of the iteration
$\zeta_{aff} = x_{aff}^T s_{aff} + z_{aff}^T w_{aff}$, the affine complementarity
$\mu = \zeta_{aff}/2n$, the average complementarity
$\sigma = (\zeta_{aff}/\zeta_{start})^3$

Therefore, the computations involved in solving the centering step are

$$\begin{aligned}
\rho &= \Theta^{-1}(\sigma\mu(X^{-1} - Z^{-1})e - X^{-1}X_{aff}S_{aff}e + Z^{-1}Z_{aff}W_{aff}e) \\
\Delta y_c &= (A\Theta^{-1}A^T)^{-1}A\rho \\
\Delta x_c &= \Theta^{-1}A^T\Delta y_c - \rho \\
\Delta z_c &= -\Delta x_c \\
\Delta w_c &= \sigma\mu Z^{-1}e - Z^{-1}Z_{aff}W_{aff}e - Z^{-1}W_{aff}\Delta z_c \\
\Delta s_c &= \sigma\mu X^{-1}e - X^{-1}X_{aff}S_{aff}e - X^{-1}S_{aff}\Delta x_c
\end{aligned}$$

Then

$$\begin{aligned}
&(\Delta x, \Delta y, \Delta s, \Delta z, \Delta w) = \\
&\quad (\Delta x_{aff}, \Delta y_{aff}, \Delta s_{aff}, \Delta z_{aff}, \Delta w_{aff}) \\
&\quad +(\Delta x_c, \Delta y_c, \Delta s_c, \Delta z_c, \Delta w_c)
\end{aligned}$$

$$\begin{aligned}
&(x^{k+1}, y^{k+1}, s^{k+1}, z^{k+1}, w^{k+1}) = \\
&\quad (x^k, y^k, s^k, z^k, w^k) \\
&\quad +\alpha(\Delta x, \Delta y, \Delta s, \Delta z, \Delta w)
\end{aligned}$$

where, as before, α is the *step length* assigned a value as large as possible but not so large that an x_i^{k+1}, s_i^{k+1}, z_i^{k+1}, or w_i^{k+1} is "too close" to zero.

The algebra in this section has been simplified by assuming that *all* variables have finite upper bounds. If the number of variables with finite upper bounds $n_u < n$, you need to change the algebra to reflect that the Z and W matrices have dimension $n_u \times 1$ or $n_u \times n_u$. Other computations need slight modification. For example, the average complementarity is

$$\mu = x_{aff}^T s_{aff}/n + z_{aff}^T w_{aff}/n_u$$

An important point is that any upper bounds can be handled by specializing the algorithm and *not* by generating the constraints $x \leq u$ and adding these to the main primal constraints $Ax = b$.

Network Models

The following are descriptions of some typical NPSC models.

Production, Inventory, and Distribution (Supply Chain) Problems

One common class of network models is the production-inventory-distribution or supply-chain problem. The diagram in Figure 2.1 illustrates this problem. The subscripts on the Production, Inventory, and Sales nodes indicate the time period. By replicating sections of the model, the notion of time can be included.

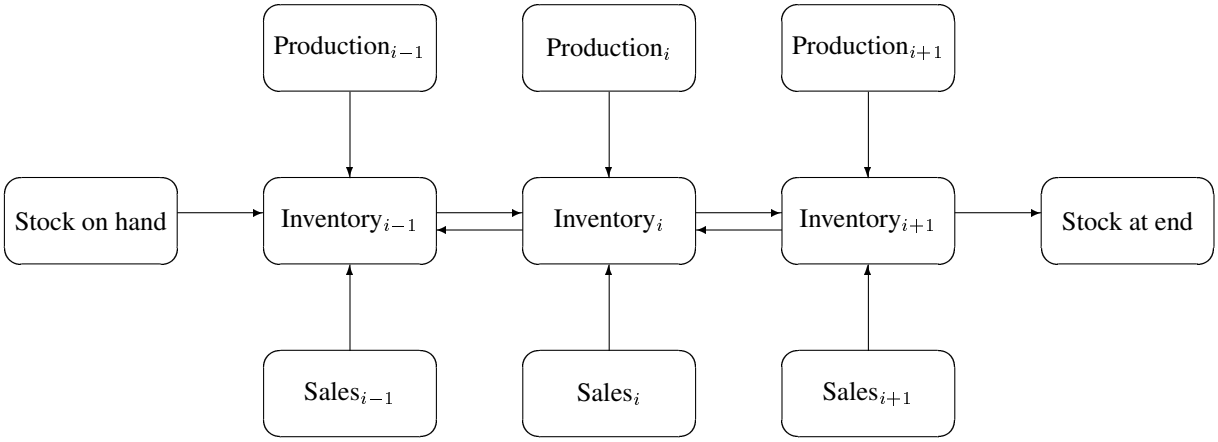


Figure 2.1. Production-Inventory-Distribution Problem

In this type of model, the nodes can represent a wide variety of facilities. Several examples are suppliers, spot markets, importers, farmers, manufacturers, factories, parts of a plant, production lines, waste disposal facilities, workstations, warehouses, coolstores, depots, wholesalers, export markets, ports, rail junctions, airports, road intersections, cities, regions, shops, customers, and consumers. The diversity of this selection demonstrates how rich the potential applications of this model are.

Depending upon the interpretation of the nodes, the objectives of the modeling exercise can vary widely. Some common types of objectives are

- to reduce collection or purchase costs of raw materials
- to reduce inventory holding or backorder costs. Warehouses and other storage facilities sometimes have capacities, and there can be limits on the amount of goods that can be placed on backorder.
- to decide where facilities should be located and what the capacity of these should be. Network models have been used to help decide where factories, hospitals, ambulance and fire stations, oil and water wells, and schools should be sited.

- to determine the assignment of resources (machines, production capability, workforce) to tasks, schedules, classes, or files
- to determine the optimal distribution of goods or services. This usually means minimizing transportation costs and reducing transit time or distances covered.
- to find the shortest path from one location to another
- to ensure that demands (for example, production requirements, market demands, contractual obligations) are met
- to maximize profits from the sale of products or the charge for services
- to maximize production by identifying bottlenecks

Some specific applications are

- car distribution models. These help determine which models and numbers of cars should be manufactured in which factories and where to distribute cars from these factories to zones in the United States in order to meet customer demand at least cost.
- models in the timber industry. These help determine when to plant and mill forests, schedule production of pulp, paper, and wood products, and distribute products for sale or export.
- military applications. The nodes can be theaters, bases, ammunition dumps, logistical suppliers, or radar installations. Some models are used to find the best ways to mobilize personnel and supplies and to evacuate the wounded in the least amount of time.
- communications applications. The nodes can be telephone exchanges, transmission lines, satellite links, and consumers. In a model of an electrical grid, the nodes can be transformers, powerstations, watersheds, reservoirs, dams, and consumers. The effect of high loads or outages might be of concern.

Proportionality Constraints

In many models, you have the characteristic that a flow through an arc must be proportional to the flow through another arc. Side constraints are often necessary to model that situation. Such constraints are called *proportionality constraints* and are useful in models where production is subject to refining or modification into different materials. The amount of each output, or any waste, evaporation, or reduction can be specified as a proportion of input.

Typically, the arcs near the supply nodes carry raw materials and the arcs near the demand nodes carry refined products. For example, in a model of the milling industry, the flow through some arcs may represent quantities of wheat. After the wheat is processed, the flow through other arcs might be flour. For others it might be bran. The side constraints model the relationship between the amount of flour or bran produced as a proportion of the amount of wheat milled. Some of the wheat can end up as neither flour, bran, nor any useful product, so this waste is drained away via arcs to a waste node.

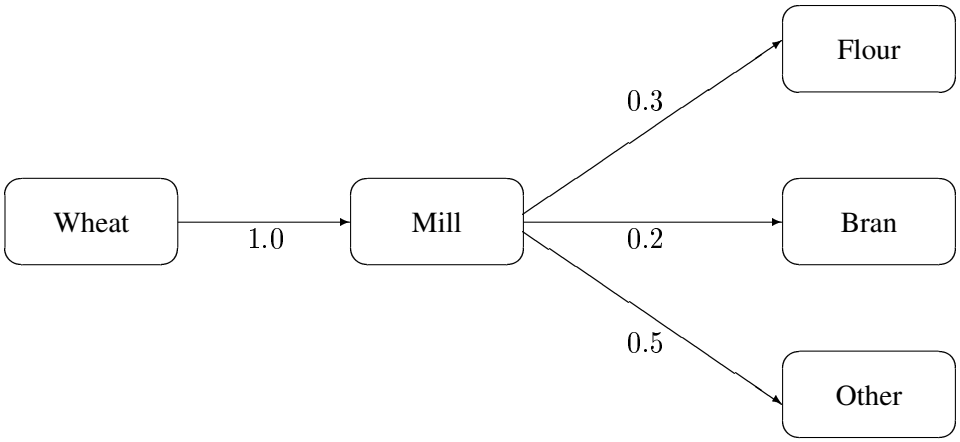


Figure 2.2. Proportionality Constraints

In order for arcs to be specified in side constraints, they must be named. By default, PROC INTPOINT names arcs using the names of the nodes at the head and tail of the arc. An arc is named with its tail node name followed by an underscore and its head node name. For example, an arc from node *from* to node *to* is called *from_to*.

Consider the network fragment in Figure 2.2. The arc Wheat_Mill conveys the wheat milled. The cost of flow on this arc is the milling cost. The capacity of this arc is the capacity of the mill. The lower flow bound on this arc is the minimum quantity that must be milled for the mill to operate economically. The constraints

$$0.3\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Flour} = 0.0$$
$$0.2\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Bran} = 0.0$$

force every unit of wheat that is milled to produce 0.3 units of flour and 0.2 units of bran. Note that it is not necessary to specify the constraint

$$0.5\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Other} = 0.0$$

since flow conservation implies that any flow that does not traverse through Mill_Flour or Mill_Bran must be conveyed through Mill_Other. And, computationally, it is better if this constraint is not specified, since there is one less side constraint and fewer problems with numerical precision. Notice that the sum of the proportions must equal 1.0 exactly; otherwise, flow conservation is violated.

Blending Constraints

Blending or quality constraints can also influence the recipes or proportions of ingredients that are mixed. For example, different raw materials can have different properties. In an application of the oil industry, the amount of products that are obtained could be different for each type of crude oil. Furthermore, fuel might have a minimum octane requirement or limited sulphur or lead content, so that a blending of crudes is needed to produce the product.

The network fragment in Figure 2.3 shows an example of this.

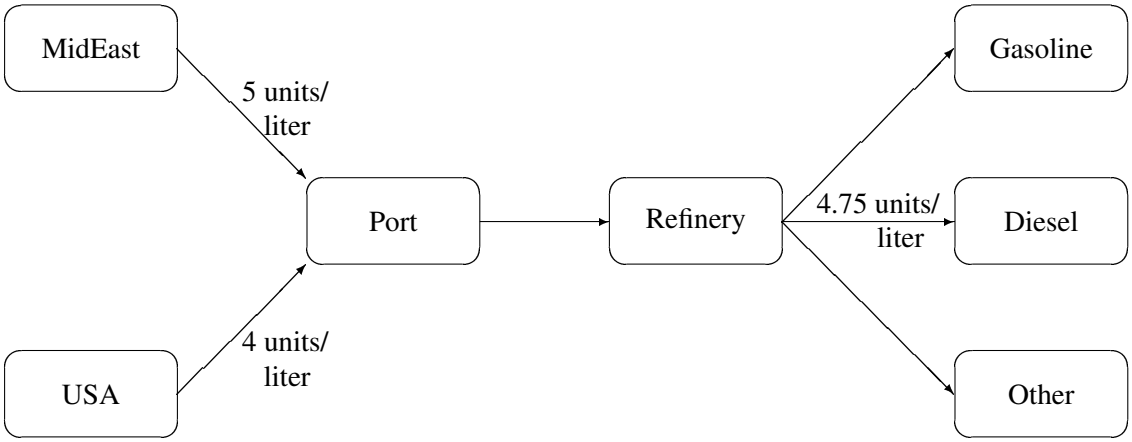


Figure 2.3. Blending Constraints

The arcs MidEast_Port and USA_Port convey crude oil from the two sources. The arc Port_Refinery represents refining while the arcs Refinery_Gasoline and Refinery_Diesel carry the gas and diesel produced. The proportionality constraints

$$0.4\,\mathsf{Port\_Refinery} - \mathsf{Refinery\_Gasoline} = 0.0$$
$$0.2\,\mathsf{Port\_Refinery} - \mathsf{Refinery\_Diesel} = 0.0$$

capture the restrictions for producing gasoline and diesel from crude. Suppose that only crude from the Middle East is used, then the resulting diesel would contain 5 units of sulphur per liter. If only crude from the U.S.A. is used, the resulting diesel would contain 4 units of sulphur per liter. Diesel can have at most 4.75 units of sulphur per liter. Some crude from the U.S.A. must be used if Middle East crude is used in order to meet the 4.75 sulphur per liter limit. The side constraint to model this requirement is

$$5\,\mathsf{MidEast\_Port} + 4\,\mathsf{USA\_Port} - 4.75\,\mathsf{Port\_Refinery} \leq 0.0$$

Since $\mathsf{Port\_Refinery} = \mathsf{MidEast\_Port} + \mathsf{USA\_Port}$, flow conservation allows this constraint to be simplified to

$$1\,\mathsf{MidEast\_Port} - 3\,\mathsf{USA\_Port} \leq 0.0$$

If, for example, 120 units of crude from the Middle East is used, then at least 40 units of crude from the U.S.A. must be used. The preceding constraint is simplified because you assume that the sulphur concentration of diesel is proportional to the sulphur concentration of the crude mix. If this is not the case, the relation

$$0.2\,\text{Port\_Refinery} = \text{Refinery\_Diesel}$$

is used to obtain

$$5\,\text{MidEast\_Port} + 4\,\text{USA\_Port} - 4.75\,(1.0/0.2\,\text{Refinery\_Diesel}) \leq 0.0$$

which equals

$$5\,\text{MidEast\_Port} + 4\,\text{USA\_Port} - 23.75\,\text{Refinery\_Diesel} \leq 0.0$$

An example similar to this oil industry problem is solved in the section "Introductory NPSC Example" on page 55.

Multicommodity Problems

Side constraints are also used in models in which there are capacities on transportation or some other shared resource, or there are limits on overall production or demand in multicommodity, multidivisional, or multiperiod problems. Each commodity, division, or period can have a separate network coupled to one main system by the side constraints. Side constraints are used to combine the outputs of subdivisions of a problem (either commodities, outputs in distinct time periods, or different process streams) to meet overall demands or to limit overall production or expenditures. This method is more desirable than doing separate *local* optimizations for individual commodity, process, or time networks and then trying to establish relationships between each when determining an overall policy if the *global* constraint is not satisfied. Of course, to make models more realistic, side constraints may be necessary in the local problems.

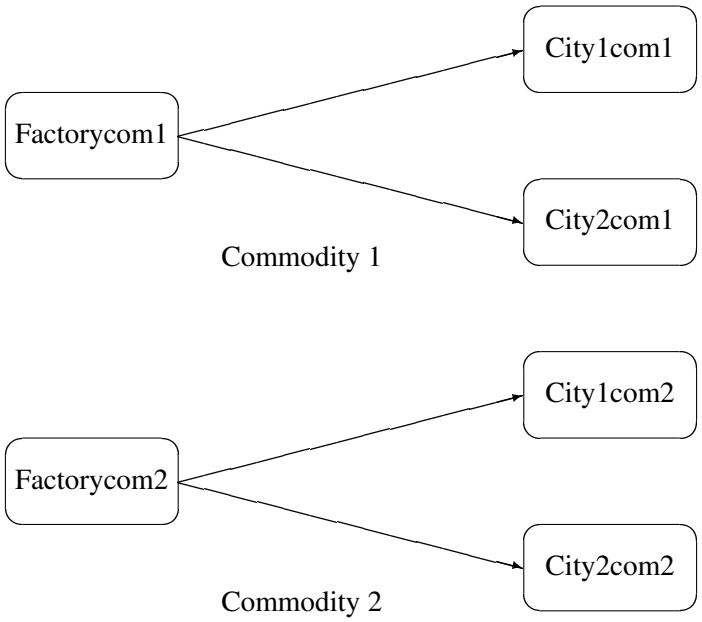


Figure 2.4. Multicommodity Problem

Figure 2.4 shows two network fragments. They represent identical production and distribution sites of two different commodities. Suffix *com1* represents commodity 1 and suffix *com2* represents commodity 2. The nodes Factorycom1 and Factorycom2 model the same factory, and nodes City1com1 and City1com2 model the same location, city 1. Similarly, City2com1 and City2com2 are the same location, city 2. Suppose that commodity 1 occupies 2 cubic meters, commodity 2 occupies 3 cubic meters, the truck dispatched to city 1 has a capacity of 200 cubic meters, and the truck dispatched to city 2 has a capacity of 250 cubic meters. How much of each commodity can be loaded onto each truck? The side constraints for this case are

$$2\,\text{Factorycom1\_City1com1} + 3\,\text{Factorycom2\_City1com2} \leq 200$$
$$2\,\text{Factorycom1\_City2com1} + 3\,\text{Factorycom2\_City2com2} \leq 250$$

Large Modeling Strategy

In many cases, the flow through an arc might actually represent the flow or movement of a commodity from place to place or from time period to time period. However, sometimes an arc is included in the network as a method of capturing some aspect of the problem that you would not normally think of as part of a network model. There is no commodity movement associated with that arc. For example, in a multiprocess, multiproduct model (Figure 2.5), there might be subnetworks for each process and each product. The subnetworks can be joined together by a set of arcs that have flows that represent the amount of product j produced by process i. To model an upper-limit constraint on the total amount of product j that can be produced, direct all arcs carrying product j to a single node and from there through a single arc. The capacity of this arc is the upper limit of product j production. It is preferable to model this structure in the network rather than to include it in the side constraints because the efficiency of the optimizer may be less affected by a reasonable increase in the size of the network rather than increasing the number or complicating side constraints.

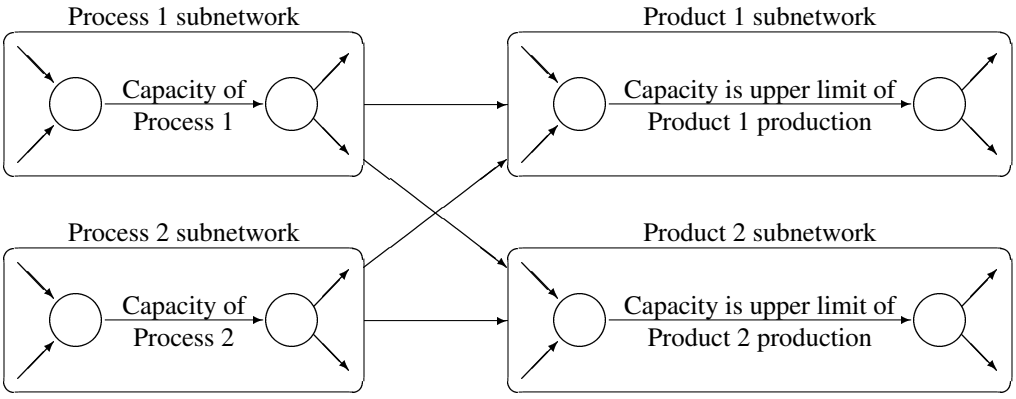


Figure 2.5. Multiprocess, Multiproduct Example

When starting a project, it is often a good strategy to use a small network formulation and then use that model as a framework upon which to add detail. For example, in the multiprocess, multiproduct model, you might start with the network depicted in

Figure 2.5. Then, for example, the process subnetwork can be enhanced to include the distribution of products. Other phases of the operation could be included by adding more subnetworks. Initially, these subnetworks can be single nodes, but in subsequent studies they can be expanded to include greater detail.

Advantages of Network Models over LP Models

Many linear programming problems have large embedded network structures. Such problems often result when modeling manufacturing processes, transportation or distribution networks, or resource allocation, or when deciding where to locate facilities. Often, some commodity is to be moved from place to place, so the more natural formulation in many applications is that of a constrained network rather than a linear program.

Using a network diagram to visualize a problem makes it possible to capture the important relationships in an easily understood picture form. The network diagram aids the communication between model builder and model user, making it easier to comprehend how the model is structured, how it can be changed, and how results can be interpreted.

If a network structure is embedded in a linear program, the problem is an NPSC (see the section "Mathematical Description of NPSC" on page 36). When the network part of the problem is large compared to the nonnetwork part, especially if the number of side constraints is small, it is worthwhile to exploit this structure to describe the model. Rather than generating the data for the flow conservation constraints, generate instead the data for the nodes and arcs of the network.

Flow Conservation Constraints

The constraints $Fx = b$ in NPSC (see the section "Mathematical Description of NPSC" on page 36) are referred to as the nodal flow conservation constraints. These constraints algebraically state that the sum of the flow through arcs directed toward a node plus that node's supply, if any, equals the sum of the flow through arcs directed away from that node plus that node's demand, if any. The flow conservation constraints are implicit in the network model and should not be specified explicitly in side constraint data when using PROC INTPOINT to solve NPSC problems.

Nonarc Variables

Nonarc variables can be used to simplify side constraints. For example, if a sum of flows appears in many constraints, it may be worthwhile to equate this expression with a nonarc variable and use this in the other constraints. This keeps the constraint coefficient matrix sparse. By assigning a nonarc variable a nonzero objective function, it is then possible to incur a cost for using resources above some lowest feasible limit. Similarly, a profit (a negative objective function coefficient value) can be made if all available resources are not used.

In some models, nonarc variables are used in constraints to absorb excess resources or supply needed resources. Then, either the excess resource can be used or the needed resource can be supplied to another component of the model.

For example, consider a multicommodity problem of making television sets that have either 19- or 25-inch screens. In their manufacture, three and four chips, respectively, are used. Production occurs at two factories during March and April. The supplier of chips can supply only 2,600 chips to factory 1 and 3,750 chips to factory 2 each month. The names of arcs are in the form Prod*n_s_m*, where *n* is the factory number, *s* is the screen size, and *m* is the month. For example, Prod1_25_Apr is the arc that conveys the number of 25-inch TVs produced in factory 1 during April. You might have to determine similar systematic naming schemes for your application.

As described, the constraints are

$$\begin{aligned}
3\,\text{Prod1_19_Mar} + 4\,\text{Prod1_25_Mar} &\leq 2600\\
3\,\text{Prod2_19_Mar} + 4\,\text{Prod2_25_Mar} &\leq 3750\\
3\,\text{Prod1_19_Apr} + 4\,\text{Prod1_25_Apr} &\leq 2600\\
3\,\text{Prod2_19_Apr} + 4\,\text{Prod2_25_Apr} &\leq 3750
\end{aligned}$$

If there are chips that could be obtained for use in March but not used for production in March, why not keep these unused chips until April? Furthermore, if the March excess chips at factory 1 could be used either at factory 1 or factory 2 in April, the model becomes

$$\begin{aligned}
3\,\text{Prod1_19_Mar} + 4\,\text{Prod1_25_Mar} + \text{F1_Unused_Mar} &= 2600\\
3\,\text{Prod2_19_Mar} + 4\,\text{Prod2_25_Mar} + \text{F2_Unused_Mar} &= 3750\\
3\,\text{Prod1_19_Apr} + 4\,\text{Prod1_25_Apr} - \text{F1_Kept_Since_Mar} &= 2600\\
3\,\text{Prod2_19_Apr} + 4\,\text{Prod2_25_Apr} - \text{F2_Kept_Since_Mar} &= 3750\\
\text{F1_Unused_Mar} + \text{F2_Unused_Mar} \text{ (continued)}&\\
- \text{F1_Kept_Since_Mar} - \text{F2_Kept_Since_Mar} &\geq 0.0
\end{aligned}$$

where F1_Kept_Since_Mar is the number of chips used during April at factory 1 that were obtained in March at either factory 1 or factory 2, and F2_Kept_Since_Mar is the number of chips used during April at factory 2 that were obtained in March. The last constraint ensures that the number of chips used during April that were obtained in March does not exceed the number of chips not used in March. There may be a cost to hold chips in inventory. This can be modeled having a positive objective function coefficient for the nonarc variables F1_Kept_Since_Mar and F2_Kept_Since_Mar. Moreover, nonarc variable upper bounds represent an upper limit on the number of chips that can be held in inventory between March and April.

See Example 2.1 through Example 2.5, which use this TV problem. The use of nonarc variables as described previously is illustrated.

Introduction

Getting Started: NPSC Problems

To solve NPSC problems using PROC INTPOINT, you save a representation of the network and the side constraints in three SAS data sets. These data sets are then passed to PROC INTPOINT for solution. There are various forms that a problem's data can take. You can use any one or a combination of several of these forms.

The NODEDATA= data set contains the names of the supply and demand nodes and the supply or demand associated with each. These are the elements in the column vector b in the NPSC problem (see the section "Mathematical Description of NPSC" on page 36).

The ARCDATA= data set contains information about the variables of the problem. Usually these are arcs, but there can be data related to nonarc variables in the ARCDATA= data set as well.

An arc is identified by the names of its tail node (where it originates) and head node (where it is directed). Each observation can be used to identify an arc in the network and, optionally, the cost per flow unit across the arc, the arc's capacity, lower flow bound, and name. These data are associated with the matrix F and the vectors c, l, and u in the NPSC problem (see the section "Mathematical Description of NPSC" on page 36).

Note: Although F is a node-arc incidence matrix, it is specified in the ARCDATA= data set by arc definitions. Do not explicitly specify these flow conservation constraints as constraints of the problem.

In addition, the ARCDATA= data set can be used to specify information about nonarc variables, including objective function coefficients, lower and upper value bounds, and names. These data are the elements of the vectors d, m, and v in the NPSC problem (see the section "Mathematical Description of NPSC" on page 36). Data for an arc or nonarc variable can be given in more than one observation.

Supply and demand data also can be specified in the ARCDATA= data set. In such a case, the NODEDATA= data set may not be needed.

The CONDATA= data set describes the side constraints and their right-hand sides. These data are elements of the matrices H and Q and the vector r. Constraint types are also specified in the CONDATA= data set. You can include in this data set upper bound values or capacities, lower flow or value bounds, and costs or objective function coefficients. It is possible to give all information about some or all nonarc variables in the CONDATA= data set.

An arc is identified in this data set by its name. If you specify an arc's name in the ARCDATA= data set, then this name is used to associate data in the CONDATA= data set with that arc. Each arc also has a default name that is the name of the tail and head node of the arc concatenated together and separated by an underscore character; tail_head, for example.

If you use the dense side constraint input format (described in the section "CONDATA= Data Set" on page 101), and want to use the default arc names, these arc names are names of SAS variables in the VAR list of the CONDATA= data set.

If you use the sparse side constraint input format (see the section "CONDATA= Data Set" on page 101) and want to use the default arc names, these arc names are values of the COLUMN list variable of the CONDATA= data set.

PROC INTPOINT reads the data from the NODEDATA= data set, the ARCDATA= data set, and the CONDATA= data set. Error checking is performed, and the model is converted into an equivalent LP. This LP is preprocessed. Preprocessing is optional but highly recommended. Preprocessing analyzes the model and tries to determine before optimization whether variables can be "fixed" to their optimal values. Knowing that, the model can be modified and these variables dropped out. It can be determined that some constraints are redundant. Sometimes, preprocessing succeeds in reducing the size of the problem, thereby making the subsequent optimization easier and faster.

The optimal solution to the equivalent LP is then found. This LP is converted back to the original NPSC problem, and the optimum for this is derived from the optimum of the equivalent LP. If the problem was preprocessed, the model is now post-processed, where fixed variables are reintroduced. The solution can be saved in the CONOUT= data set.

Introductory NPSC Example

Consider the following transshipment problem for an oil company. Crude oil is shipped to refineries where it is processed into gasoline and diesel fuel. The gasoline and diesel fuel are then distributed to service stations. At each stage, there are shipping, processing, and distribution costs. Also, there are lower flow bounds and capacities.

In addition, there are two sets of side constraints. The first set is that two times the crude from the Middle East cannot exceed the throughput of a refinery plus 15 units. (The phrase "plus 15 units" that finishes the last sentence is used to enable some side constraints in this example to have a nonzero rhs.) The second set of constraints are necessary to model the situation that one unit of crude mix processed at a refinery yields three-fourths of a unit of gasoline and one-fourth of a unit of diesel fuel.

Because there are two products that are not independent in the way in which they flow through the network, an NPSC is an appropriate model for this example (see Figure 2.6). The side constraints are used to model the limitations on the amount of Middle Eastern crude that can be processed by each refinery and the conversion proportions of crude to gasoline and diesel fuel.

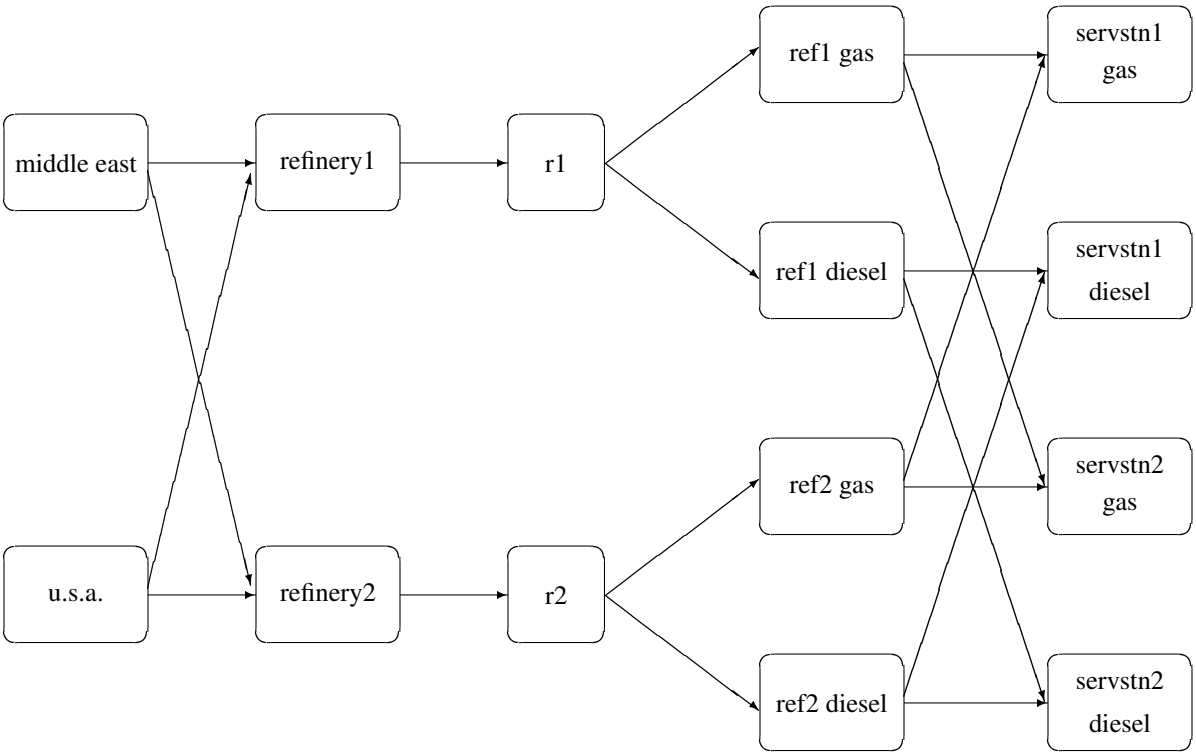


Figure 2.6. Oil Industry Example

To solve this problem with PROC INTPOINT, save a representation of the model in three SAS data sets. In the NODEDATA= data set, you name the supply and demand nodes and give the associated supplies and demands. To distinguish demand nodes from supply nodes, specify demands as negative quantities. For the oil example, the NODEDATA= data set can be saved as follows:

```
title  'Oil Industry Example';
title3 'Setting Up Nodedata = Noded For PROC INTPOINT';
data noded;
   input   _node_&$15. _sd_;
   datalines;
middle east       100
u.s.a.             80
servstn1 gas      -95
servstn1 diesel   -30
servstn2 gas      -40
servstn2 diesel   -15
;
```

The ARCDATA= data set contains the rest of the information about the network. Each observation in the data set identifies an arc in the network and gives the cost per flow unit across the arc, the capacities of the arc, the lower bound on flow across the arc, and the name of the arc.

```
title3 'Setting Up Arcdata = Arcd1 For PROC INTPOINT';
data arcd1;
   input _from_&$11. _to_&$15. _cost_ _capac_ _lo_ _name_ $;
   datalines;
middle east    refinery 1        63     95   20    m_e_ref1
middle east    refinery 2        81     80   10    m_e_ref2
u.s.a.         refinery 1        55      .    .    .
u.s.a.         refinery 2        49      .    .    .
refinery 1     r1               200    175   50    thruput1
refinery 2     r2               220    100   35    thruput2
r1             ref1 gas           .    140    .    r1_gas
r1             ref1 diesel        .     75    .    .
r2             ref2 gas           .    100    .    r2_gas
r2             ref2 diesel        .     75    .    .
ref1 gas       servstn1 gas      15     70    .    .
ref1 gas       servstn2 gas      22     60    .    .
ref1 diesel    servstn1 diesel   18      .    .    .
ref1 diesel    servstn2 diesel   17      .    .    .
ref2 gas       servstn1 gas      17     35    5    .
ref2 gas       servstn2 gas      31      .    .    .
ref2 diesel    servstn1 diesel   36      .    .    .
ref2 diesel    servstn2 diesel   23      .    .    .
;
```

Finally, the CONDATA= data set contains the side constraints for the model:

```
title3 'Setting Up Condata = Cond1 For PROC INTPOINT';
data cond1;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
         _type_ $ _rhs_;
   datalines;
-2   .   1 .   . . >= -15
 . -2   . .   1 . GE -15
 .   . -3 4   . . EQ   0
 .   .   . . -3 4  =   0
;
```

Note that the SAS variable names in the CONDATA= data set are the names of arcs given in the ARCDATA= data set. These are the arcs that have nonzero constraint coefficients in side constraints. For example, the proportionality constraint that specifies that one unit of crude at each refinery yields three-fourths of a unit of gasoline and one-fourth of a unit of diesel fuel is given for refinery 1 in the third observation and for refinery 2 in the last observation. The third observation requires that each unit of flow on the arc thruput1 equals three-fourths of a unit of flow on the arc r1_gas. Because all crude processed at refinery 1 flows through thruput1 and all gasoline produced at refinery 1 flows through r1_gas, the constraint models the situation. It proceeds similarly for refinery 2 in the last observation.

To find the minimum cost flow through the network that satisfies the supplies, demands, and side constraints, invoke PROC INTPOINT as follows:

```
proc intpoint
   bytes=1000000
   nodedata=noded        /* the supply and demand data */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
```

The following messages, which appear on the SAS log, summarize the model as read by PROC INTPOINT and note the progress toward a solution.

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 180 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of side constraint coefficients= 8 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
      Point algorithm.
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 16 .
NOTE: Number of >= constraints= 2 .
NOTE: Number of constraint coefficients= 44 .
NOTE: Number of variables= 18 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 5.
NOTE: After preprocessing, number of >= constraints= 2.
NOTE: The preprocessor eliminated 11 constraints from the
      problem.
NOTE: The preprocessor eliminated 25 constraint
      coefficients from the problem.
NOTE: After preprocessing, number of variables= 8.
NOTE: The preprocessor eliminated 10 variables from the
      problem.
NOTE: 2 columns, 0 rows and 2 coefficients were added to
      the problem to handle unrestricted variables,
      variables that are split, and constraint slack or
      surplus variables.
NOTE: There are 13 nonzero elements in A * A transpose.
NOTE: Of the 7 rows and columns, 2 are sparse.
NOTE: There are 6 nonzero superdiagonal elements in the
      sparse rows of the factored A * A transpose. This
      includes fill-in.
NOTE: There are 2 operations of the form
      u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
      sparse rows of A * A transpose.
NOTE: Bound feasibility attained by iteration 1.
NOTE: Dual feasibility attained by iteration 1.
NOTE: Constraint feasibility attained by iteration 2.
```

```
NOTE: The Primal-Dual Predictor-Corrector Interior Point
      algorithm performed 7 iterations.
NOTE: Objective = 50875.01279.
NOTE: The data set WORK.SOLUTION has 18 observations and
      10 variables.
NOTE: There were 18 observations read from the data set
      WORK.ARCD1.
NOTE: There were 6 observations read from the data set
      WORK.NODED.
NOTE: There were 4 observations read from the data set
      WORK.COND1.
```

The first set of messages shows the size of the problem. The next set of messages provides statistics on the size of the equivalent LP problem. The number of variables may not equal the number of arcs if the problem has nonarc variables. This example has none. To convert a network to the equivalent LP problem, a flow conservation constraint must be created for each node (including an excess or bypass node, if required). This explains why the number of equality constraints and the number of constraint coefficients differ from the number of equality side constraints and the number of coefficients in all side constraints.

If the preprocessor was successful in decreasing the problem size, some messages will report how well it did. In this example, the model size was cut approximately in half!

The next set of messages describes aspects of the interior point algorithm. Of particular interest are those concerned with the Cholesky factorization of AA^T where A is the coefficient matrix of the final LP. It is crucial to preorder the rows and columns of this matrix to prevent *fill-in* and reduce the number of row operations to undertake the factorization. See the section "Interior Point Algorithmic Details" on page 39 for a more extensive explanation.

Unlike PROC LP, which displays the solution and other information as output, PROC INTPOINT saves the optimum in the output SAS data set that you specify. For this example, the solution is saved in the SOLUTION data set. It can be displayed with the PRINT procedure as

```
title3 'Optimum';
proc print data=solution;
   var _from_ _to_ _cost_ _capac_ _lo_ _name_
       _supply_ _demand_ _flow_ _fcost_;
   sum _fcost_;
   run;
```

Oil Industry Example

Optimum

Obs	_from_	_to_	_cap_	_lo_	_name_	_SUPPLY	_DEMAND	_FLOW_	_FCOST_
1	refinery 1	r1	200	175	50 thruput1	.	.	145.000	28999.98
2	refinery 2	r2	220	100	35 thruput2	.	.	35.000	7700.02
3	r1	ref1 diesel	0	75	0	.	.	36.250	0.00
4	r1	ref1 gas	0	140	0 r1_gas	.	.	108.750	0.00
5	r2	ref2 diesel	0	75	0	.	.	8.750	0.00
6	r2	ref2 gas	0	100	0 r2_gas	.	.	26.250	0.00
7	middle east	refinery 1	63	95	20 m_e_ref1	100	.	80.000	5039.99
8	u.s.a.	refinery 1	55	99999999	0	80	.	65.000	3575.00
9	middle east	refinery 2	81	80	10 m_e_ref2	100	.	20.000	1620.02
10	u.s.a.	refinery 2	49	99999999	0	80	.	15.000	735.00
11	ref1 diesel	servstn1 diesel	18	99999999	0	.	30	30.000	540.00
12	ref2 diesel	servstn1 diesel	36	99999999	0	.	30	0.000	0.01
13	ref1 gas	servstn1 gas	15	70	0	.	95	68.750	1031.26
14	ref2 gas	servstn1 gas	17	35	5	.	95	26.250	446.24
15	ref1 diesel	servstn2 diesel	17	99999999	0	.	15	6.250	106.25
16	ref2 diesel	servstn2 diesel	23	99999999	0	.	15	8.750	201.24
17	ref1 gas	servstn2 gas	22	60	0	.	40	40.000	879.99
18	ref2 gas	servstn2 gas	31	99999999	0	.	40	0.000	0.01
									50875.01

Figure 2.7. CONOUT=SOLUTION

Notice that, in CONOUT=SOLUTION (Figure 2.7), the optimal flow through each arc in the network is given in the variable named _FLOW_, and the cost of flow through each arc is given in the variable _FCOST_.

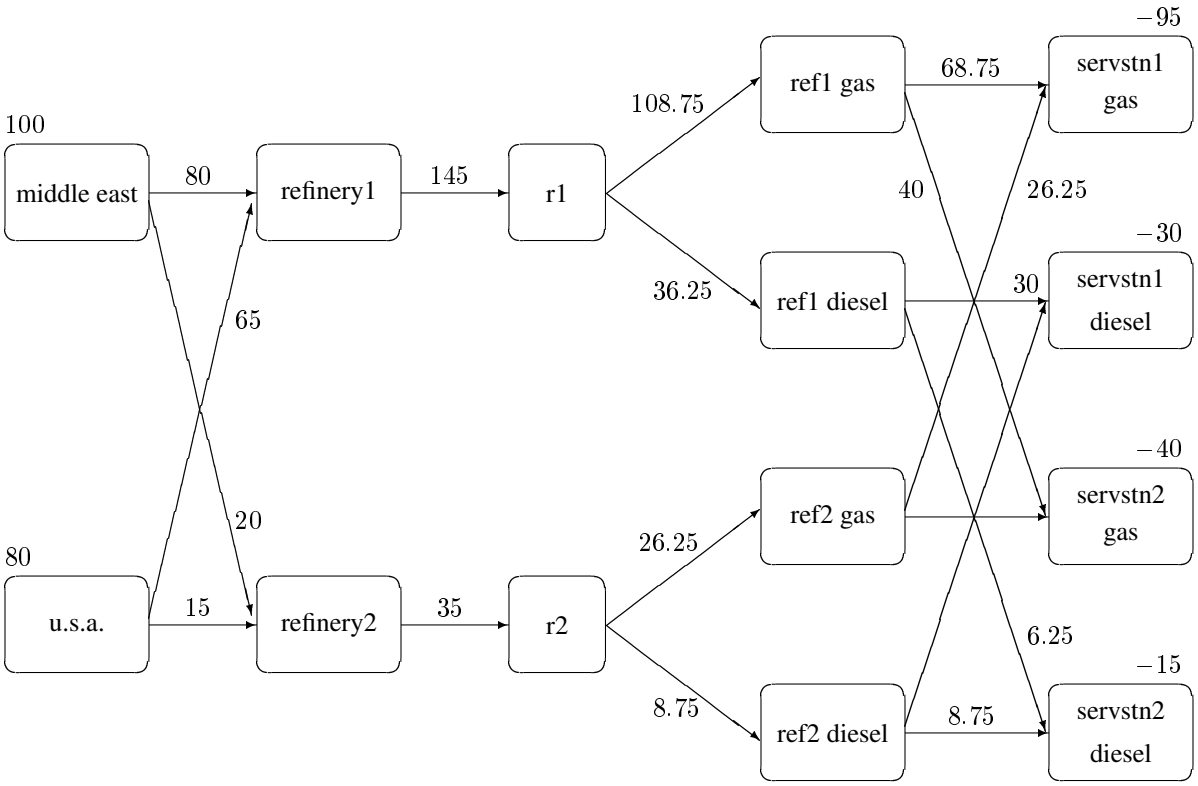


Figure 2.8. Oil Industry Solution

Getting Started: LP Problems

Data for an LP problem resembles the data for side constraints and nonarc variables supplied to PROC INTPOINT when solving an NPSC problem. It is also very similar to the data required by the LP procedure.

To solve LP problems using PROC INTPOINT, you save a representation of the LP variables and the constraints in one or two SAS data sets. These data sets are then passed to PROC INTPOINT for solution. There are various forms that a problem's data can take. You can use any one or a combination of several of these forms.

The ARCDATA= data set contains information about the LP variables of the problem. Although this data set is called ARCDATA, it contains data for no arcs. Instead, all data in this data set are related to LP variables. This data set has no SAS variables containing values that are node names.

The ARCDATA= data set can be used to specify information about LP variables, including objective function coefficients, lower and upper value bounds, and names. These data are the elements of the vectors d, m, and v in problem (LP). Data for an LP variable can be given in more than one observation.

The CONDATA= data set describes the constraints and their right-hand sides. These data are elements of the matrix Q and the vector r.

Constraint types are also specified in the CONDATA= data set. You can include in this data set LP variable data such as upper bound values, lower value bounds, and objective function coefficients. It is possible to give all information about some or all LP variables in the CONDATA= data set.

Because PROC INTPOINT evolved from PROC NETFLOW, another procedure in SAS/OR software that was originally designed to solve models with networks, the ARCDATA= data set is always expected. If the ARCDATA= data set is not specified, by default the last data set created before PROC INTPOINT is invoked is assumed to be the ARCDATA= data set. However, these characteristics of PROC INTPOINT are not helpful when an LP problem is being solved and all data is provided in a single data set specified by the CONDATA= data set, and that data set is not the last data set created before PROC INTPOINT starts. In this case, you must specify that the ARCDATA= data set and the CONDATA= data set are both equal to the input data set. PROC INTPOINT then knows that an LP problem is to be solved and that the data reside in one data set.

An LP variable is identified in this data set by its name. If you specify an LP variable's name in the ARCDATA= data set, then this name is used to associate data in the CONDATA= data set with that LP variable.

If you use the dense constraint input format (described in the section "CONDATA= Data Set" on page 101), these LP variable names are names of SAS variables in the VAR list of the CONDATA= data set.

If you use the sparse constraint input format (described in the section "CONDATA= Data Set" on page 101), these LP variable names are values of the SAS variables in the COLUMN list of the CONDATA= data set.

PROC INTPOINT reads the data from the ARCDATA= data set (if there is one) and the CONDATA= data set. Error checking is performed, and the LP is preprocessed. Preprocessing is optional but highly recommended. The preprocessor analyzes the model and tries to determine before optimization whether LP variables can be "fixed" to their optimal values. Knowing that, the model can be modified and these LP variables dropped out. Some constraints may be found to be redundant. Sometimes, preprocessing succeeds in reducing the size of the problem, thereby making the subsequent optimization easier and faster.

The optimal solution is then found for the resulting LP. If the problem was preprocessed, the model is now post-processed, where fixed LP variables are reintroduced. The solution can be saved in the CONOUT= data set.

Introductory LP Example

Consider the linear programming problem in the section "An Introductory Example" on page 164. The SAS data set in that section is created the same way here:

```
title  'Linear Programming Example';
title3 'Setting Up Condata = dcon1 For PROC INTPOINT';
data dcon1;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2
         _type_ $ _rhs_;
   datalines;
profit            -175 -165 -205  0  0  0 300 300 max     .
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0  eq     0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0  eq     0
heating_o_conv    .390 .300 .430  0  0 -1   0   0  eq     0
recipe_1             0    0    0  0 .3 .7  -1   0  eq     0
recipe_2             0    0    0 .2  0 .8   0  -1  eq     0
available          110  165   80  .  .  .   .   . upperbd .
;
```

To solve this problem, use

```
proc intpoint
   bytes=1000000
   condata=dcon1
   conout=solutn1;
   run;
```

Note how it is possible to use an input SAS data set of PROC LP and, without requiring any changes to be made to the data set, to use that as an input data set for PROC INTPOINT.

The following messages that appear on the SAS log summarize the model as read by PROC INTPOINT and note the progress toward a solution

```
NOTE: Number of variables= 8 .
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 5 .
NOTE: Number of >= constraints= 0 .
NOTE: Number of constraint coefficients= 18 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 0.
NOTE: After preprocessing, number of >= constraints= 0.
NOTE: The preprocessor eliminated 5 constraints from the
      problem.
NOTE: The preprocessor eliminated 18 constraint
      coefficients from the problem.
NOTE: After preprocessing, number of variables= 0.
NOTE: The preprocessor eliminated 8 variables from the
      problem.
WARNING: Optimization is unnecessary as the problem no
         longer has any variables and rows.
NOTE: Preprocessing could have caused that.
NOTE: Objective = 1544.
NOTE: The data set WORK.SOLUTN1 has 8 observations and 6
```

```
      variables.
NOTE: There were 7 observations read from the data set
      WORK.DCON1.
```

Notice that the preprocessor succeeded in fixing *all* LP variables to their optimal values, eliminating the need to do any actual optimization.

Unlike PROC LP, which displays the solution and other information as output, PROC INTPOINT saves the optimum in the output SAS data set you specify. For this example, the solution is saved in the SOLUTION data set. It can be displayed with PROC PRINT as

```
title3 'LP Optimum';
proc print data=solutn1;
   var _name_ _objfn_ _upperbd _lowerbd _value_ _fcost_;
   sum _fcost_;
   run;
```

Notice that in the CONOUT=SOLUTION (Figure 2.9) the optimal value through each variable in the LP is given in the variable named _VALUE_, and that the cost of value for each variable is given in the variable _FCOST_.

Linear Programming Example

LP Optimum

Obs	_NAME_	_OBJFN_	_UPPERBD	_LOWERBD	_VALUE_	_FCOST_
1	a_heavy	-165	165	0	0.00	0
2	a_light	-175	110	0	110.00	-19250
3	brega	-205	80	0	80.00	-16400
4	heatingo	0	99999999	0	77.30	0
5	jet_1	300	99999999	0	60.65	18195
6	jet_2	300	99999999	0	63.33	18999
7	naphthai	0	99999999	0	21.80	0
8	naphthal	0	99999999	0	7.45	0
						=======
						1544

Figure 2.9. CONOUT=SOLUTN1

The same model can be specified in the sparse format as in the following scon2 data set. This format enables you to omit the zero coefficients.

```
title3 'Setting Up Condata = scon2 For PROC INTPOINT';
data scon2;
   format _type_ $8. _col_ $8. _row_ $16.;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
max       .              profit                  .
eq        .              napha_l_conv            .
eq        .              napha_i_conv            .
eq        .              heating_oil_conv        .
eq        .              recipe_1                .
eq        .              recipe_2                .
upperbd   .              available               .
```

```
.        a_light       profit              -175
.        a_light       napha_l_conv         .035
.        a_light       napha_i_conv         .100
.        a_light       heating_oil_conv     .390
.        a_light       available             110
.        a_heavy       profit              -165
.        a_heavy       napha_l_conv         .030
.        a_heavy       napha_i_conv         .075
.        a_heavy       heating_oil_conv     .300
.        a_heavy       available             165
.        brega         profit              -205
.        brega         napha_l_conv         .045
.        brega         napha_i_conv         .135
.        brega         heating_oil_conv     .430
.        brega         available             80
.        naphthal      napha_l_conv          -1
.        naphthal      recipe_2               .2
.        naphthai      napha_i_conv          -1
.        naphthai      recipe_1               .3
.        heatingo      heating_oil_conv      -1
.        heatingo      recipe_1               .7
.        heatingo      recipe_2               .8
.        jet_1         profit               300
.        jet_1         recipe_1              -1
.        jet_2         profit               300
.        jet_2         recipe_2              -1
;
```

To find the minimum cost solution, invoke PROC INTPOINT (note the SPARSECONDATA option which must be specified) as follows:

```
proc intpoint
   bytes=1000000
   sparsecondata
   condata=scon2
   conout=solutn2;
   run;
```

A data set that can be used as the ARCDATA= data set can be initialized as follows:

```
data vars3;
   input _name_ $ profit available;
   datalines;
a_heavy  -165 165
a_light  -175 110
brega    -205  80
heatingo    0   .
jet_1     300   .
jet_2     300   .
naphthai    0   .
naphthal    0   .
;
```

The following CONDATA= data set is the original dense format CONDATA= dcon1 data set after the LP variable's nonconstraint information has been removed. (You could have left some or all of that information in CONDATA as PROC INTPOINT "merges" data, but doing that and checking for consistency takes time.)

```
data dcon3;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2
         _type_ $ _rhs_;
   datalines;
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0  eq     0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0  eq     0
heating_o_conv    .390 .300 .430  0  0 -1   0   0  eq     0
recipe_1             0    0    0  0 .3 .7  -1   0  eq     0
recipe_2             0    0    0 .2  0 .8   0  -1  eq     0
;
```

Note: You must now specify the MAXIMIZE option; otherwise, PROC INTPOINT will optimize to the minimum (which, incidentally, has a total objective = -3539.25). You must indicate that the SAS variable profit in the ARCDATA=vars3 data set has values that are objective function coefficients, by specifying the OBJFN statement. The UPPERBD must be specified as the SAS variable available that has as values upper bounds:

```
proc intpoint
     maximize        /* ***** necessary ***** */
     bytes=1000000
     arcdata=vars3
     condata=dcon3
     conout=solutn3;
   objfn profit;
   upperbd available;
   run;
```

The ARCDATA=vars3 data set can become more concise by noting that the model variables heatingo, naphthai, and naphthal have zero objective function coefficients (the default) and default upper bounds, so those observations need not be present:

```
data vars4;
   input _name_ $ profit available;
   datalines;
a_heavy  -165 165
a_light  -175 110
brega    -205  80
jet_1     300   .
jet_2     300   .
;
```

The CONDATA=dcon3 data set can become more concise by noting that all the constraints have the same type (eq) and zero (the default) rhs values. This model is a good candidate for using the DEFCONTYPE= option.

The DEFCONTYPE= option can be useful not only when *all* constraints have the same type as is the case here, but also when *most* constraints have the same type and you want to change the default type from $\leq$ to $=$ or $\geq$. The essential constraint type data in the CONDATA= data set is that which overrides the DEFCONTYPE= type you specified.

```
data dcon4;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2;
   datalines;
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0
heating_o_conv    .390 .300 .430  0  0 -1   0   0
recipe_1             0    0    0  0 .3 .7  -1   0
recipe_2             0    0    0 .2  0 .8   0  -1
;

proc intpoint
     maximize defcontype=eq
     arcdata=vars3
     condata=dcon3
     conout=solutn3;
   objfn profit;
   upperbd available;
   run;
```

Here are several different ways of using the ARCDATA= data set and a sparse format CONDATA= data set for this LP. The following CONDATA= data set is the result of removing the profit and available data from the original sparse format CONDATA=scon2 data set.

```
data scon5;
   format _type_ $8. _col_ $8. _row_ $16. ;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
eq       .             napha_l_conv              .
eq       .             napha_i_conv              .
eq       .             heating_oil_conv          .
eq       .             recipe_1                  .
eq       .             recipe_2                  .
.        a_light       napha_l_conv           .035
.        a_light       napha_i_conv           .100
.        a_light       heating_oil_conv       .390
.        a_heavy       napha_l_conv           .030
.        a_heavy       napha_i_conv           .075
.        a_heavy       heating_oil_conv       .300
```

```
.        brega          napha_l_conv          .045
.        brega          napha_i_conv          .135
.        brega          heating_oil_conv      .430
.        naphthal       napha_l_conv           -1
.        naphthal       recipe_2                .2
.        naphthai       napha_i_conv           -1
.        naphthai       recipe_1                .3
.        heatingo       heating_oil_conv       -1
.        heatingo       recipe_1                .7
.        heatingo       recipe_2                .8
.        jet_1          recipe_1               -1
.        jet_2          recipe_2               -1
;

proc intpoint
     maximize
     sparsecondata
     arcdata=vars3      /* or arcdata=vars4 */
     condata=scon5
     conout=solutn5;
   objfn profit;
   upperbd available;
   run;
```

The CONDATA=scon5 data set can become more concise by noting that all the constraints have the same type (eq) and zero (the default) rhs values. Use the DEFCONTYPE= option again. Once the first five observations of the CONDATA=scon5 data set are removed, the _type_ variable has values that are missing in all of the remaining observations. Therefore, this variable can be removed.

```
data scon6;
   input _col_ $ _row_&$16. _coef_;
   datalines;
a_light  napha_l_conv          .035
a_light  napha_i_conv          .100
a_light  heating_oil_conv      .390
a_heavy  napha_l_conv          .030
a_heavy  napha_i_conv          .075
a_heavy  heating_oil_conv      .300
brega    napha_l_conv          .045
brega    napha_i_conv          .135
brega    heating_oil_conv      .430
naphthal napha_l_conv           -1
naphthal recipe_2                .2
naphthai napha_i_conv           -1
naphthai recipe_1                .3
heatingo heating_oil_conv       -1
heatingo recipe_1                .7
heatingo recipe_2                .8
jet_1    recipe_1               -1
jet_2    recipe_2               -1
;
```

```
proc intpoint
     maximize
     defcontype=eq
     sparsecondata
     arcdata=vars4
     condata=scon6
     conout=solutn6;
   objfn profit;
   upperbd available;
   run;
```

Typical PROC INTPOINT Run

You start PROC INTPOINT by giving the PROC INTPOINT statement. You can specify many options in the PROC INTPOINT statement to control the procedure, or you can rely on default settings and specify very few options. However, there are some options you must specify:

- You must specify the BYTES= parameter indicating the size of the working memory that the procedure is allowed to use. This option has no default.
- In many instances (and certainly when solving NPSC problems), you need to specify the ARCDATA= data set. This option has a default (which is the SAS data set that was created last before PROC INTPOINT began running), but that may need to be overridden.
- The CONDATA= data set must also be specified if the problem is NPSC and has side constraints, or if it is an LP problem.
- When solving a network problem, you have to specify the NODEDATA= data set, if some model data is given in such a data set.

Some options, while optional, are frequently required. To have the optimal solution output to a SAS data set, you have to specify the CONOUT= data set. You may want to indicate reasons why optimization should stop (for example, you can indicate the maximum number of iterations that can be performed), or you might want to alter stopping criteria so that optimization does not stop prematurely. Some options enable you to control other aspects of the interior point algorithm. Specifying certain values for these options can reduce the time it takes to solve a problem.

The SAS variable lists should be given next. If you have SAS variables in the input data sets that have special names (for example, a SAS variable in the ARCDATA= data set named _TAIL_ that has tail nodes of arcs as values), it may not be necessary to have many or any variable lists. If you do not specify a TAIL variable list, PROC INTPOINT will search the ARCDATA= data set for a SAS variable named _TAIL_.

What usually follows is a RUN statement, which indicates that all information that you, the user, need to supply to PROC INTPOINT has been given, and the procedure is to start running. This also happens if you specify a statement in your SAS program that PROC INTPOINT does not recognize as one of its own, the next DATA step or procedure.

The QUIT statement indicates that PROC INTPOINT must immediately finish.

For example, a PROC INTPOINT run might look something like this:

```
proc intpoint
    bytes=   /* working memory size */
    arcdata= /* data set */
    condata= /* data set */
    /* other options */
;
variable list specifications;  /* if necessary */
run;     /* start running, read data, */
         /* and do the optimization.  */
```

Syntax: INTPOINT Procedure

Below are statements used in PROC INTPOINT, listed in alphabetical order as they appear in the text that follows.

PROC INTPOINT *options* ;
CAPACITY *variable* ;
COEF *variables* ;
COLUMN *variable* ;
COST *variable* ;
DEMAND *variable* ;
HEADNODE *variable* ;
ID *variables* ;
LO *variable* ;
NAME *variable* ;
NODE *variable* ;
QUIT;
RHS *variable* ;
ROW *variables* ;
RUN;
SUPDEM *variable* ;
SUPPLY *variable* ;
TAILNODE *variable* ;
TYPE *variable* ;
VAR *variables* ;

Functional Summary

Table 2.1 outlines the options that can be specified in the INTPOINT procedure. All options are specified in the PROC INTPOINT statement.

Table 2.1. Functional Summary

Description	Statement	Option
Input Data Set Options:		
arcs input data set	PROC INTPOINT	ARCDATA=
nodes input data set	PROC INTPOINT	NODEDATA=
constraint input data set	PROC INTPOINT	CONDATA=
Output Data Set Option:		
constrained solution data set	PROC INTPOINT	CONOUT=
Data Set Read Options:		
CONDATA has sparse data format	PROC INTPOINT	SPARSECONDATA
default constraint type	PROC INTPOINT	DEFCONTYPE=
special COLUMN variable value	PROC INTPOINT	TYPEOBS=
special COLUMN variable value	PROC INTPOINT	RHSOBS=
used to interpret arc and variable names	PROC INTPOINT	NAMECTRL=
no nonarc data in ARCDATA	PROC INTPOINT	ARCS_ONLY_ARCDATA
data for an arc found once in ARCDATA	PROC INTPOINT	ARC_SINGLE_OBS
data for a constraint found once in CONDATA	PROC INTPOINT	CON_SINGLE_OBS
data for a coefficient found once in CONDATA	PROC INTPOINT	NON_REPLIC=
data is grouped, exploited during data read	PROC INTPOINT	GROUPED=
Problem Size Specification Options:		
approximate number of nodes	PROC INTPOINT	NNODES=
approximate number of arcs	PROC INTPOINT	NARCS=
approximate number of variables	PROC INTPOINT	NNAS=
approximate number of coefficients	PROC INTPOINT	NCOEFS=
approximate number of constraints	PROC INTPOINT	NCONS=
Network Options:		
default arc cost, objective function coefficient	PROC INTPOINT	DEFCOST=
default arc capacity, variable upper bound	PROC INTPOINT	DEFCAPACITY=
default arc flow and variable lower bound	PROC INTPOINT	DEFMINFLOW=
network's only supply node	PROC INTPOINT	SOURCE=
SOURCE's supply capability	PROC INTPOINT	SUPPLY=
network's only demand node	PROC INTPOINT	SINK=
SINK's demand	PROC INTPOINT	DEMAND=
convey excess supply/demand through network	PROC INTPOINT	THRUNET
find max flow between SOURCE and SINK	PROC INTPOINT	MAXFLOW
cost of bypass arc, MAXFLOW problem	PROC INTPOINT	BYPASSDIVIDE=
find shortest path from SOURCE to SINK	PROC INTPOINT	SHORTPATH

Description	Statement	Option
Interior Point Algorithm Options:		
factorization method	PROC INTPOINT	FACT_METHOD=
allowed amount of dual infeasibility	PROC INTPOINT	TOLDINF=
allowed amount of primal infeasibility	PROC INTPOINT	TOLPINF=
allowed total amount of dual infeasibility	PROC INTPOINT	TOLTOTDINF=
allowed total amount of primal infeasibility	PROC INTPOINT	TOLTOTPINF=
cut-off tolerance for Cholesky factorization	PROC INTPOINT	CHOLTINYTOL=
density threshold for Cholesky processing	PROC INTPOINT	DENSETHR=
step-length multiplier	PROC INTPOINT	PDSTEPMULT=
preprocessing type	PROC INTPOINT	PRSLTYPE=
print optimization progress on SAS log	PROC INTPOINT	PRINTLEVEL2=
ratio test zero tolerance	PROC INTPOINT	RTTOL=
Interior Point Algorithm Stopping Criteria:		
maximum number of interior point iterations	PROC INTPOINT	MAXITERB=
primal-dual (duality) gap tolerance	PROC INTPOINT	PDGAPTOL=
stop because of complementarity	PROC INTPOINT	STOP_C=
stop because of duality gap	PROC INTPOINT	STOP_DG=
stop because of $infeas_b$	PROC INTPOINT	STOP_IB=
stop because of $infeas_c$	PROC INTPOINT	STOP_IC=
stop because of $infeas_d$	PROC INTPOINT	STOP_ID=
stop because of complementarity	PROC INTPOINT	AND_STOP_C=
stop because of duality gap	PROC INTPOINT	AND_STOP_DG=
stop because of $infeas_b$	PROC INTPOINT	AND_STOP_IB=
stop because of $infeas_c$	PROC INTPOINT	AND_STOP_IC=
stop because of $infeas_d$	PROC INTPOINT	AND_STOP_ID=
stop because of complementarity	PROC INTPOINT	KEEPGOING_C=
stop because of duality gap	PROC INTPOINT	KEEPGOING_DG=
stop because of $infeas_b$	PROC INTPOINT	KEEPGOING_IB=
stop because of $infeas_c$	PROC INTPOINT	KEEPGOING_IC=
stop because of $infeas_d$	PROC INTPOINT	KEEPGOING_ID=
stop because of complementarity	PROC INTPOINT	AND_KEEPGOING_C=
stop because of duality gap	PROC INTPOINT	AND_KEEPGOING_DG=
stop because of $infeas_b$	PROC INTPOINT	AND_KEEPGOING_IB=
stop because of $infeas_c$	PROC INTPOINT	AND_KEEPGOING_IC=
stop because of $infeas_d$	PROC INTPOINT	AND_KEEPGOING_ID=
Memory Control Options:		
issue memory usage messages to SAS log	PROC INTPOINT	MEMREP
number of bytes to use for main memory	PROC INTPOINT	BYTES=
Miscellaneous Options:		
infinity value	PROC INTPOINT	INFINITY=
maximization instead of minimization	PROC INTPOINT	MAXIMIZE

Description	Statement	Option
zero tolerance - optimization	PROC INTPOINT	ZERO2=
zero tolerance - real number comparisons	PROC INTPOINT	ZEROTOL=
suppress similar SAS log messages	PROC INTPOINT	VERBOSE=
scale problem data	PROC INTPOINT	SCALE=
write optimization time to SAS log	PROC INTPOINT	OPTIM_TIMER

PROC INTPOINT Statement

PROC INTPOINT *options* **;**

This statement invokes the procedure. The following options can be specified in the PROC INTPOINT statement.

Data Set Options

This section briefly describes all the input and output data sets used by PROC INTPOINT. The ARCDATA= data set, the NODEDATA= data set, and the CONDATA= data set can contain SAS variables that have special names, for instance _CAPAC_, _COST_, and _HEAD_. PROC INTPOINT looks for such variables if you do not give explicit variable list specifications. If a SAS variable with a special name is found and that SAS variable is not in another variable list specification, PROC INTPOINT determines that values of the SAS variable are to be interpreted in a special way. By using SAS variables that have special names, you may not need to have any variable list specifications.

ARCDATA=*SAS-data-set*
names the data set that contains arc and, optionally, nonarc variable information and nodal supply/demand data. The ARCDATA= data set must be specified in all PROC INTPOINT statements when solving NPSC problems.

If your problem is an LP, the ARCDATA= data set is optional. You can specify LP variable information such as objective function coefficients, and lower and upper bounds.

CONDATA=*SAS-data-set*
names the data set that contains the side constraint data. The data set can also contain other data such as arc costs, capacities, lower flow bounds, nonarc variable upper and lower bounds, and objective function coefficients. PROC INTPOINT needs a CONDATA= data set to solve a constrained problem. See the section "CONDATA= Data Set" on page 101 for more information.

If your problem is an LP, this data set contains the constraint data, and can also contain other data such as objective function coefficients, and lower and upper bounds. PROC INTPOINT needs a CONDATA= data set to solve an LP.

CONOUT=*SAS-data-set*
COUT=*SAS-data-set*
names the output data set that receives an optimal solution. See the section "CONOUT= Data Set" on page 110 for more information.

If PROC INTPOINT is outputting observations to the output data set and you want this to stop, press the keys used to stop SAS procedures.

NODEDATA=*SAS-data-set*
names the data set that contains the node supply and demand specifications. You do not need observations in the NODEDATA= data set for transshipment nodes. (Transshipment nodes neither supply nor demand flow.) All nodes are assumed to be transshipment nodes unless supply or demand data indicate otherwise. It is acceptable for some arcs to be directed toward supply nodes or away from demand nodes.

This data set is used only when you are solving network problems (not when solving LP problems), in which case the use of the NODEDATA= data set is optional provided that, if the NODEDATA= data set is not used, supply and demand details are specified by other means. Other means include using the MAXFLOW or SHORTPATH option, SUPPLY or DEMAND variable list (or both) in the ARCDATA= data set, and the SOURCE=, SUPPLY=, SINK=, or DEMAND= option in the PROC INTPOINT statement.

General Options

The following is a list of options you can use with PROC INTPOINT. The options are listed in alphabetical order.

ARCS_ONLY_ARCDATA
indicates that data for arcs only are in the ARCDATA= data set. When PROC INTPOINT reads the data in the ARCDATA= data set, memory would not be wasted to receive data for nonarc variables. The read might then be performed faster. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

ARC_SINGLE_OBS
indicates that for all arcs and nonarc variables, data for each arc or nonarc variable is found in only one observation of the ARCDATA= data set. When reading the data in the ARCDATA= data set, PROC INTPOINT knows that the data in an observation is for an arc or a nonarc variable that has not had data previously read and that needs to be checked for consistency. The read might then be performed faster.

When solving an LP, specifying the ARC_SINGLE_OBS option indicates that for all LP variables, data for each LP variable is found in only one observation of the ARCDATA= data set. When reading the data in the ARCDATA= data set, PROC INTPOINT knows that the data in an observation is for an LP variable that has not had data previously read and that needs to be checked for consistency. The read might then be performed faster.

If you specify ARC_SINGLE_OBS, PROC INTPOINT automatically works as if GROUPED=ARCDATA is also specified.

See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

BYPASSDIVIDE=*b*
BYPASSDIV=*b*
BPD=*b*

should be used only when the MAXFLOW option has been specified; that is, PROC INTPOINT is solving a maximal flow problem. PROC INTPOINT prepares to solve maximal flow problems by setting up a bypass arc. This arc is directed from the SOURCE= to the SINK= and will eventually convey flow equal to INFINITY minus the maximal flow through the network. The cost of the bypass arc must be great enough to drive flow through the network, rather than through the bypass arc. Also, the cost of the bypass arc must be greater than the eventual total cost of the maximal flow, which can be nonzero if some network arcs have nonzero costs. The cost of the bypass is set to the value of the INFINITY= option. Valid values for the BYPASSDIVIDE= option must be greater than or equal to 1.1.

If there are no nonzero costs of arcs in the MAXFLOW problem, the cost of the bypass arc is set to 1.0 (-1.0 if maximizing) if you do not specify the BYPASSDIVIDE= option. The default value for the BYPASSDIVIDE= option (in the presence of nonzero arc costs) is 100.0.

BYTES=*b*

indicates the size of the main working memory (in bytes) that PROC INTPOINT will allocate. Specifying this option is mandatory. The working memory is used to store all the arrays and buffers used by PROC INTPOINT. If this memory has a size smaller than what is required to store all arrays and buffers, PROC INTPOINT uses various schemes that page information between auxiliary memory (often your machine's disk) and RAM.

For small problems, specify BYTES=100000. For large problems (those with hundreds of thousands or millions of variables), BYTES=1000000 might do. For solving problems of that size, if you are running on a machine with an inadequate amount of RAM, PROC INTPOINT's performance will suffer since it will be forced to page or to rely on virtual memory.

If you specify the MEMREP option, PROC INTPOINT will issue messages on the SAS log informing you of its memory usage; that is, how much memory is required to prevent paging, and details about the amount of paging that must be performed, if applicable.

CON_SINGLE_OBS

improves how the CONDATA= data set is read. How it works depends on whether the CONDATA has a dense or sparse format.

If the CONDATA= data set has the dense format, specifying CON_SINGLE_OBS indicates that, for each constraint, data for each can be found in only one observation of the CONDATA= data set.

If the CONDATA= data set has a sparse format, and data for each arc, nonarc variable, or LP variable can be found in only one observation of the CONDATA, then specify

the CON_SINGLE_OBS option. If there are n SAS variables in the ROW and COEF list, then each arc or nonarc can have at most n constraint coefficients in the model. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

DEFCAPACITY=*c*
DC=*c*
requests that the default arc capacity and the default nonarc variable value upper bound (or for LP problems, the default LP variable value upper bound) be *c*. If this option is not specified, then DEFCAPACITY=INFINITY.

DEFCONTYPE=*c*
DEFTYPE=*c*
DCT=*c*
specifies the default constraint type. This default constraint type is either *less than or equal to* or is the type indicated by DEFCONTYPE=*c*. Valid values for this option are

LE, le, or <=	for *less than or equal to*
EQ, eq, or =	for *equal to*
GE, ge, or >=	for *greater than or equal to*

The values do not need to be enclosed in quotes.

DEFCOST=*c*
requests that the default arc cost and the default nonarc variable objective function coefficient (or for an LP, the default LP variable objective function coefficient) be *c*. If this option is not specified, then DEFCOST=0.0.

DEFMINFLOW=*m*
DMF=*m*
requests that the default lower flow bound through arcs and the default lower value bound of nonarc variables (or for an LP, the default lower value bound of LP variables) be *m*. If a value is not specified, then DEFMINFLOW=0.0.

DEMAND=*d*
specifies the demand at the SINK node specified by the SINK= option. The DEMAND= option should be used only if the SINK= option is given in the PROC INTPOINT statement and neither the SHORTPATH option nor the MAXFLOW option is specified. If you are solving a minimum cost network problem and the SINK= option is used to identify the sink node, and the DEMAND= option is not specified, then the demand at the sink node is made equal to the network's total supply.

GROUPED=*grouped*
PROC INTPOINT can take a much shorter time to read data if the data have been grouped prior to the PROC INTPOINT call. This enables PROC INTPOINT to conclude that, for instance, a new NAME list variable value seen in the ARCDATA= data set grouped by the values of the NAME list variable before PROC INTPOINT was called is new. PROC INTPOINT does not need to check that the NAME has been read in a previous observation. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

- GROUPED=ARCDATA indicates that the ARCDATA= data set has been grouped by values of the NAME list variable. If _NAME_ is the name of the NAME list variable, you could use

  ```
  proc sort data=arcdata; by _name_;
  ```

  prior to calling PROC INTPOINT. Technically, you do not have to sort the data, only to ensure that all similar values of the NAME list variable are grouped together. If you specify the ARCS_ONLY_ARCDATA option, PROC INTPOINT automatically works as if GROUPED=ARCDATA is also specified.

- GROUPED=CONDATA indicates that the CONDATA= data set has been grouped.

  If the CONDATA= data set has a dense format, GROUPED=CONDATA indicates that the CONDATA= data set has been grouped by values of the ROW list variable. If _ROW_ is the name of the ROW list variable, you could use

  ```
  proc sort data=condata; by _row_;
  ```

  prior to calling PROC INTPOINT. Technically, you do not have to sort the data, only to ensure that all similar values of the ROW list variable are grouped together. If you specify the CON_SINGLE_OBS option, or if there is no ROW list variable, PROC INTPOINT automatically works as if GROUPED=CONDATA has been specified.

  If the CONDATA= data set has the sparse format, GROUPED=CONDATA indicates that CONDATA has been grouped by values of the COLUMN list variable. If _COL_ is the name of the COLUMN list variable, you could use

  ```
  proc sort data=condata; by _col_;
  ```

  prior to calling PROC INTPOINT. Technically, you do not have to sort the data, only to ensure that all similar values of the COLUMN list variable are grouped together.

- GROUPED=BOTH indicates that both GROUPED=ARCDATA and GROUPED=CONDATA are TRUE.

- GROUPED=NONE indicates that the data sets have not been grouped, that is, neither GROUPED=ARCDATA nor GROUPED=CONDATA is TRUE. This is the default, but it is much better if GROUPED=ARCDATA, or GROUPED=CONDATA, or GROUPED=BOTH.

A data set like

```
... _XXXXX_ ....
      bbb
      bbb
      aaa
      ccc
      ccc
```

is a candidate for the GROUPED= option. Similar values are grouped together. When PROC INTPOINT is reading the ith observation, either the value of the _XXXXX_ variable is the same as the $(i-1)$st (that is, the previous observation's) _XXXXX_ value, or it is a new _XXXXX_ value not seen in any previous observation. This also means that if the ith _XXXXX_ value is different from the $(i-1)$st _XXXXX_ value, the value of the $(i-1)$st _XXXXX_ variable will not be seen in any observations $i, i+1, \ldots$.

INFINITY=*i*
INF=*i*

is the largest number used by PROC INTPOINT in computations. A number too small can adversely affect the solution process. You should avoid specifying an enormous value for the INFINITY= option because numerical roundoff errors can result. If a value is not specified, then INFINITY=99999999. The INFINITY= option cannot be assigned a value less than 9999.

MAXFLOW
MF

specifies that PROC INTPOINT solve a maximum flow problem. In this case, the PROC INTPOINT procedure finds the maximum flow from the node specified by the SOURCE= option to the node specified by the SINK= option. PROC INTPOINT automatically assigns an INFINITY= option supply to the SOURCE= option node and the SINK= option is assigned the INFINITY= option demand. In this way, the MAXFLOW option sets up a maximum flow problem as an equivalent minimum cost problem.

You can use the MAXFLOW option when solving any flow problem (not necessarily a maximum flow problem) when the network has one supply node (with infinite supply) and one demand node (with infinite demand). The MAXFLOW option can be used in conjunction with all other options (except SHORTPATH, SUPPLY=, and DEMAND=) and capabilities of PROC INTPOINT.

MAXIMIZE
MAX

specifies that PROC INTPOINT find the maximum cost flow through the network. If both the MAXIMIZE and the SHORTPATH options are specified, the solution obtained is the longest path between the SOURCE= and SINK= nodes. Similarly, MAXIMIZE and MAXFLOW together cause PROC INTPOINT to find the minimum flow between these two nodes; this is zero if there are no nonzero lower flow bounds. If solving an LP, specifying the MAXIMIZE option is necessary if you want the maximal optimal solution found instead of the minimal optimum.

MEMREP

indicates that information on the memory usage and paging schemes (if necessary) is reported by PROC INTPOINT on the SAS log.

NAMECTRL=*i*

is used to interpret arc and nonarc variable names in the CONDATA= data set. In the ARCDATA= data set, an arc is identified by its tail and head node. In the CONDATA= data set, arcs are identified by names. You can give a name to an arc by having a

NAME list specification that indicates a SAS variable in the ARCDATA= data set that has names of arcs as values.

PROC INTPOINT requires that arcs that have information about them in the CONDATA= data set have names, but arcs that do not have information about them in the CONDATA= data set can also have names. Unlike a nonarc variable whose name uniquely identifies it, an arc can have several different names. An arc has a default name in the form *tail_head*, that is, the name of the arc's tail node followed by an underscore and the name of the arc's head node.

In the CONDATA= data set, if the dense data format is used (described in the section "CONDATA= Data Set" on page 101), a name of an arc or a nonarc variable is the *name* of a SAS variable listed in the VAR list specification. If the sparse data format of the CONDATA= data set is used, a name of an arc or a nonarc variable is a *value* of the SAS variable listed in the COLUMN list specification.

The NAMECTRL= option is used when a name of an arc or a nonarc variable in the CONDATA= data set (either a VAR list variable name or a value of the COLUMN list variable) is in the form *tail_head* and there exists an arc with these end nodes. If *tail_head* has not already been tagged as belonging to an arc or nonarc variable in the ARCDATA= data set, PROC INTPOINT needs to know whether *tail_head* is the name of the arc or the name of a nonarc variable.

If you specify NAMECTRL=1, a name that is not defined in the ARCDATA= data set is assumed to be the name of a nonarc variable. NAMECTRL=2 treats *tail_head* as the name of the arc with these endnodes, provided no other name is used to associate data in the CONDATA= data set with this arc. If the arc does have other names that appear in the CONDATA= data set, *tail_head* is assumed to be the name of a nonarc variable. If you specify NAMECTRL=3, *tail_head* is assumed to be a name of the arc with these end nodes, whether the arc has other names or not. The default value of NAMECTRL is 3.

If the dense format is used for the CONDATA= data set, there are two circumstances that affect how this data set is read:

1. if you are running SAS Version 6, or a previous version to that, or if you are running SAS Version 7 onward and you specify

   ```
   options validvarname=v6;
   ```

   in your SAS session. Let's refer to this as *case 1*.
2. if you are running SAS Version 7 onward and you do not specify

   ```
   options validvarname=v6;
   ```

   in your SAS session. Let's refer to this as *case 2*.

For *case 1*, the SAS System converts SAS variable names in a SAS program to uppercase. The VAR list variable names are uppercased. Because of this, PROC

INTPOINT automatically uppercases names of arcs and nonarc variables or LP variables (the values of the NAME list variable) in the ARCDATA= data set. The names of arcs and nonarc variables or LP variables (the values of the NAME list variable) appear uppercased in the CONOUT= data set.

Also, if the dense format is used for the CONDATA= data set, be careful with default arc names (names in the form tailnode_headnode). Node names (values in the TAILNODE and HEADNODE list variables) in the ARCDATA= data set are not automatically uppercased by PROC INTPOINT. Consider the following code.

```
data arcdata;
   input _from_ $ _to_ $ _name $ ;
   datalines;
from to1 .
from to2 arc2
TAIL TO3 .
;
data densecon;
   input from_to1 from_to2 arc2 tail_to3;
   datalines;
2 3 3 5
;
proc intpoint
   arcdata=arcdata condata=densecon;
   run;
```

The SAS System does not uppercase character string values within SAS data sets. PROC INTPOINT never uppercases node names, so the arcs in observations 1, 2, and 3 in the preceding ARCDATA= data set have the default names from_to1, from_to2, and TAIL_TO3, respectively. When the dense format of the CONDATA= data set is used, PROC INTPOINT does uppercase values of the NAME list variable, so the name of the arc in the second observation of the ARCDATA= data set is ARC2. Thus, the second arc has two names: its default from_to2 and the other that was specified ARC2.

As the SAS System uppercases program code, you must think of the input statement

```
input from_to1 from_to2 arc2 tail_to3;
```

as really being

```
INPUT FROM_TO1 FROM_TO2 ARC2 TAIL_TO3;
```

The SAS variables named FROM_TO1 and FROM_TO2 are *not* associated with any of the arcs in the preceding ARCDATA= data set. The values FROM_TO1 and FROM_TO2 are different from all of the arc names from_to1, from_to2, TAIL_TO3, and ARC2. FROM_TO1 and FROM_TO2 could end up being the names of two nonarc variables.

The SAS variable named ARC2 is the name of the second arc in the ARCDATA= data set, even though the name specified in the ARCDATA= data set looks like

arc2. The SAS variable named TAIL_TO3 is the default name of the third arc in the ARCDATA= data set.

For *case 2*, the SAS System does not convert SAS variable names in a SAS program to uppercase. The VAR list variable names are not uppercased. PROC INTPOINT does not automatically uppercase names of arcs and nonarc variables or LP variables (the values of the NAME list variable) in the ARCDATA= data set. PROC INTPOINT does not uppercase any SAS variable names, data set values, or indeed anything. Therefore, PROC INTPOINT respects case, and characters in the data if compared must have the right case if you mean them to be the same. Note how the input statement in the data step that initialized the data set densecon below is specified in the following code:

```
data arcdata;
   input _from_ $ _to_ $ _name $ ;
   datalines;
from to1 .
from to2 arc2
TAIL TO3 .
;
data densecon;
   input from_to1 from_to2 arc2 TAIL_TO3;
   datalines;
2 3 3 5
;
proc intpoint
   arcdata=arcdata condata=densecon;
   run;
```

NARCS=*n*
specifies the approximate number of arcs. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

NCOEFS=*n*
specifies the approximate number of constraint coefficients. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

NCONS=*n*
specifies the approximate number of constraints. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

NNAS=*n*
specifies the approximate number of nonarc variables. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

NNODES=*n*
specifies the approximate number of nodes. See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

NON_REPLIC=*non_replic*
prevents PROC INTPOINT from doing unnecessary checks of data previously read.

- NON_REPLIC=COEFS indicates that each constraint coefficient is specified *once* in the CONDATA= data set.
- NON_REPLIC=NONE indicates that constraint coefficients can be specified more than once in the CONDATA= data set. NON_REPLIC=NONE is the default.

See the section "How to Make the Data Read of PROC INTPOINT More Efficient" on page 120.

OPTIM_TIMER

indicates that the procedure is to issue a message to the SAS log giving the CPU time spent doing optimization. This includes the time spent preprocessing, performing optimization, and postprocessing. Not counted in that time is the rest of the procedure execution, which includes reading the data and creating output SAS data sets.

The time spent optimizing can be small compared to the total CPU time used by the procedure. This is especially true when the problem is quite small (e.g., fewer than 10,000 variables).

RHSOBS=*charstr*

specifies the keyword that identifies a right-hand-side observation when using the sparse format for data in the CONDATA= data set. The keyword is expected as a value of the SAS variable in the CONDATA= data set named in the COLUMN list specification. The default value of the RHSOBS= option is _RHS_ or _rhs_. If *charstr* is not a valid SAS variable name, enclose it in quotes.

SCALE=*scale*

indicates that the NPSC side constraints or the LP constraints are to be scaled. Scaling is useful when some coefficients are either much larger or much smaller than other coefficients. Scaling might make all coefficients have values that have a smaller range, and this can make computations more stable numerically. Try the SCALE= option if PROC INTPOINT is unable to solve a problem because of numerical instability. Specify

- SCALE=ROW, SCALE=CON, or SCALE=CONSTRAINT if you want the largest absolute value of coefficients in each constraint to be about 1.0
- SCALE=COL, SCALE=COLUMN, or SCALE=NONARC if you want NPSC nonarc variable columns or LP variable columns to be scaled so that the absolute value of the largest constraint coefficient of that variable is near to 1
- SCALE=BOTH if you want the largest absolute value of coefficients in each constraint, and the absolute value of the largest constraint coefficient of an NPSC nonarc variable or LP variable to be near to 1. This is the default.
- SCALE=NONE if no scaling should be done

SHORTPATH
SP

specifies that PROC INTPOINT solve a shortest path problem. The INTPOINT procedure finds the shortest path between the nodes specified in the SOURCE= option

and the SINK= option. The costs of arcs are their *lengths*. PROC INTPOINT automatically assigns a supply of one flow unit to the SOURCE= node, and the SINK= node is assigned to have a one flow unit demand. In this way, the SHORTPATH option sets up a shortest path problem as an equivalent minimum cost problem.

If a network has one supply node (with supply of one unit) and one demand node (with demand of one unit), you could specify the SHORTPATH option, with the SOURCE= and SINK= nodes, even if the problem is not a shortest path problem. You then should not provide any supply or demand data in the NODEDATA= data set or the ARCDATA= data set.

SINK=*sinkname*
SINKNODE=*sinkname*

identifies the demand node. The SINK= option is useful when you specify the MAXFLOW option or the SHORTPATH option and you need to specify toward which node the shortest path or maximum flow is directed. The SINK= option also can be used when a minimum cost problem has only one demand node. Rather than having this information in the ARCDATA= data set or the NODEDATA= data set, use the SINK= option with an accompanying DEMAND= specification for this node. The SINK= option must be the name of a head node of at least one arc; thus, it must have a character value. If the value of the SINK= option is not a valid SAS character variable name (if, for example, it contains embedded blanks), it must be enclosed in quotes.

SOURCE=*sourcename*
SOURCENODE=*sourcename*

identifies a supply node. The SOURCE= option is useful when you specify the MAXFLOW or the SHORTPATH option and need to specify from which node the shortest path or maximum flow originates. The SOURCE= option also can be used when a minimum cost problem has only one supply node. Rather than having this information in the ARCDATA= data set or the NODEDATA= data set, use the SOURCE= option with an accompanying SUPPLY= amount of supply at this node. The SOURCE= option must be the name of a tail node of at least one arc; thus, it must have a character value. If the value of the SOURCE= option is not a valid SAS character variable name (if, for example, it contains embedded blanks), it must be enclosed in quotes.

SPARSECONDATA
SCDATA

indicates that the CONDATA= data set has data in the sparse data format. Otherwise, it is assumed that the data are in the dense format.

Note: If the SPARSECONDATA option is not specified, and you are running SAS software Version 6 or you have specified

```
options validvarname=v6;
```

all NAME list variable values in the ARCDATA= data set are uppercased. See the section "Case Sensitivity" on page 111.

SUPPLY=*s*

specifies the supply at the source node specified by the SOURCE= option. The SUPPLY= option should be used only if the SOURCE= option is given in the PROC INTPOINT statement and neither the SHORTPATH option nor the MAXFLOW option is specified. If you are solving a minimum cost network problem and the SOURCE= option is used to identify the source node and the SUPPLY= option is not specified, then by default the supply at the source node is made equal to the network's total demand.

THRUNET

tells PROC INTPOINT to force through the network any excess supply (the amount by which total supply exceeds total demand) or any excess demand (the amount by which total demand exceeds total supply) as is required. If a network problem has unequal total supply and total demand and the THRUNET option is not specified, PROC INTPOINT drains away the excess supply or excess demand in an optimal manner. The consequences of specifying or not specifying THRUNET are discussed in the section "Balancing Total Supply and Total Demand" on page 119.

TYPEOBS=*charstr*

specifies the keyword that identifies a type observation when using the sparse format for data in the CONDATA= data set. The keyword is expected as a value of the SAS variable in the CONDATA= data set named in the COLUMN list specification. The default value of the TYPEOBS= option is _TYPE_ or _type_. If *charstr* is not a valid SAS variable name, enclose it in quotes.

VERBOSE=*v*

limits the number of similar messages that are displayed on the SAS log.

For example, when reading the ARCDATA= data set, PROC INTPOINT might have cause to issue the following message many times:

```
ERROR: The HEAD list variable value in obs i in ARCDATA is
       missing and the TAIL list variable value of this obs
       is nonmissing. This is an incomplete arc specification.
```

If there are many observations that have this fault, messages that are similar are issued for only the first VERBOSE= such observations. After the ARCDATA= data set has been read, PROC INTPOINT will issue the message

```
NOTE: More messages similar to the ones immediately above
      could have been issued but were suppressed as
      VERBOSE=v.
```

If observations in the ARCDATA= data set have this error, PROC INTPOINT stops and you have to fix the data. Imagine that this error is only a warning and PROC INTPOINT proceeded to other operations such as reading the CONDATA= data set. If PROC INTPOINT finds there are numerous errors when reading that data set, the number of messages issued to the SAS log are also limited by the VERBOSE= option.

When PROC INTPOINT finishes and messages have been suppressed, the message

```
NOTE: To see all messages, specify VERBOSE=vmin.
```

is issued. The value of *vmin* is the smallest value that should be specified for the VERBOSE= option so that *all* messages are displayed if PROC INTPOINT is run again with the same data and everything else (except VERBOSE=*vmin*) unchanged.

The default value for the VERBOSE= option is 12.

ZERO2=*z*
Z2=*z*

specifies the zero tolerance level used when determining whether the final solution has been reached. ZERO2= is also used when outputting the solution to the CONOUT= data set. Values within *z* of zero are set to 0.0, where *z* is the value of the ZERO2= option. Flows close to the lower flow bound or capacity of arcs are reassigned those exact values. If there are nonarc variables, values close to the lower or upper value bound of nonarc variables are reassigned those exact values. When solving an LP problem, values close to the lower or upper value bound of LP variables are reassigned those exact values.

The ZERO2= option works when determining whether optimality has been reached or whether an element in the vector $(\Delta x^k, \Delta y^k, \Delta s^k)$ is less than or greater than zero. It is crucial to know that when determining the maximal value for the step length α in the formula

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

See the description of the PDSTEPMULT= option for more details on this computation.

Two values are deemed to be close if one is within *z* of the other. The default value for the ZERO2= option is 0.000001. Any value specified for the ZERO2= option that is < 0.0 or > 0.0001 is not valid.

ZEROTOL=*z*

specifies the zero tolerance used when PROC INTPOINT must compare any real number with another real number, or zero. For example, if x and y are real numbers, then for x to be considered greater than y, x must be at least $y + z$. The ZEROTOL= option is used throughout any PROC INTPOINT run.

ZEROTOL=*z* controls the way PROC INTPOINT performs all double precision comparisons; that is, whether a double precision number is equal to, not equal to, greater than (or equal to), or less than (or equal to) zero or some other double precision number. A double precision number is deemed to be the same as another such value if the absolute difference between them is less than or equal to the value of the ZEROTOL= option.

The default value for the ZEROTOL= option is $1.0\text{E}{-14}$. You can specify the ZEROTOL= option in the INTPOINT statement. Valid values for the ZEROTOL= option must be > 0.0 and < 0.0001. Do not specify a value too close to zero as this defeats the purpose of the ZEROTOL= option. Neither should the value be too large, as comparisons might be incorrectly performed.

Interior Point Algorithm Options

FACT_METHOD=*f*

enables you to choose the type of algorithm used to factorize and solve the main linear systems at each iteration of the interior point algorithm.

FACT_METHOD=LEFT_LOOKING is new for SAS 9.1.2. It uses algorithms described in George, Liu, and Ng (2001). Left looking is one of the main methods used to perform Cholesky optimization and, along with some recently developed implementation approaches, can be faster and require less memory than other algorithms.

Specify FACT_METHOD=USE_OLD if you want the procedure to use the only factorization available prior to SAS 9.1.2.

TOLDINF=*t*
RTOLDINF=*t*

specifies the allowed amount of dual infeasibility. In the section "Interior Point Algorithmic Details" on page 39, the vector $infeas_d$ is defined. If all elements of this vector are $\leq t$, the solution is considered dual feasible. $infeas_d$ is replaced by a zero vector, making computations faster. This option is the dual equivalent to the TOLPINF= option. Increasing the value of the TOLDINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLPINF=*t*
RTOLPINF=*t*

specifies the allowed amount of primal infeasibility. This option is the primal equivalent to the TOLDINF= option. In the section "Interior Point: Upper Bounds" on page 43, the vector $infeas_b$ is defined. In the section "Interior Point Algorithmic Details" on page 39, the vector $infeas_c$ is defined. If all elements in these vectors are $\leq t$, the solution is considered primal feasible. $infeas_b$ and $infeas_c$ are replaced by zero vectors, making computations faster. Increasing the value of the TOLPINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLTOTDINF=*t*
RTOLTOTDINF=*t*

specifies the allowed total amount of dual infeasibility. In the section "Interior Point Algorithmic Details" on page 39, the vector $infeas_d$ is defined. If $\sum_{i=1}^{n} infeas_{di} \leq t$, the solution is considered dual feasible. $infeas_d$ is replaced by a zero vector, making computations faster. This option is the dual equivalent to the TOLTOTPINF= option. Increasing the value of the TOLTOTDINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLTOTPINF=*t*
RTOLTOTPINF=*t*

specifies the allowed total amount of primal infeasibility. This option is the primal equivalent to the TOLTOTDINF= option. In the section "Interior Point: Upper

Bounds" on page 43, the vector $infeas_b$ is defined. In the section "Interior Point Algorithmic Details" on page 39, the vector $infeas_c$ is defined. If $\sum_{i=1}^{n} infeas_{bi} \leq t$ and $\sum_{i=1}^{m} infeas_{ci} \leq t$, the solution is considered primal feasible. $infeas_b$ and $infeas_c$ are replaced by zero vectors, making computations faster. Increasing the value of the TOLTOTPINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

CHOLTINYTOL=*c*
RCHOLTINYTOL=*c*

specifies the cut-off tolerance for Cholesky factorization of the $A\Theta A^{-1}$. If a diagonal value drops below *c*, the row is essentially treated as dependent and is ignored in the factorization. Valid values for *c* are between $1.0\text{E}{-30}$ and $1.0\text{E}{-6}$. The default value is $1.0\text{E}{-8}$.

DENSETHR=*d*
RDENSETHR=*d*

specifies the density threshold for Cholesky factorization. When the symbolic factorization encounters a column of L (where L is the remaining unfactorized submatrix) that has DENSETHR= proportion of nonzeros and the remaining part of L is at least 12×12, the remainder of L is treated as dense. In practice, the lower right part of the Cholesky triangular factor L is quite dense and it can be computationally more efficient to treat it as 100% dense. The default value for *d* is 0.7. A specification of $d \leq 0.0$ causes all dense processing; $d \geq 1.0$ causes all sparse processing.

PDSTEPMULT=*p*
RPDSTEPMULT=*p*

specifies the step-length multiplier. The maximum feasible step-length chosen by the interior point algorithm is multiplied by the value of the PDSTEPMULT= option. This number must be less than 1 to avoid moving beyond the barrier. An actual step-length greater than 1 indicates numerical difficulties. Valid values for *p* are between 0.01 and 0.999999. The default value is 0.99995.

In the section "Interior Point Algorithmic Details" on page 39, the solution of the next iteration is obtained by moving along a direction from the current iteration's solution:

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where α is the maximum feasible step-length chosen by the interior point algorithm. If $\alpha \leq 1$, then α is reduced slightly by multiplying it by *p*. α is a value as large as possible but ≤ 1.0 and not so large that an x_i^{k+1} or s_i^{k+1} of some variable i is "too close" to zero.

PRSLTYPE=*p*
IPRSLTYPE=*p*

Preprocessing the linear programming problem often succeeds in allowing some variables and constraints to be temporarily eliminated from the resulting LP that must be solved. This reduces the solution time and possibly also the chance that the optimizer will run into numerical difficulties. The task of preprocessing is inexpensive to do.

You control how much preprocessing to do by specifying PRSLTYPE=*p*, where *p* can be -1, 0, 1, 2, or 3:

-1 Do not perform preprocessing. For most problems, specifying PRSLTYPE=-1 is *not* recommended.

0 Given upper and lower bounds on each variable, the greatest and least contribution to the row activity of each variable is computed. If these are within the limits set by the upper and lower bounds on the row activity, then the row is redundant and can be discarded. Otherwise, whenever possible, the bounds on any of the variables are tightened. For example, if all coefficients in a constraint are positive and all variables have zero lower bounds, then the row's smallest contribution is zero. If the rhs value of this constraint is zero, then if the constraint type is $=$ or $\leq$, all the variables in that constraint are fixed to zero. These variables and the constraint are removed. If the constraint type is $\geq$, the constraint is redundant. If the rhs is negative and the constraint is $\leq$, the problem is infeasible. If just one variable in a row is not fixed, the row to used to impose an implicit upper or lower bound on the variable and then this row is eliminated. The preprocessor also tries to tighten the bounds on constraint right-hand sides.

1 When there are exactly two unfixed variables with coefficients in an equality constraint, one variable is solved in terms of the other. The problem will have one less variable. The new matrix will have at least two fewer coefficients and one less constraint. In other constraints where both variables appear, two coefficients are combined into one. PRSLTYPE=0 reductions are also done.

2 It may be possible to determine that an equality constraint is not constraining a variable. That is, if all variables are nonnegative, then $x - \sum_i y_i = 0$ does not constrain x, since it must be nonnegative if all the y_i's are nonnegative. In this case, x is eliminated by subtracting this equation from all others containing x. This is useful when the only other entry for x is in the objective function. This reduction is performed if there is at most one other nonobjective coefficient. PRSLTYPE=0 reductions are also done.

3 All possible reductions are performed. PRSLTYPE=3 is the default.

Preprocessing is iterative. As variables are fixed and eliminated, and constraints are found to be redundant and they too are eliminated, and as variable bounds and constraint right-hand sides are tightened, the LP to be optimized is modified to reflect these changes. Another iteration of preprocessing of the modified LP may reveal more variables and constraints that are eliminated, or tightened.

PRINTLEVEL2=*p*

is used when you want to see PROC INTPOINT's progress to the optimum. PROC INTPOINT will produce a table on the SAS log. A row of the table is generated during each iteration and may consist of values of

- the affine step complementarity
- the complementarity of the solution for the next iteration

- the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43)
- the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39)
- the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39)

As optimization progresses, the values in all columns should converge to zero. If you specify PRINTLEVEL2=2, all columns will appear in the table. If PRINTLEVEL2=1 is specified, only the affine step complementarity and the complementarity of the solution for the next iteration will appear. Some time is saved by not calculating the infeasibility values.

PRINTLEVEL2=2 is specified in all PROC INTPOINT runs in the section "Examples: INTPOINT Procedure" on page 132.

RTTOL=*r*

specifies the zero tolerance used during the ratio test of the interior point algorithm. The ratio test determines α, the maximum feasible step length.

Valid values for r are greater than $1.0\text{E}{-}14$. The default value is $1.0\text{E}{-}10$.

In the section "Interior Point Algorithmic Details" on page 39, the solution of the next iteration is obtained by moving along a direction from the current iteration's solution:

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where α is the maximum feasible step-length chosen by the interior point algorithm. If $\alpha \leq 1$, then α is reduced slightly by multiplying it by the value of the PDSTEPMULT= option. α is a value as large as possible but ≤ 1.0 and not so large that an x_i^{k+1} or s_i^{k+1} of some variable i is negative. When determining α, only negative elements of Δx and Δs are important.

RTTOL=r indicates a number close to zero so that another number n is considered truly negative if $n \leq -r$. Even though $n < 0$, if $n > -r$, n may be too close to zero and may have the wrong sign due to rounding error.

Interior Point Algorithm Options: Stopping Criteria

MAXITERB=*m*
IMAXITERB=*m*

specifies the maximum number of iterations that the interior point algorithm can perform. The default value for *m* is 100. One of the most remarkable aspects of the interior point algorithm is that for most problems, it usually needs to do a small number of iterations, *no matter the size of the problem*.

PDGAPTOL=*p*
RPDGAPTOL=*p*

specifies the primal-dual gap or duality gap tolerance. Duality gap is defined in the section "Interior Point Algorithmic Details" on page 39. If the relative gap

$(duality\ gap/(c^T x))$ between the primal and dual objectives is smaller than the value of the PDGAPTOL= option and both the primal and dual problems are feasible, then PROC INTPOINT stops optimization with a solution that is deemed optimal. Valid values for *p* are between 1.0E−12 and 1.0E−1. The default is 1.0E−7.

STOP_C=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is <= *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

STOP_DG=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is <= *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

STOP_IB=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is <= *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

STOP_IC=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is <= *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

STOP_ID=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is <= *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

AND_STOP_C=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is <= *s*, *and* the other conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

AND_STOP_DG=*s*
is used to determine whether optimization should stop. At the beginning of each iteration, if the duality gap (the value of the Duality_gap column in the table produced

when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $<= s$, *and* the other conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

AND_STOP_IB=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $<= s$, *and* the other conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

AND_STOP_IC=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $<= s$, *and* the other conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

AND_STOP_ID=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $<= s$, *and* the other conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 126.

KEEPGOING_C=*s*

is used to determine whether optimization should stop. When a stopping condition is met, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

KEEPGOING_DG=*s*

is used to determine whether optimization should stop. When a stopping condition is met, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

KEEPGOING_IB=*s*

is used to determine whether optimization should stop. When a stopping condition is met, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

KEEPGOING_IC=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

KEEPGOING_ID=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

AND_KEEPGOING_C=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, *and* the other conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

AND_KEEPGOING_DG=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, *and* the other conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

AND_KEEPGOING_IB=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=2) is $> s$, *and* the other conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

AND_KEEPGOING_IC=*s*
is used to determine whether optimization should stop. When a stopping condition is met, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $> s$, *and* the other conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

AND_KEEPGOING_ID=*s*

is used to determine whether optimization should stop. When a stopping condition is met, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $>$ *s*, *and* the other conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 126.

CAPACITY Statement

CAPACITY *variable* **;**

CAPAC *variable* **;**

UPPERBD *variable* **;**

The CAPACITY statement identifies the SAS variable in the ARCDATA= data set that contains the maximum feasible flow or capacity of the network arcs. If an observation contains nonarc variable information, the CAPACITY list variable is the upper value bound for the nonarc variable named in the NAME list variable in that observation.

When solving an LP, the CAPACITY statement identifies the SAS variable in the ARCDATA= data set that contains the maximum feasible value of the LP variables.

The CAPACITY list variable must have numeric values. It is not necessary to have a CAPACITY statement if the name of the SAS variable is _CAPAC_, _UPPER_, _UPPERBD, or _HI_.

COEF Statement

COEF *variables* **;**

The COEF list is used with the sparse input format of the CONDATA= data set. The COEF list can contain more than one SAS variable, each of which must have numeric values. If the COEF statement is not specified, the CONDATA= data set is searched and SAS variables with names beginning with _COE are used. The number of SAS variables in the COEF list must be no greater than the number of SAS variables in the ROW list.

The values of the COEF list variables in an observation can be interpreted differently than these variables' values in other observations. The values can be coefficients in the side constraints, costs and objective function coefficients, bound data, constraint type data, or rhs data. If the COLUMN list variable has a value that is a name of an arc or a nonarc variable, the ith COEF list variable is associated with the constraint or special row name named in the ith ROW list variable. Otherwise, the COEF list variables indicate type values, rhs values, or missing values.

When solving an LP, the values of the COEF list variables in an observation can be interpreted differently than these variables' values in other observations. The values can be coefficients in the constraints, objective function coefficients, bound data,

constraint type data, or rhs data. If the COLUMN list variable has a value that is a name of an LP variable, the ith COEF list variable is associated with the constraint or special row name named in the ith ROW list variable. Otherwise, the COEF list variables indicate type values, rhs values, or missing values.

COLUMN Statement

COLUMN *variable* **;**

The COLUMN list is used with the sparse input format of the CONDATA= data set.

This list consists of one SAS variable in the CONDATA= data set that has as values the names of arc variables, nonarc variables, or missing values. When solving an LP, this list consists of one SAS variable in the CONDATA= data set that has as values the names of LP variables, or missing values. Some, if not all, of these values also can be values of the NAME list variables of the ARCDATA= data set. The COLUMN list variable can have other special values (Refer to the TYPEOBS= and RHSOBS= options). If the COLUMN list is not specified after the PROC INTPOINT statement, the CONDATA= data set is searched and a SAS variable named _COLUMN_ is used. The COLUMN list variable must have character values.

COST Statement

COST *variable* **;**

OBJFN *variable* **;**

The COST statement identifies the SAS variable in the ARCDATA= data set that contains the per unit flow cost through an arc. If an observation contains nonarc variable information, the value of the COST list variable is the objective function coefficient of the nonarc variable named in the NAME list variable in that observation.

If solving an LP, the COST statement identifies the SAS variable in the ARCDATA= data set that contains the per unit objective function coefficient of an LP variable named in the NAME list variable in that observation.

The COST list variable must have numeric values. It is not necessary to specify a COST statement if the name of the SAS variable is _COST_ or _LENGTH_.

DEMAND Statement

DEMAND *variable* **;**

The DEMAND statement identifies the SAS variable in the ARCDATA= data set that contains the demand at the node named in the corresponding HEADNODE list variable. The DEMAND list variable must have numeric values. It is not necessary to have a DEMAND statement if the name of this SAS variable is _DEMAND_. See the section "Missing S Supply and Missing D Demand Values" on page 114 for cases when the SUPDEM list variable values can have other values. There should be no DEMAND statement if you are solving an LP.

HEADNODE Statement

HEADNODE *variable* **;**

HEAD *variable* **;**

TONODE *variable* **;**

TO *variable* **;**

The HEADNODE statement specifies the SAS variable that must be present in the ARCDATA= data set that contains the names of nodes toward which arcs are directed. It is not necessary to have a HEADNODE statement if the name of the SAS variable is _HEAD_ or _TO_. The HEADNODE variable must have character values.

There should be no HEAD statement if you are solving an LP.

ID Statement

ID *variables* **;**

The ID statement specifies SAS variables containing values for pre- and post-optimal processing and analysis. These variables are not processed by PROC INTPOINT but are read by the procedure and written in the CONOUT= data set. For example, imagine a network used to model a distribution system. The SAS variables listed on the ID statement can contain information on the type of vehicle, the transportation mode, the condition of the road, the time to complete the journey, the name of the driver, or other ancillary information useful for report writing or describing facets of the operation that do not have bearing on the optimization. The ID variables can be character, numeric, or both.

If no ID list is specified, the procedure forms an ID list of all SAS variables not included in any other implicit or explicit list specification. If the ID list is specified, any SAS variables in the ARCDATA= data set not in any list are dropped and do not appear in the CONOUT= data set.

LO Statement

LO *variable* **;**

LOWERBD *variable* **;**

MINFLOW *variable* **;**

The LO statement identifies the SAS variable in the ARCDATA= data set that contains the minimum feasible flow or lower flow bound for arcs in the network. If an observation contains nonarc variable information, the LO list variable has the value of the lower bound for the nonarc variable named in the NAME list variable. If solving an LP, the LO statement identifies the SAS variable in the ARCDATA= data set that contains the lower value bound for LP variables. The LO list variables must have numeric values. It is not necessary to have a LO statement if the name of this SAS variable is _LOWER_, _LO_, _LOWERBD, or _MINFLOW.

NAME Statement

NAME *variable* **;**

ARCNAME *variable* **;**

VARNAME *variable* **;**

Each arc and nonarc variable in an NPSC, or each variable in an LP, that has data in the CONDATA= data set must have a unique name. This variable is identified in the ARCDATA= data set. The NAME list variable must have character values (see the NAMECTRL= option in the PROC INTPOINT statement for more information). It is not necessary to have a NAME statement if the name of this SAS variable is _NAME_.

NODE Statement

NODE *variable* **;**

The NODE list variable, which must be present in the NODEDATA= data set, has names of nodes as values. These values must also be values of the TAILNODE list variable, the HEADNODE list variable, or both. If this list is not explicitly specified, the NODEDATA= data set is searched for a SAS variable with the name _NODE_. The NODE list variable must have character values.

QUIT Statement

QUIT ;

The QUIT statement indicates that PROC INTPOINT is to stop immediately. The solution is not saved in the CONOUT= data set. The QUIT statement has no options.

RHS Statement

RHS *variable* **;**

The RHS variable list is used when the dense format of the CONDATA= data set is used. The values of the SAS variable specified in the RHS list are constraint right-hand-side values. If the RHS list is not specified, the CONDATA= data set is searched and a SAS variable with the name _RHS_ is used. The RHS list variable must have numeric values. If there is no RHS list and no SAS variable named _RHS_, all constraints are assumed to have zero right-hand-side values.

ROW Statement

ROW *variables* **;**

The ROW list is used when either the sparse or the dense format of the CONDATA= data set is being used. SAS variables in the ROW list have values that are constraint or special row names. The SAS variables in the ROW list must have character values.

If the dense data format is used, there must be only one SAS variable in this list. In this case, if a ROW list is not specified, the CONDATA= data set is searched and the

SAS variable with the name _ROW_ or _CON_ is used. If that search fails to find a suitable SAS variable, data for each constraint must reside in only one observation.

If the sparse data format is used and the ROW statement is not specified, the CONDATA= data set is searched and SAS variables with names beginning with _ROW or _CON are used. The number of SAS variables in the ROW list must not be less than the number of SAS variables in the COEF list. The ith ROW list variable is paired with the ith COEF list variable. If the number of ROW list variables is greater than the number of COEF list variables, the last ROW list variables have no COEF partner. These ROW list variables that have no corresponding COEF list variable are used in observations that have a TYPE list variable value. All ROW list variable values are tagged as having the type indicated. If there is no TYPE list variable, all ROW list variable values are constraint names.

RUN Statement

RUN ;

The RUN statement causes optimization to be started. The RUN statement has no options. If PROC INTPOINT is called and is not terminated because of an error or a QUIT statement, and you have not used a RUN statement, a RUN statement is assumed implicitly as the last statement of PROC INTPOINT. Therefore, PROC INTPOINT reads that data, performs optimization, and saves the optimal solution in the CONOUT= data set.

SUPDEM Statement

SUPDEM *variable* **;**

The SAS variable in this list, which must be present in the NODEDATA= data set, contains supply and demand information for the nodes in the NODE list. A positive SUPDEM list variable value s $(s > 0)$ denotes that the node named in the NODE list variable can supply s units of flow. A negative SUPDEM list variable value $-d$ $(d > 0)$ means that this node demands d units of flow. If a SAS variable is not explicitly specified, a SAS variable with the name _SUPDEM_ or _SD_ in the NODEDATA= data set is used as the SUPDEM variable. If a node is a transshipment node (neither a supply nor a demand node), an observation associated with this node need not be present in the NODEDATA= data set. If present, the SUPDEM list variable value must be zero or a missing value. See the section "Missing S Supply and Missing D Demand Values" on page 114 for cases when the SUPDEM list variable values can have other values.

SUPPLY Statement

SUPPLY *variable* **;**

The SUPPLY statement identifies the SAS variable in the ARCDATA= data set that contains the supply at the node named in that observation's TAILNODE list variable. If a tail node does not supply flow, use zero or a missing value for the observation's SUPPLY list variable value. If a tail node has supply capability, a missing value indicates that the supply quantity is given in another observation. It is not necessary

to have a SUPPLY statement if the name of this SAS variable is _SUPPLY_. See the section "Missing S Supply and Missing D Demand Values" on page 114 for cases when the SUPDEM list variable values can have other values. There should be no SUPPLY statement if you are solving an LP.

TAILNODE Statement

TAILNODE *variable* **;**

TAIL *variable* **;**

FROMNODE *variable* **;**

FROM *variable* **;**

The TAILNODE statement specifies the SAS variable that must (when solving an NPSC problem) be present in the ARCDATA= data set that has as values the names of tail nodes of arcs. The TAILNODE variable must have character values. It is not necessary to have a TAILNODE statement if the name of the SAS variable is _TAIL_ or _FROM_. If the TAILNODE list variable value is missing, it is assumed that the observation of the ARCDATA= data set contains information concerning a nonarc variable. There should be no TAILNODE statement if you are solving an LP.

TYPE Statement

TYPE *variable* **;**

CONTYPE *variable* **;**

The TYPE list, which is optional, names the SAS variable that has as values keywords that indicate either the constraint type for each constraint or the type of special rows in the CONDATA= data set. The values of the TYPE list variable also indicate, in each observation of the CONDATA= data set, how values of the VAR or COEF list variables are to be interpreted and how the type of each constraint or special row name is determined. If the TYPE list is not specified, the CONDATA= data set is searched and a SAS variable with the name _TYPE_ is used. Valid keywords for the TYPE variable are given below. If there is no TYPE statement and no other method is used to furnish type information (see the DEFCONTYPE= option), all constraints are assumed to be of the type "less than or equal to" and no special rows are used. The TYPE list variable must have character values and can be used when the data in the CONDATA= data set is in either the sparse or the dense format. If the TYPE list variable value has a * as its first character, the observation is ignored because it is a comment observation.

TYPE List Variable Values

The following are valid TYPE list variable values. The letters in boldface denote the characters that PROC INTPOINT uses to determine what type the value suggests. You need to have at least these characters. In the following list, the minimal TYPE list variable values have additional characters to aid you in remembering these values.

<	less than or equal to ($\leq$)
=	equal to (=)
>	greater than or equal to ($\geq$)
CAPAC	capacity
COST	cost
EQ	equal to
FREE	free row (used only for linear programs solved by interior point)
GE	greater than or equal to
LE	less than or equal to
LOWERBD	lower flow or value bound
LOW*blank*	lower flow or value bound
MAXIMIZE	maximize (opposite of cost)
MINIMIZE	minimize (same as cost)
OBJECTIVE	objective function (same as cost)
RHS	rhs of constraint
TYPE	type of constraint
UPPCOST	reserved for future use
UNREST	unrestricted variable (used only for linear programs solved by interior point)
UPPER	upper value bound or capacity; second letter must not be N

The valid TYPE list variable values in function order are

- **LE** less than or equal to ($\leq$)
- **EQ** equal to ($=$)
- **GE** greater than or equal to ($\geq$)
- **CO**ST
  MINIMIZE
  MAXIMIZE
  OBJECTIVE
  cost or objective function coefficient
- **CA**PAC
  UPPER
  capacity or upper value bound
- **LOW**ERBD
  LOW*blank*
  lower flow or value bound
- **R**HS rhs of constraint
- **T**YPE type of constraint

A TYPE list variable value that has the first character $*$ causes the observation to be treated as a comment. If the first character is a negative sign, then $\leq$ is the type. If the first character is a zero, then $=$ is the type. If the first character is a positive number, then $\geq$ is the type.

VAR Statement

VAR *variables* **;**

The VAR variable list is used when the dense data format is used for the CONDATA= data set. The names of these SAS variables are also names of the arc and nonarc variables that have data in the CONDATA= data set. If solving an LP, the names of these SAS variables are also names of the LP variables. If no explicit VAR list is specified, all numeric SAS variables in the CONDATA= data set that are not in other SAS variable lists are put onto the VAR list. The VAR list variables must have numeric values. The values of the VAR list variables in some observations can be interpreted differently than in other observations. The values can be coefficients in the side constraints, costs and objective function coefficients, or bound data. When solving an LP, the values of the SAS variables in the VAR list can be constraint coefficients, objective function coefficients, or bound data. How these numeric values are interpreted depends on the value of each observation's TYPE or ROW list variable value. If there are no TYPE list variables, the VAR list variable values are all assumed to be side constraint coefficients.

Details: INTPOINT Procedure

Input Data Sets

PROC INTPOINT is designed so that there are as few rules as possible that you must obey when inputting a problem's data. Raw data are acceptable. This should cut the amount of processing required to groom the data before it is input to PROC INTPOINT. Data formats are so flexible that, due to space restrictions, all possible forms for a problem's data are not shown here. Try any reasonable form for your problem's data; it should be acceptable. PROC INTPOINT will outline its objections.

You can supply the same piece of data several ways. You do not have to restrict yourself to using any particular one. If you use several ways, PROC INTPOINT checks that the data are consistent each time that the data are encountered. After all input data sets have been read, data are merged so that the problem is described completely. The observations can be in any order.

ARCDATA= Data Set

See the section "Getting Started: NPSC Problems" on page 54 and the section "Introductory NPSC Example" on page 55 for a description of this input data set.

Note: Information for an arc or nonarc variable can be specified in more than one observation. For example, consider an arc directed from node A toward node B that has a cost of 50, capacity of 100, and lower flow bound of 10 flow units. Some possible observations in the ARCDATA= data set are as follows:

```
_tail_  _head_  _cost_  _capac_  _lo_
  A       B      50        .       .
  A       B       .      100      .
  A       B       .        .     10
```

```
A       B       50       100       .
A       B        .       100      10
A       B       50        .       10
A       B       50       100      10
```

Similarly, for a nonarc variable that has an upper bound of 100, a lower bound of 10, and an objective function coefficient of 50, the _TAIL_ and _HEAD_ values are missing.

When solving an LP that has an LP variable named my_var with an upper bound of 100, a lower bound of 10, and an objective function coefficient of 50, some possible observations in the ARCDATA= data set are

```
_name_   _cost_   _capac_   _lo_
my_var     50        .        .
my_var      .      100        .
my_var      .        .       10
my_var     50      100        .
my_var      .      100       10
my_var     50        .       10
my_var     50      100       10
```

CONDATA= Data Set

Regardless of whether the data in the CONDATA= data set is in the sparse or dense format, you will receive a warning if PROC INTPOINT finds a constraint row that has no coefficients. You will also be warned if any nonarc or LP variable has no constraint coefficients.

Dense Input Format

If the dense format is used, most SAS variables in the CONDATA= data set belong to the VAR list. The names of the SAS variables belonging to this list have names of arc and nonarc variables or, if solving an LP, names of the LP variables. These names can be values of the SAS variables in the ARCDATA= data set that belong to the NAME list, or names of nonarc variables, or names in the form *tail_head*, or any combination of these three forms. Names in the form *tail_head* are default arc names, and if you use them, you must specify node names in the ARCDATA= data set (values of the TAILNODE and HEADNODE list variables).

The CONDATA= data set can have three other SAS variables belonging, respectively, to the ROW, the TYPE, and the RHS lists. The CONDATA= data set of the oil industry example in the section "Introductory NPSC Example" on page 55 uses the dense data format.

Consider the SAS code that creates a dense format CONDATA= data set that has data for three constraints. This data set was used in the section "Introductory NPSC Example" on page 55.

```
data cond1;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
      _type_ $ _rhs_;
   datalines;
-2  .  1 .  . . >= -15
 . -2  . .  1 . GE -15
 .  . -3 4  . . EQ   0
 .  .  . . -3 4  =   0
;
```

You can use nonconstraint type values to furnish data on costs, capacities, lower flow bounds (and, if there are nonarc or LP variables, objective function coefficients and upper and lower bounds). You need not have such (or as much) data in the ARCDATA= data set. The first three observations in the following data set are examples of observations that provide cost, capacity, and lower bound data.

```
data cond1b;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
         _type_ $ _rhs_;
   datalines;
63 81 200   . 220   . cost   .
95 80 175 140 100 100 capac  .
20 10  50   .  35   . lo     .
-2  .   1   .   .   . >=   -15
 . -2   .   .   1   . GE   -15
 .  .  -3   4   .   . EQ     0
 .  .   .   .  -3   4  =     0
;
```

If a ROW list variable is used, the data for a constraint can be spread over more than one observation. To illustrate, the data for the first constraint (which is called con1) and the cost and capacity data (in special rows called costrow and caprow, respectively) are spread over more than one observation in the following data set.

```
data cond1c;
   input _row_ $
         m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
         _type_ $ _rhs_;
   datalines;
costrow 63  .   .   .   .  . .      .
costrow  . 81 200   .   .  . cost   .
.        .  .   .   . 220  . cost   .
caprow   .  .   .   .   .  . capac  .
caprow  95  . 175   . 100 100 .      .
caprow   . 80 175 140   .  . .      .
lorow   20 10  50   .  35  . lo     .
con1    -2  .   1   .   .  . .      .
con1     .  .   .   .   .  . >=   -15
con2     . -2   .   .   1  . GE   -15
con3     .  .  -3   4   .  . EQ     0
con4     .  .   .   .  -3  4  =     0
;
```

Using both ROW and TYPE lists, you can use special row names. Examples of these are costrow and caprow in the last data set. It should be restated that in any of the input data sets of PROC INTPOINT, the order of the observations does not matter. However, the CONDATA= data set can be read more quickly if PROC INTPOINT knows what type of constraint or special row a ROW list variable value is. For example, when the first observation is read, PROC INTPOINT does not know whether costrow is a constraint or special row and how to interpret the value 63 for the arc with the name m_e_ref1. When PROC INTPOINT reads the second observation, it learns that costrow has cost type and that the values 81 and 200 are costs. When the entire CONDATA= data set has been read, PROC INTPOINT knows the type of all special rows and constraints. Data that PROC INTPOINT had to set aside (such as the first observation 63 value and the costrow ROW list variable value, which at the time had unknown type, but is subsequently known to be a cost special row) is reprocessed. During this second pass, if a ROW list variable value has unassigned constraint or special row type, it is treated as a constraint with DEFCONTYPE= (or DEFCONTYPE= default) type. Associated VAR list variable values are coefficients of that constraint.

Sparse Input Format

The side constraints usually become sparse as the problem size increases. When the sparse data format of the CONDATA= data set is used, only nonzero constraint coefficients must be specified. Remember to specify the SPARSECONDATA option in the PROC INTPOINT statement. With the sparse method of specifying constraint information, the names of arc and nonarc variables or, if solving an LP, the names of LP variables do not have to be valid SAS variable names.

A sparse format CONDATA= data set for the oil industry example in the section "Introductory NPSC Example" on page 55 is displayed below.

```
title 'Setting Up Condata = Cond2 for PROC INTPOINT';
data cond2;
   input _column_ $ _row1 $ _coef1 _row2 $ _coef2 ;
   datalines;
m_e_ref1  con1  -2     .    .
m_e_ref2  con2  -2     .    .
thruput1  con1   1  con3  -3
r1_gas       .   .  con3   4
thruput2  con2   1  con4  -3
r2_gas       .   .  con4   4
_type_    con1   1  con2   1
_type_    con3   0  con4   0
_rhs_     con1 -15  con2 -15
;
```

Recall that the COLUMN list variable values _type_ and _rhs_ are the default values of the TYPEOBS= and RHSOBS= options. Also, the default rhs value of constraints (con3 and con4) is zero. The third to last observation has the value _type_ for the COLUMN list variable. The _ROW1 variable value is con1, and the _COEF1_ variable has the value 1. This indicates that the constraint con1 is *greater than* or equal to type (because the value 1 is *greater than* zero). Similarly,

the data in the second to last observation's _ROW2 and _COEF2 variables indicate that con2 is an *equality* constraint (0 *equals* zero).

An alternative, using a TYPE list variable, is

```
title 'Setting Up Condata = Cond3 for PROC INTPOINT';
data cond3;
   input _column_ $ _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
m_e_ref1  con1  -2     .    . >=
m_e_ref2  con2  -2     .    . .
thruput1  con1   1  con3  -3 .
r1_gas       .   .  con3   4 .
thruput2  con2   1  con4  -3 .
r2_gas       .   .  con4   4 .
.         con3   .  con4   . eq
.         con1 -15  con2 -15 ge
;
```

If the COLUMN list variable is missing in a particular observation (the last 2 observations in the data set cond3, for instance), the constraints named in the ROW list variables all have the constraint type indicated by the value in the TYPE list variable. It is for this type of observation that you are allowed more ROW list variables than COEF list variables. If corresponding COEF list variables are not missing (for example, the last observation in the data set cond3), these values are the rhs values of those constraints. Therefore, you can specify both constraint type and rhs in the same observation.

As in the previous CONDATA= data set, if the COLUMN list variable is an arc or nonarc variable, the COEF list variable values are coefficient values for that arc or nonarc variable in the constraints indicated in the corresponding ROW list variables. If in this same observation the TYPE list variable contains a constraint type, all constraints named in the ROW list variables in that observation have this constraint type (for example, the first observation in the data set cond3). Therefore, you can specify both constraint type and coefficient information in the same observation.

Also note that DEFCONTYPE=EQ could have been specified, saving you from having to include in the data that con3 and con4 are of this type.

In the oil industry example, arc costs, capacities, and lower flow bounds are presented in the ARCDATA= data set. Alternatively, you could have used the following input data sets. The arcd2 data set has only two SAS variables. For each arc, there is an observation in which the arc's tail and head node are specified.

```
title3 'Setting Up Arcdata = Arcd2 for PROC INTPOINT';
data arcd2;
   input  _from_&$11. _to_&$15. ;
   datalines;
middle east  refinery 1
middle east  refinery 2
u.s.a.       refinery 1
u.s.a.       refinery 2
```

```
refinery 1   r1
refinery 2   r2
r1           ref1 gas
r1           ref1 diesel
r2           ref2 gas
r2           ref2 diesel
ref1 gas     servstn1 gas
ref1 gas     servstn2 gas
ref1 diesel  servstn1 diesel
ref1 diesel  servstn2 diesel
ref2 gas     servstn1 gas
ref2 gas     servstn2 gas
ref2 diesel  servstn1 diesel
ref2 diesel  servstn2 diesel
;

title 'Setting Up Condata = Cond4 for PROC INTPOINT';
data cond4;
   input _column_&$27. _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
.                              con1 -15   con2 -15    ge
.                            costrow   .      .   .  cost
.                                  .   . caprow   . capac
middle east_refinery 1          con1  -2      .   .     .
middle east_refinery 2          con2  -2      .   .     .
refinery 1_r1                   con1   1   con3  -3     .
r1_ref1 gas                        .   .   con3   4     =
refinery 2_r2                   con2   1   con4  -3     .
r2_ref2 gas                        .   .   con4   4    eq
middle east_refinery 1       costrow  63 caprow  95     .
middle east_refinery 2       costrow  81 caprow  80     .
u.s.a._refinery 1            costrow  55      .   .     .
u.s.a._refinery 2            costrow  49      .   .     .
refinery 1_r1                costrow 200 caprow 175     .
refinery 2_r2                costrow 220 caprow 100     .
r1_ref1 gas                        .   . caprow 140     .
r1_ref1 diesel                     .   . caprow  75     .
r2_ref2 gas                        .   . caprow 100     .
r2_ref2 diesel                     .   . caprow  75     .
ref1 gas_servstn1 gas        costrow  15 caprow  70     .
ref1 gas_servstn2 gas        costrow  22 caprow  60     .
ref1 diesel_servstn1 diesel  costrow  18      .   .     .
ref1 diesel_servstn2 diesel  costrow  17      .   .     .
ref2 gas_servstn1 gas        costrow  17 caprow  35     .
ref2 gas_servstn2 gas        costrow  31      .   .     .
ref2 diesel_servstn1 diesel  costrow  36      .   .     .
ref2 diesel_servstn2 diesel  costrow  23      .   .     .
middle east_refinery 1             .  20      .   .    lo
middle east_refinery 2             .  10      .   .    lo
refinery 1_r1                      .  50      .   .    lo
refinery 2_r2                      .  35      .   .    lo
ref2 gas_servstn1 gas              .   5      .   .    lo
;
```

The first observation in the cond4 data set defines con1 and con2 as *greater than or equal to* ($\geq$) constraints that both (by coincidence) have rhs values of -15. The second observation defines the special row costrow as a cost row. When costrow is a ROW list variable value, the associated COEF list variable value is interpreted as a cost or objective function coefficient. PROC INTPOINT has to do less work if constraint names and special rows are defined in observations near the top of a data set, but this is not a strict requirement. The fourth to ninth observations contain constraint coefficient data. Observations seven and nine have TYPE list variable values that indicate that constraints con3 and con4 are equality constraints. The last five observations contain lower flow bound data. Observations that have an arc or nonarc variable name in the COLUMN list variable, a nonconstraint type TYPE list variable value, and a value in (one of) the COEF list variables are valid.

The following data set is equivalent to the cond4 data set.

```
title 'Setting Up Condata = Cond5 for PROC INTPOINT';
data cond5;
   input _column_&$27. _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
middle east_refinery 1          con1  -2 costrow  63     .
middle east_refinery 2          con2  -2   lorow  10     .
refinery 1_r1                      .   .    con3  -3     =
r1_ref1 gas                   caprow 140    con3   4     .
refinery 2_r2                   con2   1    con4  -3     .
r2_ref2 gas                        .   .    con4   4    eq
.                               CON1 -15    CON2 -15    GE
ref2 diesel_servstn1 diesel        .  36 costrow    .  cost
.                                  .   .  caprow    . capac
.                              lorow   .       .    .    lo
middle east_refinery 1         caprow  95   lorow  20     .
middle east_refinery 2         caprow  80 costrow  81     .
u.s.a._refinery 1                  .   .       .  55  cost
u.s.a._refinery 2             costrow  49       .   .     .
refinery 1_r1                    con1   1  caprow 175     .
refinery 1_r1                   lorow  50 costrow 200     .
refinery 2_r2                 costrow 220  caprow 100     .
refinery 2_r2                      .  35       .   .    lo
r1_ref1 diesel                caprow2  75       .   . capac
r2_ref2 gas                        .   .  caprow 100     .
r2_ref2 diesel                caprow2  75       .   .     .
ref1 gas_servstn1 gas         costrow  15  caprow  70     .
ref1 gas_servstn2 gas         caprow2  60 costrow  22     .
ref1 diesel_servstn1 diesel        .   . costrow  18     .
ref1 diesel_servstn2 diesel   costrow  17       .   .     .
ref2 gas_servstn1 gas         costrow  17   lorow   5     .
ref2 gas_servstn1 gas              .   . caprow2  35     .
ref2 gas_servstn2 gas              .  31       .   .  cost
ref2 diesel_servstn2 diesel        .   . costrow  23     .
;
```

Converting from an NPSC to an LP Problem

If you have data for a linear programming program that has an embedded network, the steps required to change that data into a form that is acceptable by PROC INTPOINT are

1. Identify the nodal flow conservation constraints. The coefficient matrix of these constraints (a submatrix of the LP's constraint coefficient matrix) has only two nonzero elements in each column, -1 and 1.
2. Assign a node to each nodal flow conservation constraint.
3. The rhs values of conservation constraints are the corresponding node's supplies and demands. Use this information to create the NODEDATA= data set.
4. Assign an arc to each column of the flow conservation constraint coefficient matrix. The arc is directed from the node associated with the row that has the 1 element in it and directed toward to the node associated with the row that has the -1 element in it. Set up the ARCDATA= data set that has two SAS variables. This data set could resemble ARCDATA=arcd2. These will eventually be the TAILNODE and HEADNODE list variables when PROC INTPOINT is used. Each observation consists of the tail and head node of each arc.
5. Remove from the data of the linear program all data concerning the nodal flow conservation constraints.
6. Put the remaining data into a CONDATA= data set. This data set will probably resemble CONDATA=cond4 or CONDATA=cond5.

The Sparse Format Summary

The following list illustrates possible CONDATA= data set observation sparse formats. a1, b1, b2, b3 and c1 have as a _COLUMN_ variable value either the name of an arc (possibly in the form *tail_head*) or the name of a nonarc variable (if you are solving an NPSC), or the name of the LP variable (if you are solving an LP). These are collectively referred to as variable in the tables that follow.

- If there is no TYPE list variable in the CONDATA= data set, the problem must be constrained and there is no nonconstraint data in the CONDATA= data set:

	COLUMN	_ROWx_	_COEFx_	_ROWy_ (no _COEFy_) (may not be in CONDATA)
a1	variable	constraint	lhs coef	
a2	_TYPE_ or TYPEOBS=	constraint	-1 0 1	
a3	_RHS_ or RHSOBS= or missing	constraint	rhs value	constraint or missing
a4	_TYPE_ or TYPEOBS=	constraint	missing	
a5	_RHS_ or RHSOBS= or missing	constraint	missing	

Observations of the form a4 and a5 serve no useful purpose but are still allowed to make problem generation easier.

- If there are no ROW list variables in the data set, the problem has no constraints and the information is nonconstraint data. There must be a TYPE list variable and only one COEF list variable in this case. The COLUMN list variable has as values the names of arcs or nonarc variables and must not have missing values or special row names as values:

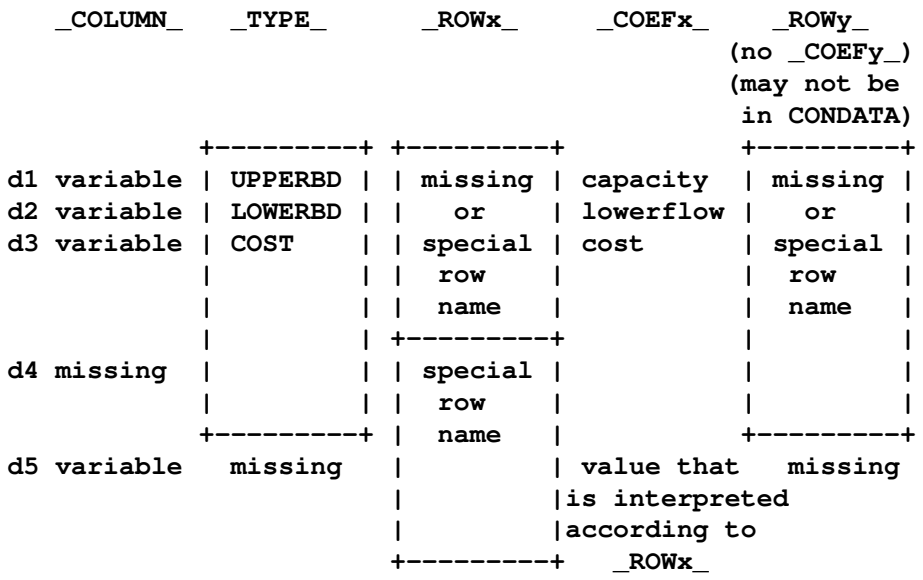
	COLUMN	_TYPE_	_COEFx_
b1	variable	UPPERBD	capacity
b2	variable	LOWERBD	lower flow
b3	variable	COST	cost

- Using a TYPE list variable for constraint data implies the following:

	COLUMN	_TYPE_	_ROWx_	_COEFx_	_ROWy_ (no _COEFy_) (may not be in CONDATA)
c1	variable	missing	c	lhs coef	
c2	_TYPE_ or TYPEOBS=	missing	o	-1 0 1	
c3	_RHS_ or missing or RHSOBS=	missing	n s t	rhs value	constraint or missing
c4	variable	con type	r	lhs coef	
c5	_RHS_ or missing or RHSOBS=	con type	a i n	rhs value	
c6	missing	TYPE	t	-1 0 1	
c7	missing	RHS		rhs value	

If the observation is in form c4 or c5, and the _COEFx_ values are missing, the constraint is assigned the type data specified in the _TYPE_ variable.

- Using a TYPE list variable for arc and nonarc variable data implies the following:



The observations of the form d1 to d5 can have ROW list variable values. Observation d4 must have ROW list variable values. The ROW value is put into the ROW name tree so that when dealing with observation d4 or d5, the COEF list variable value is interpreted according to the type of ROW list variable value. For example, the following three observations define the _ROWx_ variable values up_row, lo_row, and co_row as being an upper value bound row, lower value bound row, and cost row, respectively:

```
_COLUMN_          _TYPE_          _ROWx_          _COEFx_

   .              UPPERBD         up_row             .
variable_a        LOWERBD         lo_row         lower flow
variable_b        COST            co_row         cost
```

PROC INTPOINT is now able to correctly interpret the following observation:

```
_COLUMN_ _TYPE_ _ROW1_ _COEF1_ _ROW2_ _COEF2_ _ROW3_ _COEF3_

 var_c      .    up_row  upval  lo_row  loval  co_row   cost
```

If the TYPE list variable value is a constraint type and the value of the COLUMN list variable equals the value of the TYPEOBS= option or the default value _TYPE_, the TYPE list variable value is ignored.

NODEDATA= Data Set

See the section "Getting Started: NPSC Problems" on page 54 and the section "Introductory NPSC Example" on page 55 for a description of this input data set.

Output Data Set

For NPSC problems, the procedure determines the flow that should pass through each arc as well as the value that should be assigned to each nonarc variable. The goal is that the minimum flow bounds, capacities, lower and upper value bounds, and side constraints are not violated. This goal is reached when total cost incurred by such a flow pattern and value assignment is feasible and optimal. The solution found must also conserve flow at each node.

For LP problems, the procedure determines the value that should be assigned to each variable. The goal is that the lower and upper value bounds and the constraints are not violated. This goal is reached when the total cost incurred by such a value assignment is feasible and optimal.

The CONOUT= data set can be produced and contains a solution obtained after performing optimization.

CONOUT= Data Set

The variables in the CONOUT= data set depend on whether or not the problem has a network component. If the problem has a network component, the variables and their possible values in an observation are as follows:

FROM	a tail node of an arc. This is a missing value if an observation has information about a nonarc variable.
_TO_	a head node of an arc. This is a missing value if an observation has information about a nonarc variable.
COST	the cost of an arc or the objective function coefficient of a nonarc variable
CAPAC	the capacity of an arc or upper value bound of a nonarc variable
_LO_	the lower flow bound of an arc or lower value bound of a nonarc variable
NAME	a name of an arc or nonarc variable
SUPPLY	the supply of the tail node of the arc in the observation. This is a missing value if an observation has information about a nonarc variable.
DEMAND	the demand of the head node of the arc in the observation. This is a missing value if an observation has information about a nonarc variable.
FLOW	the flow through the arc or value of the nonarc variable
FCOST	flow cost, the product of _COST_ and _FLOW_
RCOST	the reduced cost of the arc or nonarc variable
ANUMB	the number of the arc (positive) or nonarc variable (nonpositive); used for warm starting PROC NETFLOW
TNUMB	the number of the tail node in the network basis spanning tree; used for warm starting PROC NETFLOW
STATUS	the status of the arc or nonarc variable

If the problem does not have a network component, the variables and their possible

values in an observation are as follows:

OBJFN	the objective function coefficient of a variable
_UPPERBD	the upper value bound of a variable
_LOWERBD	the lower value bound of a variable
NAME	the name of a variable
VALUE	the value of the variable
FCOST	objective function value for that variable; the product of _OBJFN_ and _VALUE_

The variables present in the ARCDATA= data set are present in a CONOUT= data set. For example, if there is a variable called tail in the ARCDATA= data set and you specified the SAS variable list

```
from tail;
```

then tail is a variable in the CONOUT= data sets instead of _FROM_. Any ID list variables also appear in the CONOUT= data sets.

Case Sensitivity

Whenever the INTPOINT procedure has to compare character strings, whether they are node names, arc names, nonarc names, LP variable names, or constraint names, if the two strings have different lengths, or on a character by character basis the character is different *or has different cases*, PROC INTPOINT judges the character strings to be different.

Not only is this rule enforced when one or both character strings are obtained as values of SAS variables in PROC INTPOINT's input data sets, it also should be obeyed if one or both character strings were originally SAS variable names, or were obtained as the values of options or statements parsed to PROC INTPOINT. For example, if the network has only one node that has supply capability, or if you are solving a MAXFLOW or SHORTPATH problem, you can indicate that node using the SOURCE= option. If you specify

```
proc intpoint source=NotableNode
```

then PROC INTPOINT looks for a value of the TAILNODE list variable that is NotableNode.

Version 6 of the SAS System converts text that makes up statements into uppercase. The name of the node searched for would be NOTABLENODE, even if this was your SAS code:

```
proc intpoint source=NotableNode
```

If you want PROC INTPOINT to behave as it did in Version 6, specify

```
options validvarname=v6;
```

If the SPARSECONDATA option is not specified, and you are running SAS software Version 6, or you are running SAS software Version 7 onward and have specified

```
options validvarname=v6;
```

all values of the SAS variables that belong to the NAME list are uppercased. This is because the SAS System has uppercased all SAS variable names, particularly those in the VAR list of the CONDATA= data set.

Entities that contain blanks must be enclosed in quotes.

Loop Arcs

Loop arcs (which are arcs directed toward nodes from which they originate) are prohibited. Rather, introduce a dummy intermediate node in loop arcs. For example, replace arc (A,A) with (A,B) and (B,A); B is the name of a new node, and it must be distinct for each loop arc.

Multiple Arcs

Multiple arcs with the same tail and head nodes are prohibited. PROC INTPOINT checks to ensure there are no such arcs before proceeding with the optimization. Introduce a new dummy intermediate node in multiple arcs. This node must be distinct for each multiple arc. For example, if some network has three arcs directed from node A toward node B, then replace one of these three with arcs (A,C) and (C,B) and replace another one with (A,D) and (D,B). C and D are new nodes added to the network.

Flow and Value Bounds

The capacity and lower flow bound of an arc can be equal. Negative arc capacities and lower flow bounds are permitted. If both arc capacities and lower flow bounds are negative, the lower flow bound must be at least as negative as the capacity. An arc (A,B) that has a negative flow of $-f$ units can be interpreted as an arc that conveys f units of flow from node B to node A.

The upper and lower value bound of a nonarc variable can be equal. Negative upper and lower bounds are permitted. If both are negative, the lower bound must be at least as negative as the upper bound.

When solving an LP, the upper and lower value bounds of an LP variable can be equal. Negative upper and lower bounds are permitted. If both are negative, the lower bound must be at least as negative as the upper bound.

In short, for any problem to be feasible, a lower bound must be $\leq$ the associated upper bound.

Tightening Bounds and Side Constraints

If any piece of data is furnished to PROC INTPOINT more than once, PROC INTPOINT checks for consistency so that no conflict exists concerning the data values. For example, if the cost of some arc is seen to be one value and as more data are read, the cost of the same arc is seen to be another value, PROC INTPOINT issues an error message on the SAS log and stops. There are two exceptions to this:

- The bounds of arcs and nonarc variables, or the bounds of LP variables, are made as tight as possible. If several different values are given for the lower flow bound of an arc, the greatest value is used. If several different values are given for the lower bound of a nonarc or LP variable, the greatest value is used. If several different values are given for the capacity of an arc, the smallest value is used. If several different values are given for the upper bound of a nonarc or LP variable, the smallest value is used.
- Several values can be given for inequality constraint right-hand sides. For a particular constraint, the lowest rhs value is used for the rhs if the constraint is of *less than or equal to* type. For a particular constraint, the greatest rhs value is used for the rhs if the constraint is of *greater than or equal to* type.

Reasons for Infeasibility

Before optimization commences, PROC INTPOINT tests to ensure that the problem is not infeasible by ensuring that, with respect to supplies, demands, and arc flow bounds, flow conservation can be obeyed at each node:

- Let *IN* be the sum of lower flow bounds of arcs directed toward a node plus the node's supply. Let *OUT* be the sum of capacities of arcs directed from that node plus the node's demand. If *IN* exceeds *OUT*, not enough flow can leave the node.
- Let *OUT* be the sum of lower flow bounds of arcs directed from a node plus the node's demand. Let *IN* be the total capacity of arcs directed toward the node plus the node's supply. If *OUT* exceeds *IN*, not enough flow can arrive at the node.

Reasons why a network problem can be infeasible are similar to those previously mentioned but apply to a set of nodes rather than for an individual node.

Consider the network illustrated in Figure 2.10.

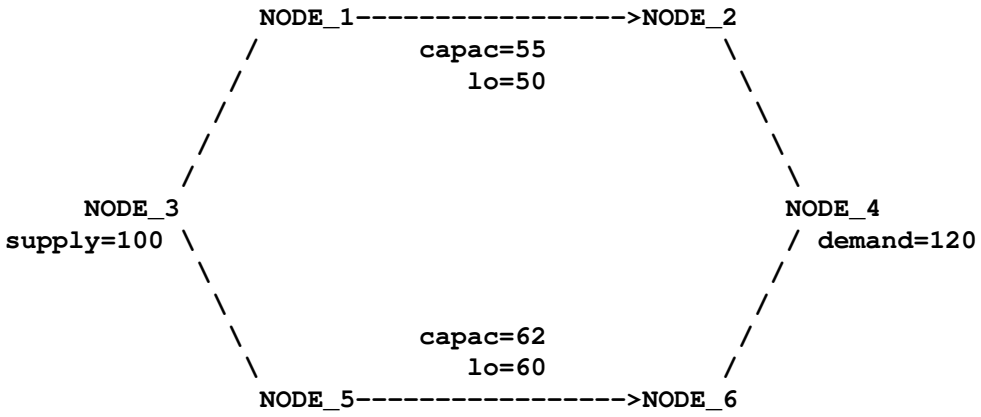


Figure 2.10. An Infeasible Network

The demand of NODE_4 is 120. That can never be satisfied because the maximal flow through arcs (NODE_1, NODE_2) and (NODE_5, NODE_6) is 117. More specifically, the implicit supply of NODE_2 and NODE_6 is only 117, which is insufficient to satisfy the demand of other nodes (real or implicit) in the network.

Furthermore, the lower flow bounds of arcs (NODE_1, NODE_2) and (NODE_5, NODE_6) are greater than the flow that can reach the tail nodes of these arcs, that, by coincidence, is the total supply of the network. The implicit demand of nodes NODE_1 and NODE_5 is 110, which is greater than the amount of flow that can reach these nodes.

Missing S Supply and Missing D Demand Values

In some models, you may want a node to be either a supply or demand node but you want the node to supply or demand the optimal number of flow units. To indicate that a node is such a supply node, use a missing S value in the SUPPLY list variable in the ARCDATA= data set or the SUPDEM list variable in the NODEDATA= data set. To indicate that a node is such a demand node, use a missing D value in the DEMAND list variable in the ARCDATA= data set or the SUPDEM list variable in the NODEDATA= data set.

Suppose the oil example in the section "Introductory NPSC Example" on page 55 is changed so that crude oil can be obtained from either the Middle East or U.S.A. in any amounts. You should specify that the node middle east is a supply node, but you do not want to stipulate that it supplies 100 units, as before. The node u.s.a. should also remain a supply node, but you do not want to stipulate that it supplies 80 units. You must specify that these nodes have missing S supply capabilities:

```
title  'Oil Industry Example';
title3 'Crude Oil can come from anywhere';
data miss_s;
   missing S;
   input   _node_&$15. _sd_;
```

```
   datalines;
middle east          S
u.s.a.               S
servstn1 gas       -95
servstn1 diesel    -30
servstn2 gas       -40
servstn2 diesel    -15
;
```

The following PROC INTPOINT run uses the same ARCDATA= and CONDATA= data sets used in the section "Introductory NPSC Example" on page 55:

```
proc intpoint
   bytes=100000
   nodedata=miss_s       /* the supply (missing S) and */
                         /* demand data                */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;

proc print;
   var _from_ _to_ _cost_ _capac_ _lo_ _flow_ _fcost_;
   sum _fcost_;
   run;
```

The following messages appear on the SAS log:

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Of these, 2 have unspecified (.S) supply capability.
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 0 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of side constraint coefficients= 8 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
      Point algorithm.
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 17 .
NOTE: Number of >= constraints= 2 .
NOTE: Number of constraint coefficients= 48 .
NOTE: Number of variables= 20 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 7.
NOTE: After preprocessing, number of >= constraints= 2.
NOTE: The preprocessor eliminated 10 constraints from the
      problem.
NOTE: The preprocessor eliminated 23 constraint
```

```
      coefficients from the problem.
NOTE: After preprocessing, number of variables= 11.


NOTE: The preprocessor eliminated 9 variables from the
      problem.
NOTE: 2 columns, 0 rows and 2 coefficients were added to
      the problem to handle unrestricted variables,
      variables that are split, and constraint slack or
      surplus variables.
NOTE: There are 16 nonzero elements in A * A transpose.
NOTE: Of the 9 rows and columns, 4 are sparse.
NOTE: There are 11 nonzero superdiagonal elements in the
      sparse rows of the factored A * A transpose. This
      includes fill-in.
NOTE: There are 5 operations of the form
      u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
      sparse rows of A * A transpose.
NOTE: Bound feasibility attained by iteration 1.
NOTE: Dual feasibility attained by iteration 1.
NOTE: Constraint feasibility attained by iteration 2.
NOTE: The Primal-Dual Predictor-Corrector Interior Point
      algorithm performed 7 iterations.
NOTE: Objective = 50075.
NOTE: The data set WORK.SOLUTION has 18 observations and
      10 variables.
NOTE: There were 18 observations read from the data set
      WORK.ARCD1.
NOTE: There were 6 observations read from the data set
      WORK.MISS_S.
NOTE: There were 4 observations read from the data set
      WORK.COND1.
```

The CONOUT= data set is shown in Figure 2.11.

Oil Industry Example

Crude Oil can come from anywhere

Obs	_from_	_to_	_cost_	_capac_	_lo_	_FLOW_	_FCOST_
1	refinery 1	r1	200	175	50	145.000	29000.00
2	refinery 2	r2	220	100	35	35.000	7700.00
3	r1	ref1 diesel	0	75	0	36.250	0.00
4	r1	ref1 gas	0	140	0	108.750	0.00
5	r2	ref2 diesel	0	75	0	8.750	0.00
6	r2	ref2 gas	0	100	0	26.250	0.00
7	middle east	refinery 1	63	95	20	20.000	1260.00
8	u.s.a.	refinery 1	55	99999999	0	125.000	6875.00
9	middle east	refinery 2	81	80	10	10.000	810.00
10	u.s.a.	refinery 2	49	99999999	0	25.000	1225.00
11	ref1 diesel	servstn1 diesel	18	99999999	0	30.000	540.00
12	ref2 diesel	servstn1 diesel	36	99999999	0	0.000	0.00
13	ref1 gas	servstn1 gas	15	70	0	68.750	1031.25
14	ref2 gas	servstn1 gas	17	35	5	26.250	446.25
15	ref1 diesel	servstn2 diesel	17	99999999	0	6.250	106.25
16	ref2 diesel	servstn2 diesel	23	99999999	0	8.750	201.25
17	ref1 gas	servstn2 gas	22	60	0	40.000	880.00
18	ref2 gas	servstn2 gas	31	99999999	0	0.000	0.00
							========
							50075.00

Figure 2.11. Missing S SUPDEM Values in NODEDATA

The optimal supplies of nodes middle east and u.s.a. are 30 and 150 units, respectively. For this example, the same optimal solution is obtained if these nodes had supplies less than these values (each supplies 1 unit, for example) and the THRUNET option was specified in the PROC INTPOINT statement. With the THRUNET option active, when total supply exceeds total demand, the specified nonmissing demand values are the lowest number of flow units that must be absorbed by the corresponding node. This is demonstrated in the following PROC INTPOINT run. The missing S is most useful when nodes are to supply optimal numbers of flow units and it turns out that for some nodes, the optimal supply is 0.

```
data miss_s_x;
   missing S;
   input   _node_&$15. _sd_;
   datalines;
middle east          1
u.s.a.               1
servstn1 gas       -95
servstn1 diesel    -30
servstn2 gas       -40
servstn2 diesel    -15
;

proc intpoint
   bytes=100000
   thrunet
   nodedata=miss_s_x     /* No supply (missing S)      */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
```

```
proc print;
   var _from_ _to_ _cost_ _capac_ _lo_ _flow_ _fcost_;
   sum _fcost_;
   run;
```

The following messages appear on the SAS log. Note that the Total supply= 2, not 0 as in the last run:

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 2 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of side constraint coefficients= 8 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
      Point algorithm.
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 17 .
NOTE: Number of >= constraints= 2 .
NOTE: Number of constraint coefficients= 48 .
NOTE: Number of variables= 20 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 7.
NOTE: After preprocessing, number of >= constraints= 2.
NOTE: The preprocessor eliminated 10 constraints from the
      problem.
NOTE: The preprocessor eliminated 23 constraint
      coefficients from the problem.
NOTE: After preprocessing, number of variables= 11.
NOTE: The preprocessor eliminated 9 variables from the
      problem.
NOTE: 2 columns, 0 rows and 2 coefficients were added to
      the problem to handle unrestricted variables,
      variables that are split, and constraint slack or
      surplus variables.
NOTE: There are 16 nonzero elements in A * A transpose.
NOTE: Of the 9 rows and columns, 4 are sparse.
NOTE: There are 11 nonzero superdiagonal elements in the
      sparse rows of the factored A * A transpose. This
      includes fill-in.
NOTE: There are 5 operations of the form
      u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
      sparse rows of A * A transpose.
NOTE: Bound feasibility attained by iteration 1.
NOTE: Dual feasibility attained by iteration 1.
NOTE: Constraint feasibility attained by iteration 2.
NOTE: The Primal-Dual Predictor-Corrector Interior Point
      algorithm performed 7 iterations.
NOTE: Objective = 50075.
```

```
NOTE: The data set WORK.SOLUTION has 18 observations and
      10 variables.
NOTE: There were 18 observations read from the data set
      WORK.ARCD1.
NOTE: There were 6 observations read from the data set
      WORK.MISS_S_X.
NOTE: There were 4 observations read from the data set
      WORK.COND1.
```

If total supply exceeds total demand, any missing S values are ignored. If total demand exceeds total supply, any missing D values are ignored.

Balancing Total Supply and Total Demand

When Total Supply Exceeds Total Demand

When total supply of a network problem exceeds total demand, PROC INTPOINT adds an extra node (called the *excess node*) to the problem and sets the demand at that node equal to the difference between total supply and total demand. There are three ways that this excess node can be joined to the network. All three ways entail PROC INTPOINT generating a set of arcs (henceforth referred to as the *generated* arcs) that are directed toward the excess node. The total amount of flow in generated arcs equals the demand of the excess node. The generated arcs originate from one of three sets of nodes.

When you specify the THRUNET option, the set of nodes that generated arcs originate from are all demand nodes, even those demand nodes with unspecified demand capability. You indicate that a node has unspecified demand capability by using a missing D value instead of an actual value for demand data (discussed in the section "Missing S Supply and Missing D Demand Values" on page 114). The value specified as the demand of a demand node is in effect a lower bound of the number of flow units that node can actually demand. For missing D demand nodes, this lower bound is zero.

If you do not specify the THRUNET option, the way in which the excess node is joined to the network depends on whether there are demand nodes with unspecified demand capability (nodes with missing D demand) or not.

If there are missing D demand nodes, these nodes are the set of nodes that generated arcs originate from. The value specified as the demand of a demand node, if not missing D, is the number of flow units that node can actually demand. For a missing D demand node, the actual demand of that node may be zero or greater.

If there are no missing D demand nodes, the set of nodes that generated arcs originate from are the set of supply nodes. The value specified as the supply of a supply node is in effect an upper bound of the number of flow units that node can actually supply. For missing S supply nodes (discussed in the section "Missing S Supply and Missing D Demand Values" on page 114), this upper bound is zero, so missing S nodes when total supply exceeds total demand are transshipment nodes, that is, nodes that neither supply nor demand flow.

When Total Supply Is Less Than Total Demand

When total supply of a network problem is less than total demand, PROC INTPOINT adds an extra node (called the *excess node*) to the problem and sets the supply at that node equal to the difference between total demand and total supply. There are three ways that this excess node can be joined to the network. All three ways entail PROC INTPOINT generating a set of arcs (henceforth referred to as the *generated* arcs) that originate from the excess node. The total amount of flow in generated arcs equals the supply of the excess node. The generated arcs are directed toward one of three sets of nodes.

When you specify the THRUNET option, the set of nodes that generated arcs are directed toward are all supply nodes, even those supply nodes with unspecified supply capability. You indicate that a node has unspecified supply capability by using a missing S value instead of an actual value for supply data (discussed in the section "Missing S Supply and Missing D Demand Values" on page 114). The value specified as the supply of a supply node is in effect a lower bound of the number of flow units that the node can actually supply. For missing S supply nodes, this lower bound is zero.

If you do not specify the THRUNET option, the way in which the excess node is joined to the network depends on whether there are supply nodes with unspecified supply capability (nodes with missing S supply) or not.

If there are missing S supply nodes, these nodes are the set of nodes that generated arcs are directed toward. The value specified as the supply of a supply node, if not missing S, is the number of flow units that the node can actually supply. For a missing S supply node, the actual supply of that node may be zero or greater.

If there are no missing S supply nodes, the set of nodes that generated arcs are directed toward are the set of demand nodes. The value specified as the demand of a demand node is in effect an upper bound of the number of flow units that node can actually demand. For missing D demand nodes (discussed in the section "Missing S Supply and Missing D Demand Values" on page 114), this upper bound is zero, so missing D nodes when total supply is less than total demand are transshipment nodes, that is, nodes that neither supply nor demand flow.

How to Make the Data Read of PROC INTPOINT More Efficient

This section contains information that is useful when you want to solve large constrained network problems. However, much of this information is also useful if you have a large linear programming problem. All of the options described in this section that are not directly applicable to networks (options such as ARCS_ONLY_ARCDATA, ARC_SINGLE_OBS, NNODES=, and NARCS=) can be specified to improve the speed at which LP data is read.

Large Constrained Network Problems

Many of the models presented to PROC INTPOINT are enormous. They can be considered large by linear programming standards; problems with thousands, even millions, of variables and constraints. When dealing with side constrained network

programming problems, models can have not only a linear programming component of that magnitude, but also a larger, possibly *much* larger, network component.

The majority of network problem's decision variables are arcs. Like an LP decision variable, an arc has an objective function coefficient, upper and lower value bounds, and a name. Arcs can have coefficients in constraints. Therefore, an arc is quite similar to an LP variable and places the same memory demands on optimization software as an LP variable. But a typical network model has many more arcs and nonarc variables than the typical LP model has variables. And arcs have tail and head nodes. Storing and processing node names require huge amounts of memory. To make matters worse, node names occupy memory at times when a large amount of other data should reside in memory as well.

While memory requirements are lower for a model with embedded network component compared with the equivalent LP *once optimization starts*, the same is usually not true *during the data read*. Even though nodal flow conservation constraints in the LP should not be specified in the constrained network formulation, the memory requirements to read the latter are greater because each arc (unlike an LP variable) originates at one node and is directed toward another.

Paging

PROC INTPOINT has facilities to read data when the available memory is insufficient to store all the data at once. PROC INTPOINT does this by allocating memory for different purposes; for example, to store an array or receive data read from an input SAS data set. After that memory has filled, the information is written to disk and PROC INTPOINT can resume filling that memory with new information. Often, information must be retrieved from disk so that data previously read can be examined or checked for consistency. Sometimes, to prevent any data from being lost, or to retain any changes made to the information in memory, the contents of the memory must be sent to disk before other information can take its place. This process of swapping information to and from disk is called paging. Paging can be very time-consuming, so it is crucial to minimize the amount of paging performed.

There are several steps you can take to make PROC INTPOINT read the data of network and linear programming models more efficiently, particularly when memory is scarce and the amount of paging must be reduced. PROC INTPOINT will then be able to tackle large problems in what can be considered reasonable amounts of time.

The Order of Observations

PROC INTPOINT is quite flexible in the ways data can be supplied to it. Data can be given by any reasonable means. PROC INTPOINT has convenient defaults that can save you work when generating the data. There can be several ways to supply the same piece of data, and some pieces of data can be given more than once. PROC INTPOINT reads everything, then merges it all together. However, this flexibility and convenience come at a price; PROC INTPOINT may not assume the data has a characteristic that, if possessed by the data, could save time and memory during the data read. Several options can indicate that the data has some exploitable characteristic.

For example, an arc cost can be specified once or several times in the ARCDATA= data set or the CONDATA= data set, or both. Every time it is given in the ARCDATA= data set, a check is made to ensure that the new value is the same as any corresponding value read in a previous observation of the ARCDATA= data set. Every time it is given in the CONDATA= data set, a check is made to ensure that the new value is the same as the value read in a previous observation of the CONDATA= data set, or previously in the ARCDATA= data set. PROC INTPOINT would save time if it knew that arc cost data would be encountered only once while reading the ARCDATA= data set, so performing the time-consuming check for consistency would not be necessary. Also, if you indicate that the CONDATA= data set contains data for constraints only, PROC INTPOINT will not expect any arc information, so memory will not be allocated to receive such data while reading the CONDATA= data set. This memory is used for other purposes and this might lead to a reduction in paging. If applicable, use the ARC_SINGLE_OBS or the CON_SINGLE_OBS option, or both, and the NON_REPLIC=COEFS specification to improve how the ARCDATA= data set and the CONDATA= data set are read.

PROC INTPOINT allows the observations in input data sets to be in any order. However, major time savings can result if you are prepared to order observations in particular ways. Time spent by the SORT procedure to sort the input data sets, particularly the CONDATA= data set, may be more than made up for when PROC INTPOINT reads them, because PROC INTPOINT has in memory information possibly used when the previous observation was read. PROC INTPOINT can assume a piece of data is either similar to that of the last observation read or is new. In the first case, valuable information such as an arc or a nonarc variable number or a constraint number is retained from the previous observation. In the last case, checking the data with what has been read previously is not necessary.

Even if you do not sort the CONDATA= data set, grouping observations that contain data for the same arc or nonarc variable or the same row pays off. PROC INTPOINT establishes whether an observation being read is similar to the observation just read.

In practice, many input data sets for PROC INTPOINT have this characteristic, because it is natural for data for each constraint to be grouped together (when using the dense format of the CONDATA= data set) or data for each column to be grouped together (when using the sparse format of the CONDATA= data set). If data for each arc or nonarc is spread over more than one observation of the ARCDATA= data set, it is natural to group these observations together.

Use the GROUPED= option to indicate whether observations of the ARCDATA= data set, the CONDATA= data set, or both, are grouped in a way that can be exploited during data read.

You can save time if the type data for each row appears near the top of the CONDATA= data set, especially if it has the sparse format. Otherwise, when reading an observation, if PROC INTPOINT does not know if a row is a constraint or special row, the data is set aside. Once the data set has been completely read, PROC INTPOINT must reprocess the data it set aside. By then, it knows the type of each constraint or row or, if its type was not provided, it is assumed to have a default type.

Better Memory Utilization

In order for PROC INTPOINT to make better utilization of available memory, you can specify options that indicate the approximate size of the model. PROC INTPOINT then knows what to expect. For example, if you indicate that the problem has no nonarc variables, PROC INTPOINT will not allocate memory to store nonarc data. That memory is better utilized for other purposes. Memory is often allocated to receive or store data of some type. If you indicate that the model does not have much data of a particular type, the memory that would otherwise have been allocated to receive or store that data can be used to receive or store data of another type.

The problem size options are as follows:

- NNODES= approximate number of nodes
- NARCS= approximate number of arcs
- NNAS= approximate number of nonarc variables or LP variables
- NCONS= approximate number of NPSC side constraints or LP constraints
- NCOEFS= approximate number of NPSC side constraint coefficients or LP constraint coefficients

These options will sometimes be referred to as Nxxxx= options.

You do not need to specify all these options for the model, but the more you do, the better. If you do not specify some or all of these options, PROC INTPOINT guesses the size of the problem by using what it already knows about the model. Sometimes PROC INTPOINT guesses the size of the model by looking at the number of observations in the ARCDATA= and the CONDATA= data sets. However, PROC INTPOINT uses rough rules of thumb, that typical models are proportioned in certain ways (for example, if there are constraints, then arcs, nonarc variables, or LP variables usually have about five constraint coefficients). If your model has an unusual shape or structure, you are encouraged to use these options.

If you do use the options and you do not know the exact values to specify, *overestimate* the values. For example, if you specify NARCS=10000 but the model has 10100 arcs, when dealing with the last 100 arcs, PROC INTPOINT might have to page out data for 10000 arcs each time one of the last arcs must be dealt with. Memory could have been allocated for all 10100 arcs without affecting (much) the rest of the data read, so NARCS=10000 could be more of a hindrance than a help.

The point of these Nxxxx= options is to indicate the model size when PROC INTPOINT does not know it. When PROC INTPOINT knows the "real" value, that value is used instead of Nxxxx= .

ARCS_ONLY_ARCDATA indicates that data for only arcs are in the ARCDATA= data set. Memory would not be wasted to receive data for nonarc variables.

Use the memory usage options:

- The BYTES= option specifies the size of PROC INTPOINT main working memory in number of bytes.

- The MEMREP option indicates that memory usage report is to be displayed on the SAS log.

Specifying an appropriate value for the BYTES= parameter is particularly important. Specify as large a number as possible, but not so large a number that will cause PROC INTPOINT (that is, the SAS System running underneath PROC INTPOINT) to run out of memory.

PROC INTPOINT reports its memory requirements on the SAS log if you specify the MEMREP option.

Use Defaults to Reduce the Amount of Data

Use the parameters that specify default values as much as possible. For example, if there are many arcs with the same cost value *c*, use DEFCOST=*c* for arcs that have that cost. Use missing values in the COST variable in the ARCDATA= data set instead of *c*. PROC INTPOINT ignores missing values, but must read, store, and process nonmissing values, even if they are equal to a default option or could have been equal to a default parameter had it been specified. Sometimes, using default parameters makes the need for some SAS variables in the ARCDATA= and the CONDATA= data sets no longer necessary, or reduces the quantity of data that must be read. The default options are

- DEFCOST= default cost of arcs, objective function of nonarc variables or LP variables
- DEFMINFLOW= default lower flow bound of arcs, lower bound of nonarc variables or LP variables
- DEFCAPACITY= default capacity of arcs, upper bound of nonarc variables or LP variables
- DEFCONTYPE= LE or DEFCONTYPE= <=
  DEFCONTYPE= EQ or DEFCONTYPE= =
  DEFCONTYPE= GE or DEFCONTYPE= >=

DEFCONTYPE=LE is the default.

The default options themselves have defaults. For example, you do not need to specify DEFCOST=0 in the PROC INTPOINT statement. You should still have missing values in the COST variable in the ARCDATA= data set for arcs that have zero costs.

If the network has only one supply node, one demand node, or both, use

- SOURCE= name of single node that has supply capability
- SUPPLY= the amount of supply at SOURCE
- SINK= name of single node that demands flow
- DEMAND= the amount of flow SINK demands

Do not specify that a constraint has zero right-hand-side values. That is the default. The only time it might be practical to specify a zero rhs is in observations of the CONDATA= data set read early so that PROC INTPOINT can infer that a row is a constraint. This could prevent coefficient data from being put aside because PROC INTPOINT did not know the row was a constraint.

Names of Things

To cut data read time and memory requirements, reduce the number of bytes in the longest node name, the longest arc name, the longest nonarc variable name, the longest LP variable name, and the longest constraint name to 8 bytes or less. The longer a name, the more bytes must be stored and compared with other names.

If an arc has no constraint coefficients, do not give it a name in the NAME list variable in the ARCDATA= data set. Names for such arcs serve no purpose.

PROC INTPOINT can have a default name for each arc. If an arc is directed from node *tailname* toward node *headname*, the default name for that arc is *tailname_headname*. If you do not want PROC INTPOINT to use these default arc names, specify NAMECTRL=1. Otherwise, PROC INTPOINT must use memory for storing node names and these node names must be searched often.

If you want to use the default *tailname_headname name*, that is, NAMECTRL=2 or NAMECTRL=3, do not use underscores in node names. If the CONDATA has a dense format and has a variable in the VAR list A_B_C_D, or if the value A_B_C_D is encountered as a value of the COLUMN list variable when reading the CONDATA= data set that has the sparse format, PROC INTPOINT first looks for a node named A. If it finds it, it looks for a node called B_C_D. It then looks for a node with the name A_B and possibly a node with name C_D. A search is then conducted for a node named A_B_C and possibly a node named D is done. Underscores could have caused PROC INTPOINT to look unnecessarily for nonexistent nodes. Searching for node names can be expensive, and the amount of memory to store node names is often large. It might be better to assign the arc name A_B_C_D directly to an arc by having that value as a NAME list variable value for that arc in the ARCDATA= data set and specify NAMECTRL=1.

Other Ways to Speed Up Data Reads

Arcs and nonarc variables, or LP variables, can have associated with them values or quantities that have no bearing on the optimization. This information is given in the ARCDATA= data set in the ID list variables. For example, in a distribution problem, information such as truck number and driver's name can be associated with each arc. This is useful when the optimal solution saved in the CONOUT= data set is analyzed. However, PROC INTPOINT needs to reserve memory to process this information when data is being read. For large problems when memory is scarce, it might be better to remove ancillary data from the ARCDATA. After PROC INTPOINT runs, use SAS software to merge this information into the CONOUT= data set that contains the optimal solution.

Stopping Criteria

There are several reasons why PROC INTPOINT stops interior point optimization. Optimization stops when

- the number of iteration equals MAXITERB=m
- the relative gap $(duality\ gap/(c^T x))$ between the primal and dual objectives is smaller than the value of the PDGAPTOL= option, and both the primal and dual problems are feasible. Duality gap is defined in the section "Interior Point Algorithmic Details" on page 39.

PROC INTPOINT may stop optimization when it detects that the rate at which the complementarity or duality gap is being reduced is too slow; that is, that there are consecutive iterations when the complementarity or duality gap has stopped getting smaller and the infeasibilities, if nonzero, have also stalled. Sometimes this indicates that the problem is infeasible.

The reasons to stop optimization outlined in the previous paragraph will be termed the *usual* stopping conditions in the following explanation.

However, when solving some problems, especially if the problems are large, the usual stopping criteria are inappropriate. PROC INTPOINT might stop optimizing prematurely. If it were allowed to perform additional optimization, a better solution would be found. On other occasions, PROC INTPOINT might do too much work. A sufficiently good solution might be reached several iterations before PROC INTPOINT eventually stops.

You can see PROC INTPOINT's progress to the optimum by specifying PRINTLEVEL2=2. PROC INTPOINT will produce a table on the SAS log. A row of the table is generated during each iteration and consists of values of the affine step complementarity, the complementarity of the solution for the next iteration, the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 43), the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 39), and the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 39). As optimization progresses, the values in all columns should converge to zero.

To tailor stopping criteria to your problem, you can use two sets of parameters: the STOP_x and the KEEPGOING_x parameters. The STOP_x parameters (STOP_C, STOP_DG, STOP_IB, STOP_IC, and STOP_ID) are used to test for some condition at the beginning of each iteration and if met, to stop optimizing immediately. The KEEPGOING_x parameters (KEEPGOING_C, KEEPGOING_DG, KEEPGOING_IB, KEEPGOING_IC, and KEEPGOING_ID) are used when PROC INTPOINT would ordinarily stop optimizing but does not if some conditions are not met.

For the sake of conciseness, a set of options might be referred to as the part of the option name they have in common followed by the suffix x. For example, STOP_C,

STOP_DG, STOP_IB, STOP_IC, and STOP_ID will collectively be referred to as STOP_x.

At the beginning of each iteration, PROC INTPOINT will test whether complementarity is <= STOP_C (provided you have specified a STOP_C parameter) and if it is, PROC INTPOINT will stop optimizing. If the duality gap is <= STOP_DG (provided you have specified a STOP_DG parameter), PROC INTPOINT will stop optimizing immediately. This is true as well for the other STOP_x parameters that are related to infeasibilities, STOP_IB, STOP_IC, and STOP_ID.

For example, if you want PROC INTPOINT to stop optimizing for the usual stopping conditions, plus the additional condition, complementarity ≤ 100 *or* duality gap ≤ 0.001, then use

```
proc intpoint stop_c=100 stop_dg=0.001
```

If you want PROC INTPOINT to stop optimizing for the usual stopping conditions, plus the additional condition, complementarity ≤ 1000 *and* duality gap ≤ 0.001 *and* constraint infeasibility ≤ 0.0001, then use

```
proc intpoint
   and_stop_c=1000 and_stop_dg=0.01 and_stop_ic=0.0001
```

Unlike the STOP_x parameters that cause PROC INTPOINT to stop optimizing when any one of them is satisfied, the corresponding AND_STOP_x parameters (AND_STOP_C, AND_STOP_DG, AND_STOP_IB, AND_STOP_IC, and AND_STOP_ID) cause PROC INTPOINT to stop only if all (more precisely, all that are specified) options are satisfied. For example, if PROC INTPOINT should stop optimizing when

- complementarity ≤ 100 *or* duality gap ≤ 0.001 or
- complementarity ≤ 1000 *and* duality gap ≤ 0.001 *and* constraint infeasibility ≤ 0.000

then use

```
proc intpoint
   stop_c=100 stop_dg=0.001
   and_stop_c=1000 and_stop_dg=0.01 and_stop_ic=0.0001
```

Just as the STOP_x parameters have AND_STOP_x partners, the KEEPGOING_x parameters have AND_KEEPGOING_x partners. The role of the KEEPGOING_x and AND_KEEPGOING_x parameters is to prevent optimization from stopping too early, even though a usual stopping criteria is met.

When PROC INTPOINT detects that it should stop optimizing for a usual stopping condition, it will perform the following tests:

- It will test whether complementarity is > KEEPGOING_C (provided you have specified a KEEPGOING_C parameter), and if it is, PROC INTPOINT will perform more optimization.
- Otherwise, PROC INTPOINT will then test whether the primal-dual gap is > KEEPGOING_DG (provided you have specified a KEEPGOING_DG parameter), and if it is, PROC INTPOINT will perform more optimization.
- Otherwise, PROC INTPOINT will then test whether the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ > KEEPGOING_IB (provided you have specified a KEEPGOING_IB parameter), and if it is, PROC INTPOINT will perform more optimization.
- Otherwise, PROC INTPOINT will then test whether the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ > KEEPGOING_IC (provided you have specified a KEEPGOING_IC parameter), and if it is, PROC INTPOINT will perform more optimization.
- Otherwise, PROC INTPOINT will then test whether the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ > KEEPGOING_ID (provided you have specified a KEEPGOING_ID parameter), and if it is, PROC INTPOINT will perform more optimization.
- Otherwise it will test whether complementarity is > AND_KEEPGOING_C (provided you have specified an AND_KEEPGOING_C parameter), *and* the primal-dual gap is > AND_KEEPGOING_DG (provided you have specified an AND_KEEPGOING_DG parameter), *and* the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ > AND_KEEPGOING_IB (provided you have specified an AND_KEEPGOING_IB parameter), *and* the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ > AND_KEEPGOING_IC (provided you have specified an AND_KEEPGOING_IC parameter), *and* the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ > AND_KEEPGOING_ID (provided you have specified an AND_KEEPGOING_ID parameter), and if it is, PROC INTPOINT will perform more optimization.

If all these tests to decide whether more optimization should be performed are false, optimization is stopped.

The following PROC INTPOINT example is used to illustrate how several stopping criteria options can be used together:

```
proc intpoint
   stop_c=1000
   and_stop_c=2000 and_stop_dg=0.01
   and_stop_ib=1 and_stop_ic=1 and_stop_id=1
   keepgoing_c=1500
   and_keepgoing_c=2500 and_keepgoing_dg=0.05
   and_keepgoing_ib=1 and_keepgoing_ic=1 and_keepgoing_id=1
```

At the beginning of each iteration, PROC INTPOINT will stop optimizing if

- complementarity $\leq$ 1000 or

- complementarity $\leq$ 2000 and duality gap $\leq$ 0.01 and the total bound, constraint, and dual infeasibilities are each $\leq$ 1

When PROC INTPOINT determines it should stop optimizing because a usual stopping condition is met, it will stop optimizing only if

- complementarity $\leq$ 1500 or
- complementarity $\leq$ 2500 and duality gap $\leq$ 0.05 and the total bound, constraint, and dual infeasibilities are each $\leq$ 1

Macro Variable _ORINTPO

The INTPOINT procedure always creates and initializes a SAS macro called _ORINTPO. After each PROC INTPOINT run, you can examine this macro by specifying **`%put &_ORINTPO;`** and see whether PROC INTPOINT ran correctly or what error or difficulty it encountered.

The value of _ORINTPO consists of five parts:

- ERROR_STATUS, indicating the existence or absence of any errors
- OPT_STATUS, the stage of the optimization, or what solution has been found
- OBJECTIVE=*objective*, the total cost or profit of the current solution. If PROC INTPOINT is solving a maximal flow problem, MAXFLOW=*maxflow*, the amount of the current solution's maximal flow, will follow. If PROC INTPOINT is solving a minimal flow problem (MAXFLOW and MAXIMIZE specified at the same time), then MINFLOW=*minflow*, the amount of the current solution's minimal flow, will follow instead.
- SOLUTION, describing the nature of the current solution
- information about the interior point algorithm. ITERATIONS=*n*, the number of iterations required to solve the problem. ITERATING_TIME=*Ti*, the time in seconds taken by the interior point algorithm to perform iterations for solving the problem. SOLUTION_TIME=*Ts*, the time in seconds taken by the procedure to presolve the problem, perform interior point iterations, and postsolve the problem.

The value of _ORINTPO is of the following form:

ERROR_STATUS=*charstr* OPT_STATUS=*charstr* OBJECTIVE=*objective*
SOLUTION=*charstr* ITERATIONS=*n* ITERATING_TIME=*Ti*
SOLUTION_TIME=*Ts*

Nontrailing blank characters that are unnecessary are removed. Ideally, at the end a PROC INTPOINT run, _ORINTPO should have the following value:

```
ERROR_STATUS=OK OPT_STATUS=OPTIMAL OBJECTIVE=x
SOLUTION=OPTIMAL ITERATIONS=x ITERATING_TIME=x SOLUTION_TIME=x
```

If the preprocessor detects that a problem is infeasible, _ORINTPO has the following value:

```
ERROR_STATUS=OK SOLUTION=INFEASIBLE
ITERATIONS=0 ITERATING_TIME=0 SOLUTION_TIME=0
```

Table 2.2 lists alternate values for the _ORINTPO value parts.

Table 2.2. PROC INTPOINT _ORINTPO Macro Values

Keyword	Value	Meaning
ERROR_STATUS	OK	no errors
	MEMORY	memory request failed
	IO	input/output error
	DATA	error in the data
	BUG	error in PROC INTPOINT
	SEMANTIC	semantic error
	SYNTAX	syntax error
	UNKNOWN	unknown error
OPT_STATUS	START	no optimization has been done
	STAGE_1	performing stage 1 optimization
	UNCON_OPT	reached unconstrained optimum, but there are side constraints
	STAGE_2	performing stage 2 optimization
	OPTIMAL	reached the optimum
OBJECTIVE	*objective*	total cost or profit
MINFLOW	*minflow*	if MAXFLOW MAXIMIZE is specified
MAXFLOW	*maxflow*	if MAXFLOW is specified
SOLUTION	NONOPTIMAL	more optimization is required
	STAGE_2_REQUIRED	reached unconstrained optimum, stage 2 optimization is required
	OPTIMAL	have determined the optimum
	INFEASIBLE	infeasible; no solution exists
	UNRESOLVED_OPTIMALITY _OR_FEASIBILITY	the optimization process stops before optimality or infeasibility can be proven
	MAXITERB_OPTION _STOPPED_OPTIMIZATION	the interior point algorithm stops after performing maximal number of iterations specified by the MAXITERB= option

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the INTPOINT procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

Examples: INTPOINT Procedure

The following examples illustrate some of the capabilities of PROC INTPOINT. These examples, together with the other SAS/OR examples, can be found in the SAS sample library.

In order to illustrate variations in the use of the INTPOINT procedure, Example 2.1 through Example 2.5 use data from a company that produces two sizes of televisions. The company makes televisions with a diagonal screen measurement of either 19 inches or 25 inches. These televisions are made between March and May at both of the company's two factories. Each factory has a limit on the total number of televisions of each screen dimension that can be made during those months.

The televisions are distributed to one of two shops, stored at the factory where they were made, and sold later or shipped to the other factory. Some sets can be used to fill backorders from the previous months. Each shop demands a number of each type of TV for the months of March through May. The following network in Figure 2.12 illustrates the model. Arc costs can be interpreted as production costs, storage costs, backorder penalty costs, inter-factory transportation costs, and sales profits. The arcs can have capacities and lower flow bounds.

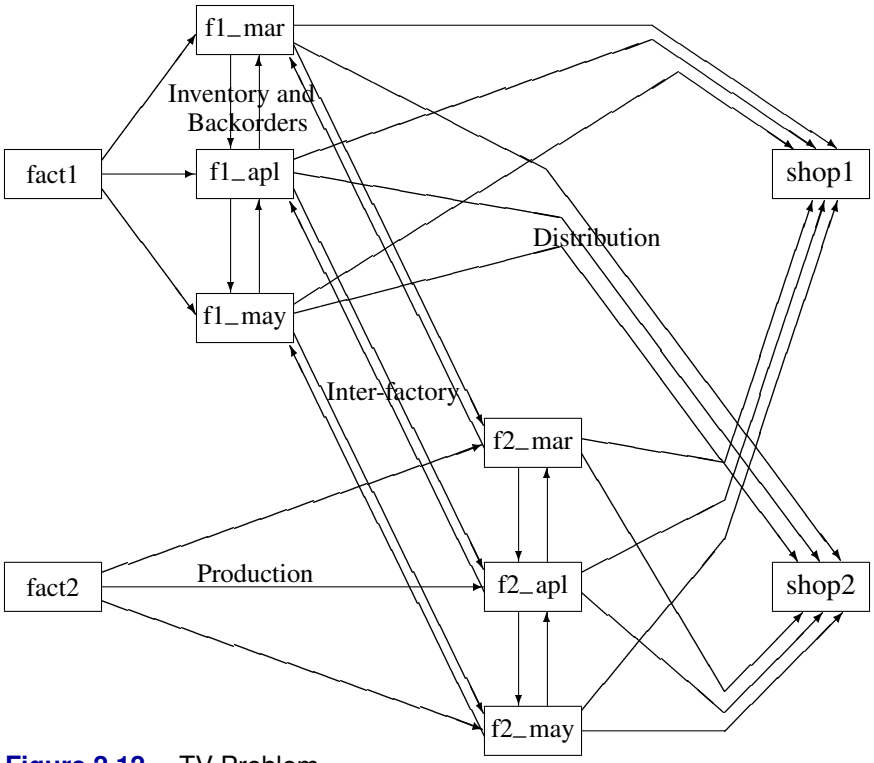


Figure 2.12. TV Problem

There are two similarly structured networks, one for the 19-inch televisions and the other for the 25-inch screen TVs. The minimum cost production, inventory, and distribution plan for both TV types can be determined in the same run of PROC INTPOINT. To ensure that node names are unambiguous, the names of nodes in the 19-inch network have suffix _1, and the node names in the 25-inch network have suffix _2.

Example 2.1. Production, Inventory, Distribution Problem

The following code shows how to save a specific problem's data in data sets and solve the model with PROC INTPOINT.

```
title 'Production Planning/Inventory/Distribution';
title2 'Minimum Cost Flow problem';
title3;

data node0;
   input _node_ $ _supdem_ ;
   datalines;
fact1_1   1000
fact2_1    850
fact1_2   1000
fact2_2   1500
shop1_1   -900
shop2_1   -900
shop1_2   -900
shop2_2  -1450
;

data arc0;
   input _tail_ $ _head_ $ _cost_ _capac_ _lo_ diagonal factory
         key_id $10. mth_made $ _name_&$17. ;
   datalines;
fact1_1  f1_mar_1  127.9  500 50 19 1 production March prod f1 19 mar
fact1_1  f1_apr_1   78.6  600 50 19 1 production April prod f1 19 apl
fact1_1  f1_may_1   95.1  400 50 19 1 production May   .
f1_mar_1 f1_apr_1   15     50  . 19 1 storage    March .
f1_apr_1 f1_may_1   12     50  . 19 1 storage    April .
f1_apr_1 f1_mar_1   28     20  . 19 1 backorder  April back f1 19 apl
f1_may_1 f1_apr_1   28     20  . 19 1 backorder  May   back f1 19 may
f1_mar_1 f2_mar_1   11      .  . 19 . f1_to_2    March .
f1_apr_1 f2_apr_1   11      .  . 19 . f1_to_2    April .
f1_may_1 f2_may_1   16      .  . 19 . f1_to_2    May   .
f1_mar_1 shop1_1  -327.65 250  . 19 1 sales      March .
f1_apr_1 shop1_1  -300    250  . 19 1 sales      April .
f1_may_1 shop1_1  -285    250  . 19 1 sales      May   .
f1_mar_1 shop2_1  -362.74 250  . 19 1 sales      March .
f1_apr_1 shop2_1  -300    250  . 19 1 sales      April .
f1_may_1 shop2_1  -245    250  . 19 1 sales      May   .
fact2_1  f2_mar_1   88.0  450 35 19 2 production March prod f2 19 mar
fact2_1  f2_apr_1   62.4  480 35 19 2 production April prod f2 19 apl
fact2_1  f2_may_1  133.8  250 35 19 2 production May   .
f2_mar_1 f2_apr_1   18     30  . 19 2 storage    March .
f2_apr_1 f2_may_1   20     30  . 19 2 storage    April .
f2_apr_1 f2_mar_1   17     15  . 19 2 backorder  April back f2 19 apl
f2_may_1 f2_apr_1   25     15  . 19 2 backorder  May   back f2 19 may
f2_mar_1 f1_mar_1   10     40  . 19 . f2_to_1    March .
f2_apr_1 f1_apr_1   11     40  . 19 . f2_to_1    April .
f2_may_1 f1_may_1   13     40  . 19 . f2_to_1    May   .
f2_mar_1 shop1_1  -297.4  250  . 19 2 sales      March .
f2_apr_1 shop1_1  -290    250  . 19 2 sales      April .
f2_may_1 shop1_1  -292    250  . 19 2 sales      May   .
f2_mar_1 shop2_1  -272.7  250  . 19 2 sales      March .
f2_apr_1 shop2_1  -312    250  . 19 2 sales      April .
f2_may_1 shop2_1  -299    250  . 19 2 sales      May   .
```

```
fact1_2  f1_mar_2  217.9  400 40 25 1 production March prod f1 25 mar
fact1_2  f1_apr_2  174.5  550 50 25 1 production April prod f1 25 apl
fact1_2  f1_may_2  133.3  350 40 25 1 production May   .
f1_mar_2 f1_apr_2   20     40  . 25 1 storage    March .
f1_apr_2 f1_may_2   18     40  . 25 1 storage    April .
f1_apr_2 f1_mar_2   32     30  . 25 1 backorder  April back f1 25 apl
f1_may_2 f1_apr_2   41     15  . 25 1 backorder  May   back f1 25 may
f1_mar_2 f2_mar_2   23      .  . 25 . f1_to_2    March .
f1_apr_2 f2_apr_2   23      .  . 25 . f1_to_2    April .
f1_may_2 f2_may_2   26      .  . 25 . f1_to_2    May   .
f1_mar_2 shop1_2  -559.76   .  . 25 1 sales      March .
f1_apr_2 shop1_2  -524.28   .  . 25 1 sales      April .
f1_may_2 shop1_2  -475.02   .  . 25 1 sales      May   .
f1_mar_2 shop2_2  -623.89   .  . 25 1 sales      March .
f1_apr_2 shop2_2  -549.68   .  . 25 1 sales      April .
f1_may_2 shop2_2  -460.00   .  . 25 1 sales      May   .
fact2_2  f2_mar_2  182.0  650 35 25 2 production March prod f2 25 mar
fact2_2  f2_apr_2  196.7  680 35 25 2 production April prod f2 25 apl
fact2_2  f2_may_2  201.4  550 35 25 2 production May   .
f2_mar_2 f2_apr_2   28     50  . 25 2 storage    March .
f2_apr_2 f2_may_2   38     50  . 25 2 storage    April .
f2_apr_2 f2_mar_2   31     15  . 25 2 backorder  April back f2 25 apl
f2_may_2 f2_apr_2   54     15  . 25 2 backorder  May   back f2 25 may
f2_mar_2 f1_mar_2   20     25  . 25 . f2_to_1    March .
f2_apr_2 f1_apr_2   21     25  . 25 . f2_to_1    April .
f2_may_2 f1_may_2   43     25  . 25 . f2_to_1    May   .
f2_mar_2 shop1_2  -567.83 500  . 25 2 sales      March .
f2_apr_2 shop1_2  -542.19 500  . 25 2 sales      April .
f2_may_2 shop1_2  -461.56 500  . 25 2 sales      May   .
f2_mar_2 shop2_2  -542.83 500  . 25 2 sales      March .
f2_apr_2 shop2_2  -559.19 500  . 25 2 sales      April .
f2_may_2 shop2_2  -489.06 500  . 25 2 sales      May   .
;

proc intpoint
   bytes=1000000
   printlevel2=2
   nodedata=node0
   arcdata=arc0
   conout=arc1;
   run;


proc print data=arc1;
   var _tail_ _head_ _cost_ _capac_ _lo_ _flow_ _fcost_
       diagonal factory key_id mth_made;
   sum _fcost_;
   run;
```

The following notes appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
```

```
       Point algorithm.
   NOTE: Number of <= constraints= 0 .
   NOTE: Number of == constraints= 21 .
   NOTE: Number of >= constraints= 0 .
   NOTE: Number of constraint coefficients= 136 .
   NOTE: Number of variables= 68 .
   NOTE: After preprocessing, number of <= constraints= 0.
   NOTE: After preprocessing, number of == constraints= 20.
   NOTE: After preprocessing, number of >= constraints= 0.
   NOTE: The preprocessor eliminated 1 constraints from the
         problem.
   NOTE: The preprocessor elimiated 9 constraint coefficients
         from the problem.
   NOTE: 0 columns, 0 rows and 0 coefficients were added to
         the problem to handle unrestricted variables,
         variables that are split, and constraint slack or
         surplus variables.
   NOTE: There are 48 nonzero elements in A * A transpose.
   NOTE: Of the 20 rows and columns, 11 are sparse.
   NOTE: There are 40 nonzero superdiagonal elements in the
         sparse rows of the factored A * A transpose. This
         includes fill-in.
   NOTE: There are 49 operations of the form
         u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
         sparse rows of A * A transpose.
Iter Complem_aff Complem-ity  Duality_gap Tot_infeasb  Tot_infeasc Tot_infeasd
   0   -1.000000   169009903     0.835362       52835        25664       38005
   1    36984291    17566249     0.914108 1649.363089   801.164462           0
   2     1982553      866890     0.413012           0 1.461734E-12           0
   3      542347      234381     0.153440           0            0           0
   4      129900       52560     0.038044           0            0           0
   5       25835       18168     0.013343           0            0           0
   6 8923.214994 2976.020480     0.002203           0            0           0
   7  928.231932  624.792307     0.000463           0            0           0
   8  218.771392   74.386900  0.000055131           0            0           0
   9   11.639195    2.197862  0.000001629           0            0           0
  10    0.089160    0.000399 2.958631E-10           0            0           0
   NOTE: The Primal-Dual Predictor-Corrector Interior Point
         algorithm performed 10 iterations.
   NOTE: Objective = -1281110.35.
   NOTE: The data set WORK.ARC1 has 64 observations and 14
         variables.
   NOTE: There were 64 observations read from the data set
         WORK.ARC0.
   NOTE: There were 8 observations read from the data set
         WORK.NODE0.
```

The solution is given in the CONOUT=arc1 data sets. In the CONOUT= data set, shown in Output 2.1.1, the variables diagonal, factory, key_id, and mth_made form an implicit ID list. The diagonal variable has one of two values, 19 or 25. factory also has one of two values, 1 or 2, to denote the factory where either production or storage occurs, from where TVs are either sold to shops or used to satisfy backorders. production, storage, sales, and backorder are values of the key_id variable.

Other values of this variable, f1_to_2 and f2_to_1, are used when flow through arcs represents the transportation of TVs between factories. The mth_made variable

has values March, April, and May, the months when TVs that are modeled as flow through an arc were made (assuming that no televisions are stored for more than one month and none manufactured in May are used to fill March backorders).

These ID variables can be used after the PROC INTPOINT run to produce reports and perform analysis on particular parts of the company's operation. For example, reports can be generated for production numbers for each factory; optimal sales figures for each shop; and how many TVs should be stored, used to fill backorders, sent to the other factory, or any combination of these, for TVs with a particular screen, those produced in a particular month, or both.

Output 2.1.1. CONOUT=ARC1

Production Planning/Inventory/Distribution
Minimum Cost Flow problem

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_FLOW_	_FCOST_	diagonal	factory	key_id	mth_made
1	fact1_1	f1_apr_1	78.60	600	50	600.000	47160.00	19	1	production	April
2	f1_mar_1	f1_apr_1	15.00	50	0	0.000	0.00	19	1	storage	March
3	f1_may_1	f1_apr_1	28.00	20	0	0.000	0.00	19	1	backorder	May
4	f2_apr_1	f1_apr_1	11.00	40	0	0.000	0.00	19	.	f2_to_1	April
5	fact1_2	f1_apr_2	174.50	550	50	550.000	95975.00	25	1	production	April
6	f1_mar_2	f1_apr_2	20.00	40	0	0.000	0.00	25	1	storage	March
7	f1_may_2	f1_apr_2	41.00	15	0	15.000	615.00	25	1	backorder	May
8	f2_apr_2	f1_apr_2	21.00	25	0	0.000	0.00	25	.	f2_to_1	April
9	fact1_1	f1_mar_1	127.90	500	50	345.000	44125.49	19	1	production	March
10	f1_apr_1	f1_mar_1	28.00	20	0	20.000	560.00	19	1	backorder	April
11	f2_mar_1	f1_mar_1	10.00	40	0	40.000	400.00	19	.	f2_to_1	March
12	fact1_2	f1_mar_2	217.90	400	40	400.000	87160.00	25	1	production	March
13	f1_apr_2	f1_mar_2	32.00	30	0	30.000	960.00	25	1	backorder	April
14	f2_mar_2	f1_mar_2	20.00	25	0	25.000	500.00	25	.	f2_to_1	March
15	fact1_1	f1_may_1	95.10	400	50	50.000	4755.00	19	1	production	May
16	f1_apr_1	f1_may_1	12.00	50	0	50.000	600.00	19	1	storage	April
17	f2_may_1	f1_may_1	13.00	40	0	0.000	0.00	19	.	f2_to_1	May
18	fact1_2	f1_may_2	133.30	350	40	40.000	5332.00	25	1	production	May
19	f1_apr_2	f1_may_2	18.00	40	0	0.000	0.00	25	1	storage	April
20	f2_may_2	f1_may_2	43.00	25	0	0.000	0.00	25	.	f2_to_1	May
21	f1_apr_1	f2_apr_1	11.00	99999999	0	30.000	330.00	19	.	f1_to_2	April
22	fact2_1	f2_apr_1	62.40	480	35	480.000	29952.00	19	2	production	April
23	f2_mar_1	f2_apr_1	18.00	30	0	0.000	0.00	19	2	storage	March
24	f2_may_1	f2_apr_1	25.00	15	0	0.000	0.00	19	2	backorder	May
25	f1_apr_2	f2_apr_2	23.00	99999999	0	0.000	0.00	25	.	f1_to_2	April
26	fact2_2	f2_apr_2	196.70	680	35	680.000	133756.00	25	2	production	April
27	f2_mar_2	f2_apr_2	28.00	50	0	0.000	0.00	25	2	storage	March
28	f2_may_2	f2_apr_2	54.00	15	0	15.000	810.00	25	2	backorder	May
29	f1_mar_1	f2_mar_1	11.00	99999999	0	0.000	0.00	19	.	f1_to_2	March
30	fact2_1	f2_mar_1	88.00	450	35	290.000	25520.00	19	2	production	March
31	f2_apr_1	f2_mar_1	17.00	15	0	0.000	0.00	19	2	backorder	April
32	f1_mar_2	f2_mar_2	23.00	99999999	0	0.000	0.00	25	.	f1_to_2	March
33	fact2_2	f2_mar_2	182.00	650	35	645.000	117390.00	25	2	production	March
34	f2_apr_2	f2_mar_2	31.00	15	0	0.000	0.00	25	2	backorder	April
35	f1_may_1	f2_may_1	16.00	99999999	0	100.000	1600.00	19	.	f1_to_2	May
36	fact2_1	f2_may_1	133.80	250	35	35.000	4683.00	19	2	production	May
37	f2_apr_1	f2_may_1	20.00	30	0	15.000	300.00	19	2	storage	April
38	f1_may_2	f2_may_2	26.00	99999999	0	0.000	0.00	25	.	f1_to_2	May
39	fact2_2	f2_may_2	201.40	550	35	35.000	7049.00	25	2	production	May
40	f2_apr_2	f2_may_2	38.00	50	0	0.000	0.00	25	2	storage	April
41	f1_mar_1	shop1_1	-327.65	250	0	155.000	-50785.73	19	1	sales	March
42	f1_apr_1	shop1_1	-300.00	250	0	250.000	-75000.00	19	1	sales	April
43	f1_may_1	shop1_1	-285.00	250	0	0.000	0.00	19	1	sales	May
44	f2_mar_1	shop1_1	-297.40	250	0	250.000	-74350.00	19	2	sales	March
45	f2_apr_1	shop1_1	-290.00	250	0	245.000	-71050.01	19	2	sales	April
46	f2_may_1	shop1_1	-292.00	250	0	0.000	0.00	19	2	sales	May
47	f1_mar_2	shop1_2	-559.76	99999999	0	0.000	0.00	25	1	sales	March
48	f1_apr_2	shop1_2	-524.28	99999999	0	0.000	0.00	25	1	sales	April
49	f1_may_2	shop1_2	-475.02	99999999	0	25.000	-11875.50	25	1	sales	May
50	f2_mar_2	shop1_2	-567.83	500	0	500.000	-283915.00	25	2	sales	March
51	f2_apr_2	shop1_2	-542.19	500	0	375.000	-203321.25	25	2	sales	April
52	f2_may_2	shop1_2	-461.56	500	0	0.000	0.00	25	2	sales	May
53	f1_mar_1	shop2_1	-362.74	250	0	250.000	-90685.00	19	1	sales	March
54	f1_apr_1	shop2_1	-300.00	250	0	250.000	-75000.00	19	1	sales	April
55	f1_may_1	shop2_1	-245.00	250	0	0.000	0.00	19	1	sales	May
56	f2_mar_1	shop2_1	-272.70	250	0	0.000	0.00	19	2	sales	March
57	f2_apr_1	shop2_1	-312.00	250	0	250.000	-78000.00	19	2	sales	April
58	f2_may_1	shop2_1	-299.00	250	0	150.000	-44850.00	19	2	sales	May
59	f1_mar_2	shop2_2	-623.89	99999999	0	455.000	-283869.95	25	1	sales	March
60	f1_apr_2	shop2_2	-549.68	99999999	0	535.000	-294078.80	25	1	sales	April
61	f1_may_2	shop2_2	-460.00	99999999	0	0.000	0.00	25	1	sales	May
62	f2_mar_2	shop2_2	-542.83	500	0	120.000	-65139.60	25	2	sales	March
63	f2_apr_2	shop2_2	-559.19	500	0	320.000	-178940.80	25	2	sales	April
64	f2_may_2	shop2_2	-489.06	500	0	20.000	-9781.20	25	2	sales	May
							==========				
							-1281110.35				

Example 2.2. Altering Arc Data

This example examines the effect of changing some of the arc costs. The backorder penalty costs are increased by 20 percent. The sales profit of 25-inch TVs sent to the shops in May is increased by 30 units. The backorder penalty costs of 25-inch TVs manufactured in May for April consumption is decreased by 30 units. The production cost of 19-inch and 25-inch TVs made in May are decreased by 5 units and 20 units, respectively. How does the optimal solution of the network after these arc cost alterations compare with the optimum of the original network?

These SAS statements produce the new NODEDATA= and ARCDATA= data sets:

```
title2 'Minimum Cost Flow problem- Altered Arc Data';
data arc2;
   set arc1;
   oldcost=_cost_;
   oldfc=_fcost_;
   oldflow=_flow_;
   if key_id='backorder'
      then _cost_=_cost_*1.2;
      else if _tail_='f2_may_2' then _cost_=_cost_-30;
   if key_id='production' & mth_made='May' then
      if diagonal=19 then _cost_=_cost_-5;
                     else _cost_=_cost_-20;
   run;

proc intpoint
   bytes=100000
   printlevel2=2
   nodedata=node0
   arcdata=arc2
   conout=arc3;
   run;

proc print data=arc3;
   var _tail_ _head_ _capac_ _lo_ _supply_ _demand_ _name_
       _cost_ _flow_ _fcost_ oldcost oldflow oldfc
       diagonal factory key_id mth_made;
       /* to get this variable order */
           sum oldfc _fcost_;
   run;
```

The following notes appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: The following messages relate to the equivalent Linear
      Programming problem solved by the Interior Point algorithm.
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 21 .
NOTE: Number of >= constraints= 0 .
NOTE: Number of constraint coefficients= 136 .
NOTE: Number of variables= 68 .
```

```
   NOTE: After preprocessing, number of <= constraints= 0.
   NOTE: After preprocessing, number of == constraints= 20.
   NOTE: After preprocessing, number of >= constraints= 0.
   NOTE: The preprocessor eliminated 1 constraints from the
         problem.
   NOTE: The preprocessor eliminated 9 constraint coefficients
         from the problem.
   NOTE: 0 columns, 0 rows and 0 coefficients were added to
         the problem to handle unrestricted variables,
         variables that are split, and constraint slack or
         surplus variables.
   NOTE: There are 48 nonzero elements in A * A transpose.
   NOTE: Of the 20 rows and columns, 11 are sparse.
   NOTE: There are 40 nonzero superdiagonal elements in the
         sparse rows of the factored A * A transpose. This
         includes fill-in.
   NOTE: There are 49 operations of the form
         u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
         sparse rows of A * A transpose.
Iter Complem_aff Complem-ity  Duality_gap Tot_infeasb  Tot_infeasc Tot_infeasd
   0   -1.000000   169822194     0.834344       52835        25664       38174
   1    37113367    17651592     0.912723 1650.135100   801.539460           0
   2     1990318      739344     0.369751           0 1.363097E-12           0
   3      358794      186448     0.125234           0            0           0
   4      116081       45454     0.033002           0            0           0
   5       26298       16467     0.012077           0            0           0
   6 8963.676317 3893.499547     0.002875           0            0           0
   7 1615.258886  718.590270     0.000532           0            0           0
   8  178.225920   47.380105  0.000035062           0            0           0
   9    6.698353    0.020987 1.5531175E-8           0            0           0
   NOTE: The Primal-Dual Predictor-Corrector Interior Point algorithm
         performed 9 iterations.
   NOTE: Objective = -1285086.46.
   NOTE: The data set WORK.ARC3 has 64 observations and 17
         variables.
   NOTE: There were 64 observations read from the data set
         WORK.ARC2.
   NOTE: There were 8 observations read from the data set
         WORK.NODE0.
```

The solution is displayed in Output 2.2.1.

Output 2.2.1. CONOUT=ARC3

Minimum Cost Flow problem- Altered arc data

Obs	_tail_	_head_	_capac_	_lo_	_SUPPLY_	_DEMAND_	_name_	_cost_	_FLOW_
1	fact1_1	f1_apr_1	600	50	1000	.	prod f1 19 apl	78.60	540.001
2	f1_mar_1	f1_apr_1	50	0	.	.		15.00	0.000
3	f1_may_1	f1_apr_1	20	0	.	.	back f1 19 may	33.60	0.000
4	f2_apr_1	f1_apr_1	40	0	.	.		11.00	0.000
5	fact1_2	f1_apr_2	550	50	1000	.	prod f1 25 apl	174.50	250.000
6	f1_mar_2	f1_apr_2	40	0	.	.		20.00	0.000
7	f1_may_2	f1_apr_2	15	0	.	.	back f1 25 may	49.20	15.000
8	f2_apr_2	f1_apr_2	25	0	.	.		21.00	0.000
9	fact1_1	f1_mar_1	500	50	1000	.	prod f1 19 mar	127.90	340.000
10	f1_apr_1	f1_mar_1	20	0	.	.	back f1 19 apl	33.60	20.000
11	f2_mar_1	f1_mar_1	40	0	.	.		10.00	40.000
12	fact1_2	f1_mar_2	400	40	1000	.	prod f1 25 mar	217.90	400.000
13	f1_apr_2	f1_mar_2	30	0	.	.	back f1 25 apl	38.40	30.000
14	f2_mar_2	f1_mar_2	25	0	.	.		20.00	25.000
15	fact1_1	f1_may_1	400	50	1000	.		90.10	114.999
16	f1_apr_1	f1_may_1	50	0	.	.		12.00	0.001
17	f2_may_1	f1_may_1	40	0	.	.		13.00	0.000
18	fact1_2	f1_may_2	350	40	1000	.		113.30	350.000
19	f1_apr_2	f1_may_2	40	0	.	.		18.00	0.000
20	f2_may_2	f1_may_2	25	0	.	.		13.00	0.000
21	f1_apr_1	f2_apr_1	99999999	0	.	.		11.00	20.000
22	fact2_1	f2_apr_1	480	35	850	.	prod f2 19 apl	62.40	480.000
23	f2_mar_1	f2_apr_1	30	0	.	.		18.00	0.000
24	f2_may_1	f2_apr_1	15	0	.	.	back f2 19 may	30.00	0.000
25	f1_apr_2	f2_apr_2	99999999	0	.	.		23.00	0.000
26	fact2_2	f2_apr_2	680	35	1500	.	prod f2 25 apl	196.70	680.000
27	f2_mar_2	f2_apr_2	50	0	.	.		28.00	0.000
28	f2_may_2	f2_apr_2	15	0	.	.	back f2 25 may	64.80	0.000
29	f1_mar_1	f2_mar_1	99999999	0	.	.		11.00	0.000
30	fact2_1	f2_mar_1	450	35	850	.	prod f2 19 mar	88.00	290.000
31	f2_apr_1	f2_mar_1	15	0	.	.	back f2 19 apl	20.40	0.000
32	f1_mar_2	f2_mar_2	99999999	0	.	.		23.00	0.000
33	fact2_2	f2_mar_2	650	35	1500	.	prod f2 25 mar	182.00	635.000
34	f2_apr_2	f2_mar_2	15	0	.	.	back f2 25 apl	37.20	0.000
35	f1_may_1	f2_may_1	99999999	0	.	.		16.00	115.000
36	fact2_1	f2_may_1	250	35	850	.		128.80	35.000
37	f2_apr_1	f2_may_1	30	0	.	.		20.00	0.000
38	f1_may_2	f2_may_2	99999999	0	.	.		26.00	335.000
39	fact2_2	f2_may_2	550	35	1500	.		181.40	35.000
40	f2_apr_2	f2_may_2	50	0	.	.		38.00	0.000
41	f1_mar_1	shop1_1	250	0	.	900		-327.65	150.000
42	f1_apr_1	shop1_1	250	0	.	900		-300.00	250.000
43	f1_may_1	shop1_1	250	0	.	900		-285.00	0.000
44	f2_mar_1	shop1_1	250	0	.	900		-297.40	250.000
45	f2_apr_1	shop1_1	250	0	.	900		-290.00	250.000
46	f2_may_1	shop1_1	250	0	.	900		-292.00	0.000
47	f1_mar_2	shop1_2	99999999	0	.	900		-559.76	0.000
48	f1_apr_2	shop1_2	99999999	0	.	900		-524.28	0.000
49	f1_may_2	shop1_2	99999999	0	.	900		-475.02	0.000
50	f2_mar_2	shop1_2	500	0	.	900		-567.83	500.000
51	f2_apr_2	shop1_2	500	0	.	900		-542.19	400.000
52	f2_may_2	shop1_2	500	0	.	900		-491.56	0.000
53	f1_mar_1	shop2_1	250	0	.	900		-362.74	250.000
54	f1_apr_1	shop2_1	250	0	.	900		-300.00	250.000
55	f1_may_1	shop2_1	250	0	.	900		-245.00	0.000
56	f2_mar_1	shop2_1	250	0	.	900		-272.70	0.000
57	f2_apr_1	shop2_1	250	0	.	900		-312.00	250.000
58	f2_may_1	shop2_1	250	0	.	900		-299.00	150.000
59	f1_mar_2	shop2_2	99999999	0	.	1450		-623.89	455.000
60	f1_apr_2	shop2_2	99999999	0	.	1450		-549.68	235.000
61	f1_may_2	shop2_2	99999999	0	.	1450		-460.00	0.000
62	f2_mar_2	shop2_2	500	0	.	1450		-542.83	110.000
63	f2_apr_2	shop2_2	500	0	.	1450		-559.19	280.000
64	f2_may_2	shop2_2	500	0	.	1450		-519.06	370.000

Obs	_FCOST_	oldcost	oldflow	oldfc	diagonal	factory	key_id	mth_made
1	42444.05	78.60	600.000	47160.00	19	1	production	April
2	0.00	15.00	0.000	0.00	19	1	storage	March
3	0.00	28.00	0.000	0.00	19	1	backorder	May
4	0.00	11.00	0.000	0.00	19	.	f2_to_1	April
5	43625.01	174.50	550.000	95975.00	25	1	production	April
6	0.00	20.00	0.000	0.00	25	1	storage	March
7	738.00	41.00	15.000	615.00	25	1	backorder	May
8	0.00	21.00	0.000	0.00	25	.	f2_to_1	April
9	43486.02	127.90	345.000	44125.49	19	1	production	March
10	672.00	28.00	20.000	560.00	19	1	backorder	April
11	400.00	10.00	40.000	400.00	19	.	f2_to_1	March
12	87160.00	217.90	400.000	87160.00	25	1	production	March
13	1152.00	32.00	30.000	960.00	25	1	backorder	April
14	500.00	20.00	25.000	500.00	25	.	f2_to_1	March
15	10361.42	95.10	50.000	4755.00	19	1	production	May
16	0.01	12.00	50.000	600.00	19	1	storage	April
17	0.00	13.00	0.000	0.00	19	.	f2_to_1	May
18	39655.00	133.30	40.000	5332.00	25	1	production	May
19	0.00	18.00	0.000	0.00	25	1	storage	April
20	0.00	43.00	0.000	0.00	25	.	f2_to_1	May
21	220.00	11.00	30.000	330.00	19	.	f1_to_2	April
22	29952.00	62.40	480.000	29952.00	19	2	production	April
23	0.00	18.00	0.000	0.00	19	2	storage	March
24	0.00	25.00	0.000	0.00	19	2	backorder	May
25	0.00	23.00	0.000	0.00	25	.	f1_to_2	April
26	133755.98	196.70	680.000	133756.00	25	2	production	April
27	0.00	28.00	0.000	0.00	25	2	storage	March
28	0.00	54.00	15.000	810.00	25	2	backorder	May
29	0.00	11.00	0.000	0.00	19	.	f1_to_2	March
30	25520.00	88.00	290.000	25520.00	19	2	production	March
31	0.00	17.00	0.000	0.00	19	2	backorder	April
32	0.00	23.00	0.000	0.00	25	.	f1_to_2	March
33	115570.02	182.00	645.000	117390.00	25	2	production	March
34	0.00	31.00	0.000	0.00	25	2	backorder	April
35	1840.00	16.00	100.000	1600.00	19	.	f1_to_2	May
36	4508.00	133.80	35.000	4683.00	19	2	production	May
37	0.00	20.00	15.000	300.00	19	2	storage	April
38	8710.00	26.00	0.000	0.00	25	.	f1_to_2	May
39	6349.00	201.40	35.000	7049.00	25	2	production	May
40	0.00	38.00	0.000	0.00	25	2	storage	April
41	-49147.54	-327.65	155.000	-50785.73	19	1	sales	March
42	-75000.00	-300.00	250.000	-75000.00	19	1	sales	April
43	-0.01	-285.00	0.000	0.00	19	1	sales	May
44	-74349.99	-297.40	250.000	-74350.00	19	2	sales	March
45	-72499.96	-290.00	245.000	-71050.01	19	2	sales	April
46	-0.00	-292.00	0.000	0.00	19	2	sales	May
47	0.00	-559.76	0.000	0.00	25	1	sales	March
48	-0.01	-524.28	0.000	0.00	25	1	sales	April
49	-0.08	-475.02	25.000	-11875.50	25	1	sales	May
50	-283915.00	-567.83	500.000	-283915.00	25	2	sales	March
51	-216875.89	-542.19	375.000	-203321.25	25	2	sales	April
52	-0.01	-461.56	0.000	0.00	25	2	sales	May
53	-90685.00	-362.74	250.000	-90685.00	19	1	sales	March
54	-75000.00	-300.00	250.000	-75000.00	19	1	sales	April
55	0.00	-245.00	0.000	0.00	19	1	sales	May
56	-0.01	-272.70	0.000	0.00	19	2	sales	March
57	-78000.00	-312.00	250.000	-78000.00	19	2	sales	April
58	-44850.00	-299.00	150.000	-44850.00	19	2	sales	May
59	-283869.94	-623.89	455.000	-283869.95	25	1	sales	March
60	-129174.79	-549.68	535.000	-294078.80	25	1	sales	April
61	0.00	-460.00	0.000	0.00	25	1	sales	May
62	-59711.36	-542.83	120.000	-65139.60	25	2	sales	March
63	-156573.27	-559.19	320.000	-178940.80	25	2	sales	April
64	-192052.10	-489.06	20.000	-9781.20	25	2	sales	May
	-1285086.45			-1281110.35				

Example 2.3. Adding Side Constraints

The manufacturer of Gizmo chips, which are parts needed to make televisions, can supply only 2,600 chips to factory 1 and 3,750 chips to factory 2 in time for production in each of the months of March and April. However, Gizmo chips will not be in short supply in May. Three chips are required to make each 19-inch TV while the 25-inch TVs require four chips each. To limit the production of televisions produced at factory 1 in March so that the TVs have the correct number of chips, a side constraint called FACT1 MAR GIZMO is used. The form of this constraint is

```
3 * prod f1 19 mar   +   4 * prod f1 25 mar   <=   2600
```

prod f1 19 mar is the name of the arc directed from the node fact1_1 toward node f1_mar_1 and, in the previous constraint, designates the flow assigned to this arc. The ARCDATA= and CONOUT= data sets have arc names in a variable called _name_.

The other side constraints (shown below) are called FACT2 MAR GIZMO, FACT1 APL GIZMO, and FACT2 APL GIZMO.

```
3 * prod f2 19 mar   +   4 * prod f2 25 mar   <=   3750
3 * prod f1 19 apl   +   4 * prod f1 25 apl   <=   2600
3 * prod f2 19 apl   +   4 * prod f2 25 apl   <=   3750
```

To maintain customer goodwill, the total number of backorders is not to exceed 50 sets. The side constraint TOTAL BACKORDER that models this restriction is

```
back f1 19 apl  +  back f1 25 apl  +
back f2 19 apl  +  back f2 25 apl  +
back f1 19 may  +  back f1 25 may  +
back f2 19 may  +  back f2 25 may   <=   50
```

The sparse CONDATA= data set format is used. All side constraints are of less than or equal type. Because this is the default type value for the DEFCONTYPE= option, type information is not necessary in the following CONDATA=con3. Also, DEFCONTYPE= <= does not have to be specified in the PROC INTPOINT statement that follows. Notice that the _column_ variable value CHIP/BO LIMIT indicates that an observation of the con3 data set contains rhs information. Therefore, specify RHSOBS='CHIP/BO LIMIT'.

```
title2 'Adding Side Constraints';
data con3;
   input _column_ &$14. _row_ &$15. _coef_ ;
   datalines;
prod f1 19 mar  FACT1 MAR GIZMO  3
prod f1 25 mar  FACT1 MAR GIZMO  4
CHIP/BO LIMIT   FACT1 MAR GIZMO  2600
prod f2 19 mar  FACT2 MAR GIZMO  3
prod f2 25 mar  FACT2 MAR GIZMO  4
CHIP/BO LIMIT   FACT2 MAR GIZMO  3750
prod f1 19 apl  FACT1 APL GIZMO  3
```

```
prod f1 25 apl  FACT1 APL GIZMO  4
CHIP/BO LIMIT   FACT1 APL GIZMO  2600
prod f2 19 apl  FACT2 APL GIZMO  3
prod f2 25 apl  FACT2 APL GIZMO  4
CHIP/BO LIMIT   FACT2 APL GIZMO  3750
back f1 19 apl  TOTAL BACKORDER  1
back f1 25 apl  TOTAL BACKORDER  1
back f2 19 apl  TOTAL BACKORDER  1
back f2 25 apl  TOTAL BACKORDER  1
back f1 19 may  TOTAL BACKORDER  1
back f1 25 may  TOTAL BACKORDER  1
back f2 19 may  TOTAL BACKORDER  1
back f2 25 may  TOTAL BACKORDER  1
CHIP/BO LIMIT   TOTAL BACKORDER  50
;
```

The four pairs of data sets that follow can be used as ARCDATA= and NODEDATA= data sets in the following PROC INTPOINT run. The set used depends on which cost information the arcs are to have.

```
ARCDATA=arc0    NODEDATA=node0
ARCDATA=arc1    NODEDATA=node0
ARCDATA=arc2    NODEDATA=node0
ARCDATA=arc3    NODEDATA=node0
```

arc0, node0, and arc1 were created in Example 2.1. The first two data sets are the original input data sets.

In the previous example, arc2 was created by modifying arc1 to reflect different arc costs. arc2 and node0 can also be used as the ARCDATA= and NODEDATA= data sets in a PROC INTPOINT run.

If you are going to continue optimization using the changed arc costs, it is probably best to use arc3 and node0 as the ARCDATA= and NODEDATA= data sets.

PROC INTPOINT is used to find the changed cost network solution that obeys the chip limit and backorder side constraints. An explicit ID list has also been specified so that the variables oldcost, oldfc, and oldflow do not appear in the subsequent output data sets:

```
proc intpoint
   bytes=1000000
   printlevel2=2
   nodedata=node0 arcdata=arc3
   condata=con3 sparsecondata rhsobs='CHIP/BO LIMIT'
   conout=arc4;
   id diagonal factory key_id mth_made;
   run;


proc print data=arc4;
   var _tail_ _head_ _cost_ _capac_ _lo_ _flow_ _fcost_;
       /* to get this variable order */
   sum _fcost_;
   run;
```

The following messages appear on the SAS log:

```
   NOTE: The following variables in ARCDATA do not belong to any
         SAS variable list. These will be ignored.
         _FLOW_
         _FCOST_
         oldcost
         oldfc
         oldflow
   NOTE: Number of nodes= 20 .
   NOTE: Number of supply nodes= 4 .
   NOTE: Number of demand nodes= 4 .
   NOTE: Total supply= 4350 , total demand= 4150 .
   NOTE: Number of arcs= 64 .
   NOTE: Number of <= side constraints= 5 .
   NOTE: Number of == side constraints= 0 .
   NOTE: Number of >= side constraints= 0 .
   NOTE: Number of side constraint coefficients= 16 .
   NOTE: The following messages relate to the equivalent
         Linear Programming problem solved by the Interior
         Point algorithm.
   NOTE: Number of <= constraints= 5 .
   NOTE: Number of == constraints= 21 .
   NOTE: Number of >= constraints= 0 .
   NOTE: Number of constraint coefficients= 152 .
   NOTE: Number of variables= 68 .
   NOTE: After preprocessing, number of <= constraints= 5.
   NOTE: After preprocessing, number of == constraints= 20.
   NOTE: After preprocessing, number of >= constraints= 0.
   NOTE: The preprocessor eliminated 1 constraints from the
         problem.
   NOTE: The preprocessor eliminated 9 constraint coefficients
         from the problem.
   NOTE: 5 columns, 0 rows and 5 coefficients were added to
         the problem to handle unrestricted variables,
         variables that are split, and constraint slack or
         surplus variables.
   NOTE: There are 74 nonzero elements in A * A transpose.
   NOTE: Of the 25 rows and columns, 14 are sparse.
   NOTE: There are 74 nonzero superdiagonal elements in the
         sparse rows of the factored A * A transpose. This
         includes fill-in.
   NOTE: There are 65 operations of the form
         u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
         sparse rows of A * A transpose.
Iter Complem_aff Complem-ity  Duality_gap Tot_infeasb  Tot_infeasc Tot_infeasd
   0   -1.000000   176663849     0.834344       52835        39643       49140
   1    51289701    21890177     0.912033 2958.813395  2220.091192 2621.647223
   2     4297808     1359558     0.517260           0  7.06244E-11   42.614836
   3      341918      246210     0.159762           0            0    7.723054
   4      124303       68295     0.049237           0            0    1.115512
   5       46970       29876     0.021786           0            0    0.482224
   6 9976.439552 6294.587840     0.004647           0            0    0.094764
   7 3266.423958 1984.437170     0.001468           0            0    0.022740
   8  472.139836  257.075141     0.000190           0            0    0.003062
   9   24.953361    6.458585  0.000004781           0            0    0.000114
  10    0.007991    0.000361 2.671196E-10           0            0           0
   NOTE: The Primal-Dual Predictor-Corrector Interior Point algorithm
         performed 10 iterations.
```

```
NOTE: Objective = -1282708.625.
NOTE: The data set WORK.ARC4 has 64 observations and 14
      variables.
NOTE: There were 64 observations read from the data set
      WORK.ARC3.
NOTE: There were 8 observations read from the data set
      WORK.NODE0.
NOTE: There were 21 observations read from the data set
      WORK.CON3.
```

Output 2.3.1. CONOUT=ARC4

Adding Side Constraints

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_FLOW_	_FCOST_
1	fact1_1	f1_apr_1	78.60	600	50	533.333	41920.00
2	f1_mar_1	f1_apr_1	15.00	50	0	0.000	0.00
3	f1_may_1	f1_apr_1	33.60	20	0	0.000	0.00
4	f2_apr_1	f1_apr_1	11.00	40	0	0.000	0.00
5	fact1_2	f1_apr_2	174.50	550	50	250.000	43625.00
6	f1_mar_2	f1_apr_2	20.00	40	0	0.000	0.00
7	f1_may_2	f1_apr_2	49.20	15	0	0.000	0.00
8	f2_apr_2	f1_apr_2	21.00	25	0	0.000	0.00
9	fact1_1	f1_mar_1	127.90	500	50	333.333	42633.33
10	f1_apr_1	f1_mar_1	33.60	20	0	20.000	672.00
11	f2_mar_1	f1_mar_1	10.00	40	0	40.000	400.00
12	fact1_2	f1_mar_2	217.90	400	40	400.000	87160.00
13	f1_apr_2	f1_mar_2	38.40	30	0	30.000	1152.00
14	f2_mar_2	f1_mar_2	20.00	25	0	25.000	500.00
15	fact1_1	f1_may_1	90.10	400	50	128.333	11562.83
16	f1_apr_1	f1_may_1	12.00	50	0	0.000	0.00
17	f2_may_1	f1_may_1	13.00	40	0	0.000	0.00
18	fact1_2	f1_may_2	113.30	350	40	350.000	39655.00
19	f1_apr_2	f1_may_2	18.00	40	0	0.000	0.00
20	f2_may_2	f1_may_2	13.00	25	0	0.000	0.00
21	f1_apr_1	f2_apr_1	11.00	99999999	0	13.333	146.67
22	fact2_1	f2_apr_1	62.40	480	35	480.000	29952.00
23	f2_mar_1	f2_apr_1	18.00	30	0	0.000	0.00
24	f2_may_1	f2_apr_1	30.00	15	0	0.000	0.00
25	f1_apr_2	f2_apr_2	23.00	99999999	0	0.000	0.00
26	fact2_2	f2_apr_2	196.70	680	35	577.500	113594.25
27	f2_mar_2	f2_apr_2	28.00	50	0	0.000	0.00
28	f2_may_2	f2_apr_2	64.80	15	0	0.000	0.00
29	f1_mar_1	f2_mar_1	11.00	99999999	0	0.000	0.00
30	fact2_1	f2_mar_1	88.00	450	35	290.000	25520.00
31	f2_apr_1	f2_mar_1	20.40	15	0	0.000	0.00
32	f1_mar_2	f2_mar_2	23.00	99999999	0	0.000	0.00
33	fact2_2	f2_mar_2	182.00	650	35	650.000	118300.00
34	f2_apr_2	f2_mar_2	37.20	15	0	0.000	0.00
35	f1_may_1	f2_may_1	16.00	99999999	0	115.000	1840.00
36	fact2_1	f2_may_1	128.80	250	35	35.000	4508.00
37	f2_apr_1	f2_may_1	20.00	30	0	0.000	0.00
38	f1_may_2	f2_may_2	26.00	99999999	0	350.000	9100.00
39	fact2_2	f2_may_2	181.40	550	35	122.500	22221.50
40	f2_apr_2	f2_may_2	38.00	50	0	0.000	0.00
41	f1_mar_1	shop1_1	-327.65	250	0	143.333	-46963.17
42	f1_apr_1	shop1_1	-300.00	250	0	250.000	-75000.00
43	f1_may_1	shop1_1	-285.00	250	0	13.333	-3800.00
44	f2_mar_1	shop1_1	-297.40	250	0	250.000	-74350.00
45	f2_apr_1	shop1_1	-290.00	250	0	243.333	-70566.67
46	f2_may_1	shop1_1	-292.00	250	0	0.000	0.00
47	f1_mar_2	shop1_2	-559.76	99999999	0	0.000	0.00
48	f1_apr_2	shop1_2	-524.28	99999999	0	0.000	0.00
49	f1_may_2	shop1_2	-475.02	99999999	0	0.000	0.00
50	f2_mar_2	shop1_2	-567.83	500	0	500.000	-283915.00
51	f2_apr_2	shop1_2	-542.19	500	0	400.000	-216876.00
52	f2_may_2	shop1_2	-491.56	500	0	0.000	0.00
53	f1_mar_1	shop2_1	-362.74	250	0	250.000	-90685.00
54	f1_apr_1	shop2_1	-300.00	250	0	250.000	-75000.00
55	f1_may_1	shop2_1	-245.00	250	0	0.000	0.00
56	f2_mar_1	shop2_1	-272.70	250	0	0.000	0.00
57	f2_apr_1	shop2_1	-312.00	250	0	250.000	-78000.00
58	f2_may_1	shop2_1	-299.00	250	0	150.000	-44850.00
59	f1_mar_2	shop2_2	-623.89	99999999	0	455.000	-283869.95
60	f1_apr_2	shop2_2	-549.68	99999999	0	220.000	-120929.60
61	f1_may_2	shop2_2	-460.00	99999999	0	0.000	0.00
62	f2_mar_2	shop2_2	-542.83	500	0	125.000	-67853.75
63	f2_apr_2	shop2_2	-559.19	500	0	177.500	-99256.23
64	f2_may_2	shop2_2	-519.06	500	0	472.500	-245255.85
							==========
							-1282708.62

Example 2.4. Using Constraints and More Alteration to Arc Data

Suppose the 25-inch screen TVs produced at factory 1 in May can be sold at either shop with an increased profit of 40 dollars each. What is the new optimal solution?

```
title2 'Using Constraints and Altering arc data';
data new_arc4;
   set arc4;
   oldcost=_cost_;
   oldflow=_flow_;
   oldfc=_fcost_;
   if _tail_='f1_may_2' & (_head_='shop1_2' | _head_='shop2_2')
      then _cost_=_cost_-40;
   run;

proc intpoint
   bytes=1000000
   printlevel2=2
   arcdata=new_arc4 nodedata=node0
   condata=con3 sparsecondata rhsobs='CHIP/BO LIMIT'
   conout=arc5;
   run;

proc print data=arc5;
   var _tail_ _head_ _cost_ _capac_ _lo_
       _supply_ _demand_ _name_
       _flow_ _fcost_ oldflow oldfc;
          /* to get this variable order */
   sum oldfc _fcost_;
   run;
```

The following messages appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: Number of <= side constraints= 5 .
NOTE: Number of == side constraints= 0 .
NOTE: Number of >= side constraints= 0 .
NOTE: Number of side constraint coefficients= 16 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
      Point algorithm.
NOTE: Number of <= constraints= 5 .
NOTE: Number of == constraints= 21 .
NOTE: Number of >= constraints= 0 .
NOTE: Number of constraint coefficients= 152 .
NOTE: Number of variables= 68 .
NOTE: After preprocessing, number of <= constraints= 5.
NOTE: After preprocessing, number of == constraints= 20.
NOTE: After preprocessing, number of >= constraints= 0.
NOTE: The preprocessor eliminated 1 constraints from the
      problem.
NOTE: The preprocessor eliminated 9 constraint coefficients
```

```
        from the problem.
   NOTE: 5 columns, 0 rows and 5 coefficients were added to
        the problem to handle unrestricted variables,
        variables that are split, and constraint slack or
        surplus variables.
   NOTE: There are 74 nonzero elements in A * A transpose.
   NOTE: Of the 25 rows and columns, 14 are sparse.
   NOTE: There are 74 nonzero superdiagonal elements in the
        sparse rows of the factored A * A transpose. This
        includes fill-in.
   NOTE: There are 65 operations of the form
        u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
        sparse rows of A * A transpose.
Iter Complem_aff Complem-ity  Duality_gap Tot_infeasb  Tot_infeasc Tot_infeasd
   0   -1.000000   178045680     0.833846       52835        39643       49592
   1    51679271    22114244     0.911781 2979.752508  2235.802470 2678.044487
   2     4360227     1397064     0.521965           0 2.084022E-11   46.964760
   3      337615      239843     0.155358           0            0    8.067907
   4      119497       59613     0.042674           0            0    1.263035
   5       30689       20758     0.015076           0            0    0.430638
   6 9107.182114 7099.343072     0.005192           0            0    0.109413
   7 3406.632390 1496.513249     0.001098           0            0    0.003935
   8  616.222707  155.883444     0.000114           0            0    0.000480
   9   23.880446    1.372116  0.000001007           0            0           0
  10    0.000755 0.000068819 -4.28512E-10           0            0           0
   NOTE: The Primal-Dual Predictor-Corrector Interior Point algorithm
        performed 10 iterations.
   NOTE: Objective = -1295661.8.
   NOTE: The data set WORK.ARC5 has 64 observations and 17
        variables.
   NOTE: There were 64 observations read from the data set
        WORK.NEW_ARC4.
   NOTE: There were 8 observations read from the data set
        WORK.NODE0.
   NOTE: There were 21 observations read from the data set
        WORK.CON3.
```

Output 2.4.1. CONOUT=ARC5

Using Constraints and Altering arc data

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_SUPPLY_	_DEMAND_
1	fact1_1	f1_apr_1	78.60	600	50	1000	.
2	f1_mar_1	f1_apr_1	15.00	50	0	.	.
3	f1_may_1	f1_apr_1	33.60	20	0	.	.
4	f2_apr_1	f1_apr_1	11.00	40	0	.	.
5	fact1_2	f1_apr_2	174.50	550	50	1000	.
6	f1_mar_2	f1_apr_2	20.00	40	0	.	.
7	f1_may_2	f1_apr_2	49.20	15	0	.	.
8	f2_apr_2	f1_apr_2	21.00	25	0	.	.
9	fact1_1	f1_mar_1	127.90	500	50	1000	.
10	f1_apr_1	f1_mar_1	33.60	20	0	.	.
11	f2_mar_1	f1_mar_1	10.00	40	0	.	.
12	fact1_2	f1_mar_2	217.90	400	40	1000	.
13	f1_apr_2	f1_mar_2	38.40	30	0	.	.
14	f2_mar_2	f1_mar_2	20.00	25	0	.	.
15	fact1_1	f1_may_1	90.10	400	50	1000	.
16	f1_apr_1	f1_may_1	12.00	50	0	.	.
17	f2_may_1	f1_may_1	13.00	40	0	.	.
18	fact1_2	f1_may_2	113.30	350	40	1000	.
19	f1_apr_2	f1_may_2	18.00	40	0	.	.
20	f2_may_2	f1_may_2	13.00	25	0	.	.
21	f1_apr_1	f2_apr_1	11.00	99999999	0	.	.
22	fact2_1	f2_apr_1	62.40	480	35	850	.
23	f2_mar_1	f2_apr_1	18.00	30	0	.	.
24	f2_may_1	f2_apr_1	30.00	15	0	.	.
25	f1_apr_2	f2_apr_2	23.00	99999999	0	.	.
26	fact2_2	f2_apr_2	196.70	680	35	1500	.
27	f2_mar_2	f2_apr_2	28.00	50	0	.	.
28	f2_may_2	f2_apr_2	64.80	15	0	.	.
29	f1_mar_1	f2_mar_1	11.00	99999999	0	.	.
30	fact2_1	f2_mar_1	88.00	450	35	850	.
31	f2_apr_1	f2_mar_1	20.40	15	0	.	.
32	f1_mar_2	f2_mar_2	23.00	99999999	0	.	.
33	fact2_2	f2_mar_2	182.00	650	35	1500	.
34	f2_apr_2	f2_mar_2	37.20	15	0	.	.
35	f1_may_1	f2_may_1	16.00	99999999	0	.	.
36	fact2_1	f2_may_1	128.80	250	35	850	.
37	f2_apr_1	f2_may_1	20.00	30	0	.	.
38	f1_may_2	f2_may_2	26.00	99999999	0	.	.
39	fact2_2	f2_may_2	181.40	550	35	1500	.
40	f2_apr_2	f2_may_2	38.00	50	0	.	.
41	f1_mar_1	shop1_1	-327.65	250	0	.	900
42	f1_apr_1	shop1_1	-300.00	250	0	.	900
43	f1_may_1	shop1_1	-285.00	250	0	.	900
44	f2_mar_1	shop1_1	-297.40	250	0	.	900
45	f2_apr_1	shop1_1	-290.00	250	0	.	900
46	f2_may_1	shop1_1	-292.00	250	0	.	900
47	f1_mar_2	shop1_2	-559.76	99999999	0	.	900
48	f1_apr_2	shop1_2	-524.28	99999999	0	.	900
49	f1_may_2	shop1_2	-515.02	99999999	0	.	900
50	f2_mar_2	shop1_2	-567.83	500	0	.	900
51	f2_apr_2	shop1_2	-542.19	500	0	.	900
52	f2_may_2	shop1_2	-491.56	500	0	.	900
53	f1_mar_1	shop2_1	-362.74	250	0	.	900
54	f1_apr_1	shop2_1	-300.00	250	0	.	900
55	f1_may_1	shop2_1	-245.00	250	0	.	900
56	f2_mar_1	shop2_1	-272.70	250	0	.	900
57	f2_apr_1	shop2_1	-312.00	250	0	.	900
58	f2_may_1	shop2_1	-299.00	250	0	.	900
59	f1_mar_2	shop2_2	-623.89	99999999	0	.	1450
60	f1_apr_2	shop2_2	-549.68	99999999	0	.	1450
61	f1_may_2	shop2_2	-500.00	99999999	0	.	1450
62	f2_mar_2	shop2_2	-542.83	500	0	.	1450
63	f2_apr_2	shop2_2	-559.19	500	0	.	1450
64	f2_may_2	shop2_2	-519.06	500	0	.	1450

Obs	_name_	_FLOW_	_FCOST_	oldflow	oldfc
1	prod f1 19 apl	533.333	41920.00	533.333	41920.00
2		0.000	0.00	0.000	0.00
3	back f1 19 may	0.000	0.00	0.000	0.00
4		0.000	0.00	0.000	0.00
5	prod f1 25 apl	250.000	43625.00	250.000	43625.00
6		0.000	0.00	0.000	0.00
7	back f1 25 may	0.000	0.00	0.000	0.00
8		0.000	0.00	0.000	0.00
9	prod f1 19 mar	333.333	42633.33	333.333	42633.33
10	back f1 19 apl	20.000	672.00	20.000	672.00
11		40.000	400.00	40.000	400.00
12	prod f1 25 mar	400.000	87160.00	400.000	87160.00
13	back f1 25 apl	30.000	1152.00	30.000	1152.00
14		25.000	500.00	25.000	500.00
15		128.333	11562.83	128.333	11562.83
16		0.000	0.00	0.000	0.00
17		0.000	0.00	0.000	0.00
18		350.000	39655.00	350.000	39655.00
19		0.000	0.00	0.000	0.00
20		0.000	0.00	0.000	0.00
21		13.333	146.67	13.333	146.67
22	prod f2 19 apl	480.000	29952.00	480.000	29952.00
23		0.000	0.00	0.000	0.00
24	back f2 19 may	0.000	0.00	0.000	0.00
25		0.000	0.00	0.000	0.00
26	prod f2 25 apl	550.000	108185.00	577.500	113594.25
27		0.000	0.00	0.000	0.00
28	back f2 25 may	0.000	0.00	0.000	0.00
29		0.000	0.00	0.000	0.00
30	prod f2 19 mar	290.000	25520.00	290.000	25520.00
31	back f2 19 apl	0.000	0.00	0.000	0.00
32		0.000	0.00	0.000	0.00
33	prod f2 25 mar	650.000	118300.00	650.000	118300.00
34	back f2 25 apl	0.000	0.00	0.000	0.00
35		115.000	1840.00	115.000	1840.00
36		35.000	4508.00	35.000	4508.00
37		0.000	0.00	0.000	0.00
38		0.000	0.00	350.000	9100.00
39		150.000	27210.00	122.500	22221.50
40		0.000	0.00	0.000	0.00
41		143.333	-46963.17	143.333	-46963.17
42		250.000	-75000.00	250.000	-75000.00
43		13.333	-3800.00	13.333	-3800.00
44		250.000	-74350.00	250.000	-74350.00
45		243.333	-70566.67	243.333	-70566.67
46		0.000	0.00	0.000	0.00
47		0.000	0.00	0.000	0.00
48		0.000	0.00	0.000	0.00
49		350.000	-180257.00	0.000	0.00
50		500.000	-283915.00	500.000	-283915.00
51		50.000	-27109.50	400.000	-216876.00
52		0.000	0.00	0.000	0.00
53		250.000	-90685.00	250.000	-90685.00
54		250.000	-75000.00	250.000	-75000.00
55		0.000	0.00	0.000	0.00
56		0.000	0.00	0.000	0.00
57		250.000	-78000.00	250.000	-78000.00
58		150.000	-44850.00	150.000	-44850.00
59		455.000	-283869.95	455.000	-283869.95
60		220.000	-120929.60	220.000	-120929.60
61		0.000	0.00	0.000	0.00
62		125.000	-67853.75	125.000	-67853.75
63		500.000	-279595.00	177.500	-99256.23
64		150.000	-77859.00	472.500	-245255.85
			==========		==========
			-1295661.80		-1282708.62

Example 2.5. Nonarc Variables in the Side Constraints

You can verify that the FACT2 MAR GIZMO constraint has a left-hand-side activity of 3,470, which is not equal to the _RHS_ of this constraint. Not all of the 3,750 chips that can be supplied to factory 2 for March production are used. It is suggested that all the possible chips be obtained in March and those not used be saved for April production. Because chips must be kept in an air-controlled environment, it costs one dollar to store each chip purchased in March until April. The maximum number of chips that can be stored in this environment at each factory is 150. In addition, a search of the parts inventory at factory 1 turned up 15 chips available for their March production.

Nonarc variables are used in the side constraints that handle the limitations of supply of Gizmo chips. A nonarc variable called f1 unused chips has as a value the number of chips that are not used at factory 1 in March. Another nonarc variable, f2 unused chips, has as a value the number of chips that are not used at factory 2 in March. f1 chips from mar has as a value the number of chips left over from March used for production at factory 1 in April. Similarly, f2 chips from mar has as a value the number of chips left over from March used for April production at factory 2 in April. The last two nonarc variables have objective function coefficients of 1 and upper bounds of 150. The Gizmo side constraints are

```
3*prod f1 19 mar + 4*prod f1 25 mar + f1 unused chips = 2615
3*prod f2 19 apl + 4*prod f2 25 apl + f2 unused chips = 3750
3*prod f1 19 apl + 4*prod f1 25 apl - f1 chips from mar = 2600
3*prod f2 19 apl + 4*prod f2 25 apl - f2 chips from mar = 3750
f1 unused chips + f2 unused chips -
f1 chips from mar - f2 chips from mar >= 0
```

The last side constraint states that the number of chips not used in March is not less than the number of chips left over from March and used in April. Here, this constraint is called CHIP LEFTOVER.

The following SAS code creates a new data set containing constraint data. It seems that most of the constraints are now equalities, so you specify DEFCONTYPE=EQ in the PROC INTPOINT statement from now on and provide constraint type data for constraints that are not "equal to" type, using the default TYPEOBS value _TYPE_ as the _COLUMN_ variable value to indicate observations that contain constraint type data. Also, from now on, the default RHSOBS value is used.

```
title2 'Nonarc Variables in the Side Constraints';
data con6;
   input _column_ &$17. _row_ &$15. _coef_ ;
   datalines;
prod f1 19 mar     FACT1 MAR GIZMO  3
prod f1 25 mar     FACT1 MAR GIZMO  4
f1 unused chips    FACT1 MAR GIZMO  1
_RHS_              FACT1 MAR GIZMO  2615
prod f2 19 mar     FACT2 MAR GIZMO  3
prod f2 25 mar     FACT2 MAR GIZMO  4
f2 unused chips    FACT2 MAR GIZMO  1
_RHS_              FACT2 MAR GIZMO  3750
```

```
prod f1 19 apl      FACT1 APL GIZMO  3
prod f1 25 apl      FACT1 APL GIZMO  4
f1 chips from mar   FACT1 APL GIZMO  -1
_RHS_               FACT1 APL GIZMO  2600
prod f2 19 apl      FACT2 APL GIZMO  3
prod f2 25 apl      FACT2 APL GIZMO  4
f2 chips from mar   FACT2 APL GIZMO  -1
_RHS_               FACT2 APL GIZMO  3750
f1 unused chips     CHIP LEFTOVER    1
f2 unused chips     CHIP LEFTOVER    1
f1 chips from mar   CHIP LEFTOVER    -1
f2 chips from mar   CHIP LEFTOVER    -1
_TYPE_              CHIP LEFTOVER    1
back f1 19 apl      TOTAL BACKORDER  1
back f1 25 apl      TOTAL BACKORDER  1
back f2 19 apl      TOTAL BACKORDER  1
back f2 25 apl      TOTAL BACKORDER  1
back f1 19 may      TOTAL BACKORDER  1
back f1 25 may      TOTAL BACKORDER  1
back f2 19 may      TOTAL BACKORDER  1
back f2 25 may      TOTAL BACKORDER  1
_TYPE_              TOTAL BACKORDER  -1
_RHS_               TOTAL BACKORDER  50
;
```

The nonarc variables f1 chips from mar and f2 chips from mar have objective function coefficients of 1 and upper bounds of 150. There are various ways in which this information can be furnished to PROC INTPOINT. If there were a TYPE list variable in the CONDATA= data set, observations could be in the form

```
_COLUMN_            _TYPE_  _ROW_ _COEF_
f1 chips from mar   objfn     .       1
f1 chips from mar   upperbd   .     150
f2 chips from mar   objfn     .       1
f2 chips from mar   upperbd   .     150
```

It is desirable to assign ID list variable values to all the nonarc variables:

```
data arc6;
   set arc5;
   drop oldcost oldfc oldflow _flow_ _fcost_ ;
   run;

data arc6_b;
   input _name_ &$17. _cost_ _capac_ factory key_id $ ;
   datalines;
f1 unused chips     .    . 1 chips
f2 unused chips     .    . 2 chips
f1 chips from mar   1 150 1 chips
f2 chips from mar   1 150 2 chips
;


proc append force
   base=arc6 data=arc6_b;
```

```
   run;

proc intpoint
   bytes=1000000
   printlevel2=2
   nodedata=node0 arcdata=arc6
   condata=con6 defcontype=eq sparsecondata
   conout=arc7;
   run;
```

The following messages appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: Number of nonarc variables= 4 .
NOTE: Number of <= side constraints= 1 .
NOTE: Number of == side constraints= 4 .
NOTE: Number of >= side constraints= 1 .
NOTE: Number of side constraint coefficients= 24 .
NOTE: The following messages relate to the equivalent
      Linear Programming problem solved by the Interior
      Point algorithm.
NOTE: Number of <= constraints= 1 .
NOTE: Number of == constraints= 25 .
NOTE: Number of >= constraints= 1 .
NOTE: Number of constraint coefficients= 160 .
NOTE: Number of variables= 72 .
NOTE: After preprocessing, number of <= constraints= 1.
NOTE: After preprocessing, number of == constraints= 24.
NOTE: After preprocessing, number of >= constraints= 1.
NOTE: The preprocessor eliminated 1 constraints from the
      problem.
NOTE: The preprocessor eliminated 9 constraint coefficients
      from the problem.
NOTE: 2 columns, 0 rows and 2 coefficients were added to
      the problem to handle unrestricted variables,
      variables that are split, and constraint slack or
      surplus variables.
NOTE: There are 78 nonzero elements in A * A transpose.
NOTE: Of the 26 rows and columns, 15 are sparse.
NOTE: There are 87 nonzero superdiagonal elements in the
      sparse rows of the factored A * A transpose. This
      includes fill-in.
NOTE: There are 104 operations of the form
      u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
      sparse rows of A * A transpose.
```

Iter	Complem_aff	Complem-ity	Duality_gap	Tot_infeasb	Tot_infeasc	Tot_infeasd
0	-1.000000	182185121	0.837584	55030	37757	47382
1	56730833	28510760	0.910015	5004.066395	3433.344938	6917.139928
2	9604878	2589633	0.665677	0	1.309672E-10	226.047223
3	328091	309636	0.191136	0	0	26.943297
4	135892	89815	0.063031	0	0	6.389904
5	62039	31604	0.022877	0	0	0
6	16881	7606.584128	0.005568	0	0	0
7	3753.426021	1918.980183	0.001408	0	0	0

```
 8  709.549939  330.027670     0.000242           0            0           0
 9  155.181759   36.697802  0.000026949           0            0           0
10    3.865957    0.421161  0.000000309           0            0           0
11    0.001557 0.000021177 1.557982E-11           0            0           0
 NOTE: The Primal-Dual Predictor-Corrector Interior Point algorithm
       performed 11 iterations.
 NOTE: Objective = -1295542.742.
 NOTE: The data set WORK.ARC7 has 68 observations and 14
       variables.
 NOTE: There were 68 observations read from the data set
       WORK.ARC6.
 NOTE: There were 8 observations read from the data set
       WORK.NODE0.
 NOTE: There were 31 observations read from the data set
       WORK.CON6.
```

The optimal solution data set, CONOUT=ARC7, is given in Output 2.5.1.

```
proc print data=arc7;
   var _tail_ _head_ _name_ _cost_ _capac_ _lo_
       _flow_ _fcost_;
   sum _fcost_;
   run;
```

The optimal value of the nonarc variable f2 unused chips is 280. This means that although there are 3,750 chips that can be used at factory 2 in March, only 3,470 are used. As the optimal value of f1 unused chips is zero, all chips available for production in March at factory 1 are used. The nonarc variable f2 chips from mar also has zero optimal value. This means that the April production at factory 2 does not need any chips that could have been held in inventory since March. However, the nonarc variable f1 chips from mar has value of 20. Thus, 3,490 chips should be ordered for factory 2 in March. Twenty of these chips should be held in inventory until April, then sent to factory 1.

Output 2.5.1. CONOUT=ARC7

Nonarc Variables in the Side Constraints

Obs	_tail_	_head_	_name_	_cost_	_capac_	_lo_	_FLOW_	_FCOST_
1	fact1_1	f1_apr_1	prod f1 19 apl	78.60	600	50	540.000	42444.00
2	f1_mar_1	f1_apr_1		15.00	50	0	0.000	0.00
3	f1_may_1	f1_apr_1	back f1 19 may	33.60	20	0	0.000	0.00
4	f2_apr_1	f1_apr_1		11.00	40	0	0.000	0.00
5	fact1_2	f1_apr_2	prod f1 25 apl	174.50	550	50	250.000	43625.00
6	f1_mar_2	f1_apr_2		20.00	40	0	0.000	0.00
7	f1_may_2	f1_apr_2	back f1 25 may	49.20	15	0	0.000	0.00
8	f2_apr_2	f1_apr_2		21.00	25	0	25.000	525.00
9	fact1_1	f1_mar_1	prod f1 19 mar	127.90	500	50	338.333	43272.83
10	f1_apr_1	f1_mar_1	back f1 19 apl	33.60	20	0	20.000	672.00
11	f2_mar_1	f1_mar_1		10.00	40	0	40.000	400.00
12	fact1_2	f1_mar_2	prod f1 25 mar	217.90	400	40	400.000	87160.00
13	f1_apr_2	f1_mar_2	back f1 25 apl	38.40	30	0	30.000	1152.00
14	f2_mar_2	f1_mar_2		20.00	25	0	25.000	500.00
15	fact1_1	f1_may_1		90.10	400	50	116.667	10511.67
16	f1_apr_1	f1_may_1		12.00	50	0	0.000	0.00
17	f2_may_1	f1_may_1		13.00	40	0	0.000	0.00
18	fact1_2	f1_may_2		113.30	350	40	350.000	39655.00
19	f1_apr_2	f1_may_2		18.00	40	0	0.000	0.00
20	f2_may_2	f1_may_2		13.00	25	0	0.000	0.00
21	f1_apr_1	f2_apr_1		11.00	99999999	0	20.000	220.00
22	fact2_1	f2_apr_1	prod f2 19 apl	62.40	480	35	480.000	29952.00
23	f2_mar_1	f2_apr_1		18.00	30	0	0.000	0.00
24	f2_may_1	f2_apr_1	back f2 19 may	30.00	15	0	0.000	0.00
25	f1_apr_2	f2_apr_2		23.00	99999999	0	0.000	0.00
26	fact2_2	f2_apr_2	prod f2 25 apl	196.70	680	35	577.500	113594.25
27	f2_mar_2	f2_apr_2		28.00	50	0	0.000	0.00
28	f2_may_2	f2_apr_2	back f2 25 may	64.80	15	0	0.000	0.00
29	f1_mar_1	f2_mar_1		11.00	99999999	0	0.000	0.00
30	fact2_1	f2_mar_1	prod f2 19 mar	88.00	450	35	290.000	25520.00
31	f2_apr_1	f2_mar_1	back f2 19 apl	20.40	15	0	0.000	0.00
32	f1_mar_2	f2_mar_2		23.00	99999999	0	0.000	0.00
33	fact2_2	f2_mar_2	prod f2 25 mar	182.00	650	35	650.000	118300.00
34	f2_apr_2	f2_mar_2	back f2 25 apl	37.20	15	0	0.000	0.00
35	f1_may_1	f2_may_1		16.00	99999999	0	115.000	1840.00
36	fact2_1	f2_may_1		128.80	250	35	35.000	4508.00
37	f2_apr_1	f2_may_1		20.00	30	0	0.000	0.00
38	f1_may_2	f2_may_2		26.00	99999999	0	0.000	0.00
39	fact2_2	f2_may_2		181.40	550	35	122.500	22221.50
40	f2_apr_2	f2_may_2		38.00	50	0	0.000	0.00
41	f1_mar_1	shop1_1		-327.65	250	0	148.333	-48601.42
42	f1_apr_1	shop1_1		-300.00	250	0	250.000	-75000.00
43	f1_may_1	shop1_1		-285.00	250	0	1.667	-475.00
44	f2_mar_1	shop1_1		-297.40	250	0	250.000	-74350.00
45	f2_apr_1	shop1_1		-290.00	250	0	250.000	-72500.00
46	f2_may_1	shop1_1		-292.00	250	0	0.000	0.00
47	f1_mar_2	shop1_2		-559.76	99999999	0	0.000	0.00
48	f1_apr_2	shop1_2		-524.28	99999999	0	0.000	0.00
49	f1_may_2	shop1_2		-515.02	99999999	0	347.500	-178969.45
50	f2_mar_2	shop1_2		-567.83	500	0	500.000	-283915.00
51	f2_apr_2	shop1_2		-542.19	500	0	52.500	-28464.98
52	f2_may_2	shop1_2		-491.56	500	0	0.000	0.00
53	f1_mar_1	shop2_1		-362.74	250	0	250.000	-90685.00
54	f1_apr_1	shop2_1		-300.00	250	0	250.000	-75000.00
55	f1_may_1	shop2_1		-245.00	250	0	0.000	0.00
56	f2_mar_1	shop2_1		-272.70	250	0	0.000	0.00
57	f2_apr_1	shop2_1		-312.00	250	0	250.000	-78000.00
58	f2_may_1	shop2_1		-299.00	250	0	150.000	-44850.00
59	f1_mar_2	shop2_2		-623.89	99999999	0	455.000	-283869.95
60	f1_apr_2	shop2_2		-549.68	99999999	0	245.000	-134671.60
61	f1_may_2	shop2_2		-500.00	99999999	0	2.500	-1250.00
62	f2_mar_2	shop2_2		-542.83	500	0	125.000	-67853.75
63	f2_apr_2	shop2_2		-559.19	500	0	500.000	-279595.00
64	f2_may_2	shop2_2		-519.06	500	0	122.500	-63584.85
65			f1 chips from mar	1.00	150	0	20.000	20.00
66			f1 unused chips	0.00	99999999	0	0.000	0.00
67			f2 chips from mar	1.00	150	0	0.000	0.00
68			f2 unused chips	0.00	99999999	0	280.000	0.00
								==========
								-1295542.74

Example 2.6. Solving an LP Problem with Data in MPS Format

In this example, PROC INTPOINT is ultimately used to solve an LP. But prior to that, there is SAS code that is used to read a MPS format file and initialize an input SAS data set. MPS was an optimization package developed for IBM computers many years ago and the format by which data had to be supplied to that system became the industry standard for other optimization software packages, including those developed recently. The MPS format is described in Murtagh (1981). If you have an LP which has data in MPS format in a file /your-directory/your-filename.dat, then the following SAS code should be run:

```
filename w '/your-directorys/your-filename.dat';
data raw;
   infile w lrecl=80 pad;
   input field1 $ 2-3 field2 $ 5-12 field3 $ 15-22
         field4  25-36 field5 $ 40-47 field6 50-61;
   run;

%sasmpsxs;

data lp;
   set;
   if _type_="FREE" then _type_="MIN";
   if lag(_type_)="*HS" then _type_="RHS";
   run;

proc sort data=lp;
   by _col_;
   run;

proc intpoint
   arcdata=lp
   condata=lp sparsecondata rhsobs=rhs grouped=condata
   conout=solutn  /* SAS data set for the optimal solution */
   bytes=20000000
   nnas=1700 ncoefs=4000 ncons=700
   printlevel2=2 memrep;
   run;

proc lp
   data=lp sparsedata
   endpause time=3600 maxit1=100000 maxit2=100000;
   run;
   show status;
   quit;
```

You will have to specify the appropriate path and file name in which your MPS format data resides.

SASMPSXS is a SAS macro provided within SAS/OR software. The MPS format resembles the sparse format of the CONDATA= data set for PROC INTPOINT. The SAS macro SASMPSXS examines the MPS data and transfers it into a SAS data set while automatically taking into account how the MPS format differs slightly from PROC INTPOINT's sparse format.

The parameters NNAS=1700, NCOEFS=4000, and NCONS=700 indicate the approximate (overestimated) number of variables, coefficients and constraints this model has. You must change these to your problems dimensions. Knowing these, PROC INTPOINT is able to utilize memory better and read the data faster. These parameters are optional.

The PROC SORT preceding PROC INTPOINT is not necessary, but sorting the SAS data set can speed up PROC INTPOINT when it reads the data. After the sort, data for each column is grouped together. GROUPED=condata can be specified.

For small problems, presorting and specifying those additional options is not going to greatly influence PROC INTPOINT's run time. However, when problems are large, presorting and specifying those additional options can be very worthwhile.

If you generate the model yourself, you will be familiar enough with it to know what to specify for the RHSOBS= parameter. If the value of the SAS variable in the COLUMN list is equal to the character string specified as the RHSOBS= option, the data in that observation is interpreted as right-hand-side data as opposed to coefficient data. If you do not know what to specify for the RHSOBS= option, you should first run PROC LP and optionally set MAXIT1=1 and MAXIT2=1. PROC LP will output a Problem Summary that includes the line

```
Rhs Variable    rhs-charstr
```

BYTES=20000000 is the size of working memory PROC INTPOINT is allowed.

The options PRINTLEVEL2=2 and MEMREP indicate that you want to see an iteration log and messages about memory usage. Specifying these options is optional.

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Chapter 3
The LP Procedure

Chapter Contents

Chapter 3
The LP Procedure

Overview: LP Procedure

The LP procedure solves linear programs, integer programs, and mixed-integer programs. It also performs parametric programming, range analysis, and reports on solution sensitivity to changes in the right-hand-side constants and price coefficients.

The LP procedure provides various control options and solution strategies. It also provides the functionality to produce various kinds of intermediate and final solution information. The procedure's interactive features enable you to take control of the problem solving process. During linear or integer iterations, for example, you can stop the procedure at intermediate stages and examine current results. If necessary, you can change options or strategies and resume the execution of the procedure.

The LP procedure is used to optimize a linear function subject to linear and integer constraints. Specifically, the LP procedure solves the general mixed-integer program of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \{\geq, =, \leq\} b \\ & \ell \leq x \leq u \\ & x_i \text{ is integer}, i \in \mathcal{S} \end{array}$$

where

- A is an $m \times n$ matrix of technological coefficients
- b is an $m \times 1$ matrix of right-hand-side (RHS) constants
- c is an $n \times 1$ matrix of objective function coefficients
- x is an $n \times 1$ matrix of structural variables
- l is an $n \times 1$ matrix of lower bounds on x
- u is an $n \times 1$ matrix of upper bounds on x
- $\mathcal{S}$ is a subset of the set of indices $\{1, \ldots, n\}$

Linear programs (when $\mathcal{S}$ is empty) are denoted by (LP). For these problems, the procedure employs the two-phase revised simplex method, which uses the Bartels-Golub update of the LU decomposed basis matrix to pivot between feasible solutions (Bartels 1971). In phase 1, PROC LP finds a basic feasible solution to (LP), while in phase 2, PROC LP finds an optimal solution, x^{opt}. The procedure implicitly handles unrestricted variables, lower-bounded variables, upper-bounded variables, and ranges on constraints. When no explicit lower bounds are specified, PROC LP assumes that all variables are bounded below by zero.

When a variable is specified as an integer variable, $\mathcal{S}$ has at least one element. The procedure then uses the branch-and-bound technique for optimization.

The relaxed problem (the problem with no integer constraints) is solved initially using the primal algorithm described previously. Constraints are added in defining the subsequent descendant problems in the branch-and-bound tree. These problems are then solved using the dual simplex algorithm. Dual pivots are referred to as phase 3 pivots.

The preprocessing option enables the procedure to identify redundant and infeasible constraints, fix variables, and reduce the feasible region before solving a problem. For linear programs, the option often can reduce the number of constraints and variables, leading to a quicker elapsed solution time and improved reliability. For integer programs, it often reduces the gap between an integer program and its relaxed linear program, which will likely lead to a reduced branch-and-bound tree and a quicker CPU time. In general, it provides users an alternative to solving large, complicated operations research problems.

The LP procedure can also analyze the sensitivity of the solution x^{opt} to changes in both the objective function and the right-hand-side constants. There are three techniques available for this analysis: sensitivity analysis, parametric programming, and range analysis. Sensitivity analysis enables you to examine the size of a perturbation to the right-hand-side or objective vector by an arbitrary change vector for which the basis of the current optimal solution remains optimal.

Parametric programming, on the other hand, enables you to specify the size of the perturbation beforehand and examine how the optimal solution changes as the desired perturbation is realized. With this technique, the procedure pivots to maintain optimality as the right-hand-side or objective vector is perturbed beyond the range for which the current solution is optimal. Range analysis is used to examine the range of each right-hand-side value or objective coefficient for which the basis of the current optimal solution remains optimal.

The LP procedure can also save both primal and dual solutions, the current tableau, and the branch-and-bound tree in SAS data sets. This enables you to generate solution reports and perform additional analyses with the SAS System. Although PROC LP reports solutions, this feature is particularly useful for reporting solutions in formats tailored to your specific needs. Saving computational results in a data set also enables you to continue executing a problem not solved because of insufficient time or other computational problems.

The LP procedure uses the Output Delivery System (ODS), a SAS subsystem that provides capabilities for displaying and controlling the output from SAS procedures. ODS enables you to modify the headers, column names, data formats, and layouts of the output tables in PROC LP.

There are no restrictions on the problem size in the LP procedure. The number of constraints and variables in a problem that PROC LP can solve depends on the host platform, the available memory, and the available disk space for utility data sets.

You can also solve linear programming problems by using PROC OPTLP. PROC

OPTLP requires a linear program to be specified using a SAS data set that adheres to the MPS format, a widely accepted format in the optimization community. You can use the SAS macro %LP2MPSD to convert typical PROC LP format data sets into MPS-format SAS data sets. The macro is available online at the SAS Customer Support Center.

Getting Started: LP Procedure

PROC LP expects the definition of one or more linear, integer, or mixed-integer programs in an input data set. There are two formats, a dense format and a sparse format, for this data set.

In the dense format, a model is expressed in a similar way as it is formulated. Each SAS variable corresponds to a model's column, and each SAS observation corresponds to a model's row. A SAS variable in the input data set is one of the following:

- a type variable
- an id variable
- a structural variable
- a right-hand-side variable
- a right-hand-side sensitivity analysis variable or
- a range variable

The type variable tells PROC LP how to interpret the observation as a part of the mathematical programming problem. It identifies and classifies objectives, constraints, and the rows that contain information of variables like types, bounds, and so on. PROC LP recognizes the following keywords as values for the type variable: MIN, MAX, EQ, LE, GE, SOSEQ, SOSLE, UNRSTRT, LOWERBD, UPPERBD, FIXED, INTEGER, BINARY, BASIC, PRICESEN, and FREE. The values of the id variable are the names of the rows in the model. The other variables identify and classify the columns with numerical values.

The sparse format to PROC LP is designed to enable you to specify only the nonzero coefficients in the description of linear programs, integer programs, and mixed-integer programs. The SAS data set that describes the sparse model must contain at least four SAS variables:

- a type variable
- a column variable
- a row variable and
- a coefficient variable

Each observation in the data set associates a type with a row or a column, or defines a coefficient or a numerical value in the model, or both. In addition to the keywords in the dense format, PROC LP also recognizes the keywords RHS, RHSSEN, and

RANGE as values of the type variable. The values of the row and column variables are the names of the rows and columns in the model. The values of the coefficient variables give the coefficients or other numerical data. The SAS data set can contain multiple pairs of row and coefficient variables. In this way, more information about the model can be specified in each observation in the data set. See the section "Sparse Data Input Format" on page 196 for further discussion.

With both the dense and sparse formats for model specification, the observation order is not important. This feature is particularly useful when using the sparse model input.

An Introductory Example

A simple blending problem illustrates the dense and sparse input formats and the use of PROC LP. A step in refining crude oil into finished oil products involves a distillation process that splits crude into various streams. Suppose there are three types of crude available: Arabian light, Arabian heavy, and Brega. These types of crude are distilled into light naphtha, intermediate naphtha, and heating oil. These in turn are blended into jet fuel using one of two recipes. What amounts of the three crudes maximize the profit from producing jet fuel? A formulation to answer this question is as follows:

$$\max \quad -175\,\text{a\_light} - 165\,\text{a\_heavy} - 205\,\text{brega} + 300\,\text{jet\_1} + 300\,\text{jet\_2}$$

$$\begin{aligned}
\text{subject to} \quad .035\,\text{a_light} + .03\,\text{a_heavy} + .045\,\text{brega} &= \text{naphthal} \\
.1\,\text{a_light} + .075\,\text{a_heavy} + .135\,\text{brega} &= \text{naphthai} \\
.39\,\text{a_light} + .3\,\text{a_heavy} + .43\,\text{brega} &= \text{heatingo} \\
.3\,\text{naphthai} + .7\,\text{heatingo} &= \text{jet_1} \\
.2\,\text{naphthal} + .8\,\text{heatingo} &= \text{jet_2} \\
\text{a_light} &\le 110 \\
\text{a_heavy} &\le 165 \\
\text{brega} &\le 80 \\
\text{a_light}, \text{a_heavy}, \text{brega}, \text{naphthai}, & \\
\text{naphthal}, \text{heatingo}, \text{jet_1}, \text{jet_2} &\ge 0
\end{aligned}$$

The following data set gives the representation of this formulation. Notice that the variable names are the structural variables, the rows are the constraints, and the coefficients are given as the values for the structural variables.

```
data;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2
         _type_ $ _rhs_;
   datalines;
profit            -175 -165 -205  0  0  0 300 300 max     .
```

```
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0  eq      0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0  eq      0
heating_o_conv    .390 .300 .430  0  0 -1   0   0  eq      0
recipe_1             0    0    0  0 .3 .7  -1   0  eq      0
recipe_2             0    0    0 .2  0 .8   0  -1  eq      0
available          110  165   80  .  .  .   .   . upperbd .
;
```

The same model can be specified in the sparse format, as follows. This format enables you to omit the zero coefficients.

```
data;
   format _type_ $8. _col_ $8. _row_ $16. ;
   input _type_ $ _col_ $ _row_ $ _coef_ ;
   datalines;
max       .             profit                 .
eq        .             napha_l_conv           .
eq        .             napha_i_conv           .
eq        .             heating_oil_conv       .
eq        .             recipe_1               .
eq        .             recipe_2               .
upperbd   .             available              .
.         a_light       profit              -175
.         a_light       napha_l_conv         .035
.         a_light       napha_i_conv         .100
.         a_light       heating_oil_conv     .390
.         a_light       available             110
.         a_heavy       profit              -165
.         a_heavy       napha_l_conv         .030
.         a_heavy       napha_i_conv         .075
.         a_heavy       heating_oil_conv     .300
.         a_heavy       available             165
.         brega         profit              -205
.         brega         napha_l_conv         .045
.         brega         napha_i_conv         .135
.         brega         heating_oil_conv     .430
.         brega         available              80
.         naphthal      napha_l_conv           -1
.         naphthal      recipe_2               .2
.         naphthai      napha_i_conv           -1
.         naphthai      recipe_1               .3
.         heatingo      heating_oil_conv       -1
.         heatingo      recipe_1               .7
.         heatingo      recipe_2               .8
.         jet_1         profit               300
.         jet_1         recipe_1              -1
.         jet_2         profit               300
.         jet_2         recipe_2              -1
.         _rhs_         recipe_1               0
;
```

Because the input order of the model into PROC LP is unimportant, this model can be specified in sparse input in arbitrary row order. Example 3.2 in the section "Examples: LP Procedure" on page 229 demonstrates this.

The dense and sparse forms of model input give you flexibility to generate models using the SAS language. The dense form of the model is solved with the statements

```
proc lp;
run;
```

The sparse form is solved with the statements

```
proc lp sparsedata;
run;
```

Example 3.1 and Example 3.2 in the section "Examples: LP Procedure" on page 229 continue with this problem.

Problem Input

As default, PROC LP uses the most recently created SAS data set as the problem input data set. However, if you want to input the problem from a specific SAS data set, use the DATA= option. For example, if the previous dense form data set has the name DENSE, the PROC LP statements can be written as

```
proc lp data=dense;
run;
```

Problem Definition Statements

In the previous dense form data set, the _ID_, _TYPE_, and _RHS_ variables are special variables in PROC LP. They stand for id variable, type variable, and right-hand-side variable. If you replace those variable names with, for example, ROWNAME, TYPE, and RHS, you need the problem definition statements (ID, TYPE and RHS) in PROC LP:

```
proc lp;
   id rowname;
   type type;
   rhs rhs;
run;
```

Other special variables for the dense format are _RHSSEN_ and _RANGE_, which identify the vectors for the right-hand-side sensitivity and range analyses. The corresponding statements are the RHSSEN and RANGE statements. (Notice that a variable name can be identical to a statement name.)

In the same way, if you replace the variables _COL_, _ROW_, _TYPE_, and _COEF_ in the previous sparse form data set by COLUMN, ROW, TYPE, and COEF, you need the problem definition statements (COL, ROW, TYPE, and COEF) in PROC LP.

```
proc lp sparsedata;
   col column;
   row row;
   type type;
   coef coef;
run;
```

In the sparse form data set, the value '_RHS_' under the variable _COL_ is a special column name, which represents the model's right-hand-side column. If you replace it by a value 'R', the PROC LP statements would be

```
proc lp sparsedata;
   rhs r;
run;
```

Other special column names for the sparse format are '_RHSSEN_' and '_RANGE_'. The corresponding statements are the RHSSEN and RANGE statements.

PROC LP is case insensitive to variable names and all character values, including the row and column names in the sparse format. The order of the problem definition statements is not important.

For the dense format, a model's row names appear as character values in a SAS data set. For the sparse format, both the row and the column names of the model appear as character values in the data set. Thus, you can put spaces or other special characters in the names. When referring to these names in the problem definition statement or other LP statements, you must use single or double quotes around them. For example, if you replace '_RHS_' by 'R H S' in the previous sparse form data set, the PROC LP statements would become

```
proc lp sparsedata;
   rhs "r h s";
run;
```

LP Options

The specifications SPARSEDATA and DATA= in the previous examples are PROC LP options. PROC LP options include

- data set options
- display control options
- interactive control options
- preprocessing options
- branch-and-bound control options
- sensitivity/parametric/ranging control options
- simplex options

Interactive Processing

Interactive control options include READPAUSE, ENDPAUSE, and so forth. You can run PROC LP interactively using those options. For example, for the blending problem example in the dense form, you can first pause the procedure before iterations start with the READPAUSE option. The PROC LP statements are

```
proc lp readpause;
run;
```

When the procedure pauses, you run the PRINT statement to display the initial technological matrix and see if the input is correct. Then you run the PIVOT statement to do one simplex pivot and pause. After that you use the SHOW statement to check the current solution status. Then you apply the RESET statement to tell the procedure to stop as soon as it finds a solution. Now you use the RUN statement to continue the execution. When the procedure stops, you run the PRINT statement again to do a price range analysis and QUIT the procedure. Use a SAS %PUT statement to display the contents of PROC LP's macro variable, _ORLP_, which contains iterations and solution information. What follows are the complete statements in batch mode:

```
proc lp readpause;
run;
print matrix(,); /* display all rows and columns. */
pivot;
show status;
reset endpause;
run;
print rangeprice;
quit;
%put &_orlp_;
```

Note: You can force PROC LP to pause during iterations by using the CTRL-BREAK key.

An Integer Programming Example

The following is a simple mixed-integer programming problem. Details can be found in Example 3.8 in the section "Examples: LP Procedure" on page 229.

```
data;
   format _row_ $10.;
   input _row_ $ choco gumdr ichoco igumdr _type_ $ _rhs_;
   datalines;
object         .25     .75   -100     -75 max         .
cooking         15      40      0       0 le      27000
color            0   56.25      0       0 le      27000
package      18.75       0      0       0 le      27000
condiments      12      50      0       0 le      27000
chocolate        1       0 -10000       0 le          0
```

```
gum             0       1       0 -10000 le          0
only_one        0       0       1       1 eq         1
binary          .       .       1       2 binary     .
;
```

The row with 'binary' type indicates that this problem is a mixed-integer program and all the integer variables are binary. The integer values of the row set an ordering for PROC LP to pick the branching variable when VARSELECT=PRIOR is chosen. Smaller values will have higher priorities. The _ROW_ variable here is an alias of the _ID_ variable.

This problem can be solved with the following statements:

```
proc lp canselect=lifo backtrack=obj varselect=far endpause;
run;
quit;
%put &_orlp_;
```

The options CANSELECT=, BACKTRACK=, and VARSELECT= specify the rules for picking the next active problem and the rule to choose the branching variable. In this example, the values LIFO, OBJ and FAR serve as the default values, so the three options can be omitted from the PROC LP statement. The following is the output from the %PUT statement:

```
STATUS=SUCCESSFUL PHASE=3 OBJECTIVE=285 P_FEAS=YES D_FEAS=YES INT_ITER=3
     INT_FEAS=2 ACTIVE=0 INT_BEST=285 PHASE1_ITER=0 PHASE2_ITER=5
     PHASE3_ITER=5
```

Figure 3.1. The Output of _ORLP_

Preprocessing

Using the PREPROCESS= option, you can apply the preprocessing techniques to pre-solve and then solve the preceding mixed-integer program:

```
proc lp preprocess=1 endpause;
run;
quit;
%put &_orlp_;
```

The preprocessing statistics are written to the SAS log file as follows:

```
NOTE: Preprocessing 1 ...
NOTE:     2 upper bounds decreased.
NOTE:     2 coefficients reduced.
NOTE: Preprocessing 2 ...
NOTE:     2 constraints eliminated.
NOTE: Preprocessing done.
```

The new output _ORLP_ is as follows:

```
STATUS=SUCCESSFUL PHASE=3 OBJECTIVE=285 P_FEAS=YES D_FEAS=YES INT_ITER=0
      INT_FEAS=1 ACTIVE=0 INT_BEST=285 PHASE1_ITER=0 PHASE2_ITER=5
      PHASE3_ITER=0
```

Figure 3.2. The Output of _ORLP_ with Preprocessing Option On

In this example, the number of integer iterations (INT_ITER=) is zero, which means that the preprocessing has reduced the gap between the relaxed linear problem and the mixed-integer program to zero.

An MPS Format Conversion Example

If your model input is in MPS input format, you can convert it to the sparse input format of PROC LP using the SAS macro function SASMPSXS. For example, if your have an MPS file called MODEL.MPS and it is stored in the directory C:\OR on a PC, the following program can help you to convert the file and solve the problem.

```
%sasmpsxs(mpsfile="c:\or\model.mps",lpdata=lp);

data;
   set lp;
   retain i 1;
   if _type_="FREE" and i=1 then
      do;
        _type_="MIN";
        i=0;
      end;
run;

proc lp sparsedata;
run;
```

In the MPS input format, all objective functions, price change rows, and free rows have the type ‘N’. The SASMPSXS macro marks them as ‘FREE’ rows. After the conversion, you must run a DATA step to identify the objective rows and price change rows. In this example, assume that the problem is one of minimization and the first ‘FREE’ row is an objective row.

Syntax: LP Procedure

Below are statements used in PROC LP, listed in alphabetical order as they appear in the text that follows.

PROC LP *options* ;
COEF *variables* ;
COL *variable* ;
ID *variable(s)* ;
IPIVOT;
PIVOT;
PRINT *options* ;
QUIT *options* ;
RANGE *variable* ;
RESET *options* ;
RHS *variables* ;
RHSSEN *variables* ;
ROW *variable(s)* ;
RUN;
SHOW *options* ;
TYPE *variable* ;
VAR *variables* ;

The TYPE, ID (or ROW), VAR, RHS, RHSSEN, and RANGE statements are used for identifying variables in the problem data set when the model is in the dense input format. In the dense input format, a model's variables appear as variables in the problem data set. The TYPE, ID (or ROW), and RHS statements can be omitted if the input data set contains variables _TYPE_, _ID_ (or _ROW_), and _RHS_; otherwise, they must be used. The VAR statement is optional. When it is omitted, PROC LP treats all numeric variables that are not explicitly or implicitly included in RHS, RHSSEN, and RANGE statements as structural variables. The RHSSEN and RANGE statements are optional statements for sensitivity and range analyses. They can be omitted if the input data set contains the _RHSSEN_ and _RANGE_ variables.

The TYPE, COL, ROW (or ID), COEF, RHS, RHSSEN, and RANGE statements are used for identifying variables in the problem data set when the model is in the sparse input format. In the sparse input format, a model's rows and columns appear as observations in the problem data set. The TYPE, COL, ROW (or ID), and COEF statements can be omitted if the input data set contains the _TYPE_ and _COL_ variables, as well as variables beginning with the prefixes _ROW (or _ID) and _COEF. Otherwise, they must be used. The RHS, RHSSEN, and RANGE statements identify the corresponding columns in the model. These statements can be omitted if there are observations that contain the RHS, RHSSEN, and RANGE types or the _RHS_, _RHSSEN_, and _RANGE_ column values.

The SHOW, RESET, PRINT, QUIT, PIVOT, IPIVOT, and RUN statements are especially useful when executing PROC LP interactively. However, they can also be used in batch mode.

Functional Summary

The statements and options available with PROC LP are summarized by purpose in the following table.

Table 3.1. Functional Summary

Description	Statement	Option
Interactive Statements:		
perform one integer pivot and pause	IPIVOT	
perform one simplex pivot and pause	PIVOT	
display information at current iteration	PRINT	
terminate processing immediately	QUIT	
reset options specified	RESET	
start or resume optimization	RUN	
show settings of options	SHOW	
Variable Lists:		
variables containing coefficients (sparse)	COEF	
variable containing column names (sparse)	COL	
alias for the ROW statement	ID	
variable (column) containing the range constant for the dense (sparse) format	RANGE	
variables (columns) containing RHS constants for the dense (sparse) format	RHS	
variables (columns) defining RHS change vectors for the dense (sparse) format	RHSSEN	
variable containing names of constraints and objective functions (names of rows) for the dense (sparse) format	ROW	
variable containing the type of each observation	TYPE	
structural variables (dense)	VAR	
Data Set Options:		
active nodes input data set	PROC LP	ACTIVEIN=
active nodes output data set	PROC LP	ACTIVEOUT=
input data set	PROC LP	DATA=
dual output data set	PROC LP	DUALOUT=
primal input data set	PROC LP	PRIMALIN=
primal output data set	PROC LP	PRIMALOUT=
sparse format data input flag	PROC LP	SPARSEDATA
tableau output data set	PROC LP	TABLEAUOUT=
Display Control Options:		
display iteration log	PROC LP	FLOW
nonzero tolerance displaying	PROC LP	FUZZ=
inverse of FLOW option	PROC LP	NOFLOW

Description	Statement	Option
inverse of PARAPRINT option	PROC LP	NOPARAPRINT
omit some displaying	PROC LP	NOPRINT
inverse of TABLEAUPRINT	PROC LP	NOTABLEAUPRINT
parametric programming displaying	PROC LP	PARAPRINT
inverse of NOPRINT	PROC LP	PRINT
iteration frequency of display	PROC LP	PRINTFREQ=
level of display desired	PROC LP	PRINTLEVEL=
display the final tableau	PROC LP	TABLEAUPRINT
Interactive Control Options:		
pause before displaying the solution	PROC LP	ENDPAUSE
pause after first feasible solution	PROC LP	FEASIBLEPAUSE
pause frequency of integer solutions	PROC LP	IFEASIBLEPAUSE=
pause frequency of integer iterations	PROC LP	IPAUSE=
inverse of ENDPAUSE	PROC LP	NOENDPAUSE
inverse of FEASIBLEPAUSE	PROC LP	NOFEASIBLEPAUSE
pause frequency of iterations	PROC LP	PAUSE=
pause if within specified proximity	PROC LP	PROXIMITYPAUSE=
pause after data is read	PROC LP	READPAUSE
Preprocessing Options:		
do not perform preprocessing	PROC LP	NOPREPROCESS
preprocessing error tolerance	PROC LP	PEPSILON=
limit preprocessing iterations	PROC LP	PMAXIT=
perform preprocessing techniques	PROC LP	PREPROCESS
Branch-and-Bound (BB) Control Options:		
perform automatic node selection technique	PROC LP	AUTO
backtrack strategy to be used	PROC LP	BACKTRACK=
branch on binary variables first	PROC LP	BINFST
active node selection strategy	PROC LP	CANSELECT=
comprehensive node selection control parameter	PROC LP	CONTROL=
backtrack related technique	PROC LP	DELTAIT=
measure for pruning BB tree	PROC LP	DOBJECTIVE=
integer tolerance	PROC LP	IEPSILON=
limit integer iterations	PROC LP	IMAXIT=
measure for pruning BB tree	PROC LP	IOBJECTIVE=
order of two branched nodes in adding to BB tree	PROC LP	LIFOTYPE=
inverse of AUTO	PROC LP	NOAUTO
inverse of BINFST	PROC LP	NOBINFST
inverse of POSTPROCESS	PROC LP	NOPOSTPROCESS
limit number of branching variables	PROC LP	PENALTYDEPTH=
measure for pruning BB tree	PROC LP	POBJECTIVE=
perform variables fixing technique	PROC LP	POSTPROCESS
percentage used in updating WOBJECTIVE	PROC LP	PWOBJECTIVE=

Description	Statement	Option
compression algorithm for storing active nodes	PROC LP	TREETYPE=
branching variable selection strategy	PROC LP	VARSELECT=
delay examination of some active nodes	PROC LP	WOBJECTIVE=
Sensitivity/Parametric/Ranging Control Options:		
inverse of RANGEPRICE	PROC LP	NORANGEPRICE
inverse of RANGERHS	PROC LP	NORANGERHS
limit perturbation of the price vector	PROC LP	PRICEPHI=
range analysis on the price coefficients	PROC LP	RANGEPRICE
range analysis on the RHS vector	PROC LP	RANGERHS
limit perturbation of the RHS vector	PROC LP	RHSPHI=
Simplex Algorithm Control Options:		
use devex method	PROC LP	DEVEX
general error tolerance	PROC LP	EPSILON=
perform goal programming	PROC LP	GOALPROGRAM
largest number used in computation	PROC LP	INFINITY=
reinversion frequency	PROC LP	INVFREQ=
reinversion tolerance	PROC LP	INVTOL=
simultaneously set MAXIT1, MAXIT2, MAXIT3 and IMAXIT values	PROC LP	MAXIT=
limit phase 1 iterations	PROC LP	MAXIT1=
limit phase 2 iterations	PROC LP	MAXIT2=
limit phase 3 iterations	PROC LP	MAXIT3=
inverse of devex	PROC LP	NODEVEX
restore basis after parametric programming	PROC LP	PARARESTORE
weight of the phase 2 objective function in phase 1	PROC LP	PHASEMIX=
multiple pricing strategy	PROC LP	PRICETYPE=
number of columns to subset in multiple pricing	PROC LP	PRICE=
limit the number of iterations randomly selecting each entering variable during phase 1	PROC LP	RANDOMPRICEMULT=
zero tolerance in ratio test	PROC LP	REPSILON=
scaling type to be performed	PROC LP	SCALE=
zero tolerance in LU decomposition	PROC LP	SMALL=
time pause limit	PROC LP	TIME=
control pivoting during LU decomposition	PROC LP	U=
RESET Statement Options:		
The RESET statement supports the same options as the PROC LP statement except for the DATA=, PRIMALIN=, and ACTIVEIN= options, and supports the following additional options:		
new variable lower bound during phase 3	RESET	LOWER=
new variable upper bound during phase 3	RESET	UPPER=

Description	Statement	Option
PRINT Statement Options:		
display the best integer solution	PRINT	BEST
display variable summary for specified columns	PRINT	COLUMN
display variable summary and price sensitivity analysis for specified columns	PRINT	COLUMN / SENSITIVITY
display variable summary for integer variables	PRINT	INTEGER
display variable summary for nonzero integer variables	PRINT	INTEGER_NONZEROS
display variable summary for integer variables with zero activity	PRINT	INTEGER_ZEROS
display submatrix for specified rows and columns	PRINT	MATRIX
display formatted submatrix for specified rows and columns	PRINT	MATRIX / PICTURE
display variable summary for continuous variables	PRINT	NONINTEGER
display variable summary for nonzero continuous variables	PRINT	NONINTEGER_NONZEROS
display variable summary for variables with nonzero activity	PRINT	NONZEROS
display price sensitivity analysis or price parametric programming	PRINT	PRICESEN
display price range analysis	PRINT	RANGEPRICE
display RHS range analysis	PRINT	RANGERHS
display RHS sensitivity analysis or RHS parametric programming	PRINT	RHSSEN
display constraint summary for specified rows	PRINT	ROW
display constraint summary and RHS sensitivity analysis for specified rows	PRINT	ROW / SENSITIVITY
display solution, variable, and constraint summaries	PRINT	SOLUTION
display current tableau	PRINT	TABLEAU
display variables with zero activity	PRINT	ZEROS
SHOW Statement Options:		
display options applied	SHOW	OPTIONS
display status of the current solution	SHOW	STATUS
QUIT Statement Option:		
save the defined output data sets and then terminate PROC LP	QUIT	/ SAVE

PROC LP Statement

PROC LP *options* **;**

This statement invokes the procedure. The following options can appear in the PROC LP statement.

Data Set Options

ACTIVEIN=*SAS-data-set*
names the SAS data set containing the active nodes in a branch-and-bound tree that is to be used to restart an integer program.

ACTIVEOUT=*SAS-data-set*
names the SAS data set in which to save the current branch-and-bound tree of active nodes.

DATA=*SAS-data-set*
names the SAS data set containing the problem data. If the DATA= option is not specified, PROC LP uses the most recently created SAS data set.

DUALOUT=*SAS-data-set*
names the SAS data set that contains the current dual solution (shadow prices) on termination of PROC LP. This data set contains the current dual solution only if PROC LP terminates successfully.

PRIMALIN=*SAS-data-set*
names the SAS data set that contains a feasible solution to the problem defined by the DATA= data set. The data set specified in the PRIMALIN= option should have the same format as a data set saved using the PRIMALOUT= option. Specifying the PRIMALIN= option is particularly useful for continuing iteration on a problem previously attempted. It is also useful for performing sensitivity analysis on a previously solved problem.

PRIMALOUT=*SAS-data-set*
names the SAS data set that contains the current primal solution when PROC LP terminates.

SPARSEDATA
tells PROC LP that the data are in the sparse input format. If this option is not specified, PROC LP assumes that the data are in the dense input format. See the section "Sparse Data Input Format" on page 196 for information about the sparse input format.

TABLEAUOUT=*SAS-data-set*
names the SAS data set in which to save the final tableau.

Display Control Options

FLOW
requests that a journal of pivot information (the Iteration Log) be displayed after every PRINTFREQ= iterations. This includes the names of the variables entering and

leaving the basis, the reduced cost of the entering variable, and the current objective value.

FUZZ=*e*
displays all numbers within *e* of zero as zeros. The default value is 1.0E−10.

NOFLOW
is the inverse of the FLOW option.

NOPARAPRINT
is the inverse of the PARAPRINT option.

NOPRINT
suppresses the display of the Variable, Constraint, and Sensitivity Analysis summaries. This option is equivalent to the PRINTLEVEL=0 option.

NOTABLEAUPRINT
is the inverse of the TABLEAUPRINT option.

PARAPRINT
indicates that the solution be displayed at each pivot when performing parametric programming.

PRINT
is the inverse of the NOPRINT option.

PRINTFREQ=*m*
indicates that after every *m*th iteration, a line in the (Integer) Iteration Log be displayed. The default value is 1.

PRINTLEVEL=*i*
indicates the amount of displaying that the procedure should perform.

PRINTLEVEL=-2	only messages to the SAS log are displayed
PRINTLEVEL=-1	is equivalent to NOPRINT unless the problem is infeasible. If it is infeasible, the infeasible rows are displayed in the Constraint Summary along with the Infeasible Information Summary.
PRINTLEVEL=0	is identical to NOPRINT
PRINTLEVEL=1	all output is displayed

The default value is 1.

TABLEAUPRINT
indicates that the final tableau be displayed.

Interactive Control Options

ENDPAUSE
requests that PROC LP pause before displaying the solution. When this pause occurs, you can enter the RESET, SHOW, PRINT, RUN, and QUIT statements.

FEASIBLEPAUSE
requests that PROC LP pause after a feasible (not necessarily integer feasible) solution has been found. At a pause, you can enter the RESET, SHOW, PRINT, PIVOT, RUN, and QUIT statements.

IFEASIBLEPAUSE=*n*
requests that PROC LP pause after every *n* integer feasible solutions. At a pause, you can enter the RESET, SHOW, PRINT, IPIVOT, PIVOT, RUN, and QUIT statements. The default value is 99999999.

IPAUSE=*n*
requests that PROC LP pause after every *n* integer iterations. At a pause, you can enter RESET, SHOW, PRINT, IPIVOT, PIVOT, RUN, and QUIT statements. The default value is 99999999.

NOENDPAUSE
is the inverse of the ENDPAUSE option.

NOFEASIBLEPAUSE
is the inverse of the FEASIBLEPAUSE option.

PAUSE=*n*
requests that PROC LP pause after every *n* iterations. At a pause, you can enter the RESET, SHOW, PRINT, IPIVOT, PIVOT, RUN, and QUIT statements. The default value is 99999999.

PROXIMITYPAUSE=*r*
causes the procedure to pause if at least one integer feasible solution has been found and the objective value of the current best integer solution can be determined to be within *r* units of the optimal integer solution. This distance, called proximity, is also displayed on the Integer Iteration Log. Note that the proximity is calculated using the minimum (maximum if the problem is maximization) objective value among the nodes that remain to be explored in the branch-and-bound tree as a bound on the value of the optimal integer solution. Following the first PROXIMITYPAUSE= pause, in order to avoid a pause at every iteration thereafter, it is recommended that you reduce this measure through the use of a RESET statement. Otherwise, if any other option or statement that causes the procedure to pause is used while the PROXIMITYPAUSE= option is in effect, pause interferences may occur. When this pause occurs, you can enter the RESET, SHOW, PRINT, IPIVOT, PIVOT, RUN, and QUIT statements. The default value is 0.

READPAUSE
requests that PROC LP pause after the data have been read and the initial basis inverted. When this pause occurs, you can enter the RESET, SHOW, PRINT, IPIVOT, PIVOT, RUN, and QUIT statements.

Preprocessing Control Options

NOPREPROCESS
is the inverse of the PREPROCESS option.

PEPSILON=*e*

specifies a positive number close to zero. This value is an error tolerance in the preprocessing. If the value is too small, any marginal changes may cause the preprocessing to repeat itself. However, if the value is too large, it may alter the optimal solution or falsely claim that the problem is infeasible. The default value is $1.0\text{E}{-8}$.

PMAXIT=*n*

performs at most *n* preprocessings. Preprocessing repeats itself if it improves some bounds or fixes some variables. However when a problem is large and dense, each preprocessing may take a significant amount of CPU time. This option limits the number of preprocessings PROC LP performs. It can also reduce the build-up of round-off errors. The default value is 100.

PREPROCESS

performs preprocessing techniques. See the section "Preprocessing" on page 204 for further discussion.

Branch-and-Bound Algorithm Control Options

AUTO, AUTO(*m,n***)**

automatically sets and adjusts the value of the CONTROL= option. Initially, it sets CONTROL=0.70, concentrating on finding an integer feasible solution or an upper bound. When an upper bound is found, it sets CONTROL=0.5, concentrating on efficiency and lower bound improvement. When the number of active problems exceeds *m*, it starts to gradually increase the value of the CONTROL= option to keep the size of active problems under control. When total active problems exceed *n*, CONTROL=1 will keep the active problems from growing further. You can alter the automatic process by resetting the value of the CONTROL= option interactively.

The default values of *m* and *n* are 20000 and 250000, respectively. You can change the two values according to your computer's space and memory capacities.

BACKTRACK=*rule*

specifies the rule used to choose the next active problem when backtracking is required. One of the following can be specified:

- BACKTRACK=*LIFO*
- BACKTRACK=*FIFO*
- BACKTRACK=*OBJ*
- BACKTRACK=*PROJECT*
- BACKTRACK=*PSEUDOC*
- BACKTRACK=*ERROR*

The default value is *OBJ*. See the section "Integer Programming" on page 205 for further discussion.

BINFST

requests that PROC LP branch on binary variables first when integer and binary variables are present. The reasoning behind this is that a subproblem will usually be fathomed or found integer feasible after less than 20% of its variables have been fixed.

Considering binary variables first attempts to reduce the size of the branch-and-bound tree. It is a heuristic technique.

CANSELECT=*rule*
specifies the rule used to choose the next active problem when backtracking is not required or used. One of the following can be specified:

- CANSELECT=*LIFO*
- CANSELECT=*FIFO*
- CANSELECT=*OBJ*
- CANSELECT=*PROJECT*
- CANSELECT=*PSEUDOC*
- CANSELECT=*ERROR*

The default value is *LIFO*. See the section "Integer Programming" on page 205 for further discussion.

CONTROL=*r*
specifies a number between 0 and 1. This option combines CANSELECT= and other rules to choose the next active problem. It takes into consideration three factors: efficiency, improving lower bounds, and improving upper bounds. When *r* is close to 0, PROC LP concentrates on improving lower bounds (upper bounds for maximization). However, the efficiency per integer iteration is usually the worst. When *r* is close to 1, PROC LP concentrates on improving upper bounds (lower bounds for maximization). In addition, the growth of active problems will be controlled and stopped at $r = 1$. When its value is around 0.5, PROC LP will be in the most efficient state in terms of CPU time and integer number of iterations. The CONTROL= option will be automatically adjusted when the AUTO option is applied.

DELTAIT=*r*
is used to modify the exploration of the branch-and-bound tree. If more than *r* integer iterations have occurred since the last integer solution was found, then the procedure uses the backtrack strategy in choosing the next node to be explored. The default value is 3 times the number of integer variables.

DOBJECTIVE=*r*
specifies that PROC LP should discard active nodes that cannot lead to an integer solution with the objective at least as small (or as large for maximizations) as the objective of the relaxed problem plus (minus) *r*. The default value is $+\infty$.

IEPSILON=*e*
requests that PROC LP consider an integer variable as having an integer value if its value is within *e* units of an integer. The default value is $1.0\text{E}{-7}$.

IMAXIT=*n*
performs at most *n* integer iterations. The default value is 100.

IOBJECTIVE=*r*
specifies that PROC LP should discard active nodes unless the node could lead to an

integer solution with the objective smaller (or larger for maximizations) than *r*. The default value is $+\infty$ for minimization ($-\infty$ for maximization).

LIFOTYPE=*c*
specifies the order in which to add the two newly branched active nodes to the LIFO list.

LIFOTYPE=0	add the node with minimum penalty first
LIFOTYPE=1	add the node with maximum penalty first
LIFOTYPE=2	add the node resulting from adding $x_i \geq \lceil x^{opt}(k)_i \rceil$ first
LIFOTYPE=3	add the node resulting from adding $x_i \leq \lfloor x^{opt}(k)_i \rfloor$ first

The default value is 0.

NOAUTO
is the inverse of the AUTO option.

NOBINFST
is the inverse of the BINFST option.

NOPOSTPROCESS
is the inverse of the POSTPROCESS option.

PENALTYDEPTH=*m*
requests that PROC LP examine *m* variables as branching candidates when VARSELECT=PENALTY. If the PENALTYDEPTH= option is not specified when VARSELECT=PENALTY, then all of the variables are considered branching candidates. The default value is the number of integer variables. See the section "Integer Programming" on page 205 for further discussion.

POBJECTIVE=*r*
specifies that PROC LP should discard active nodes that cannot lead to an integer solution with objective at least as small as $o + |o| \times r$ (at least as large as $o - |o| \times r$ for maximizations) where o is the objective of the relaxed noninteger constrained problem. The default value is $+\infty$.

POSTPROCESS
attempts to fix binary variables globally based on the relationships among the reduced cost and objective value of the relaxed problem and the objective value of the current best integer feasible solution.

PWOBJECTIVE=*r*
specifies a percentage for use in the automatic update of the WOBJECTIVE= option. If the WOBJECTIVE= option is not specified in PROC LP, then when an integer feasible solution is found, the value of the option is updated to be $b + q \times r$ where b is the best bound on the value of the optimal integer solution and q is the current proximity. Note that for maximizations, $b - q \times r$ is used. The default value is 0.95.

TREETYPE=*i*

specifies a data compression algorithm.

TREETYPE=0	no data compression
TREETYPE=1	Huffman coding compression routines
TREETYPE=2	adaptive Huffman coding compression routines
TREETYPE=3	adaptive arithmetic coding compression routines

For IP or MIP problems, the basis and bounds information of each active node is saved to a utility file. When the number of active nodes increases, the size of the utility file becomes larger and larger. If PROC LP runs into a disk problem, like "disk full ..." or "writing failure ...", you can use this option to compress the utility file. For more information on the data compression routines, refer to Nelson (1992). The default value is 0.

VARSELECT=*rule*

specifies the rule used to choose the branching variable on an integer iteration.

- VARSELECT=*CLOSE*
- VARSELECT=*PRIOR*
- VARSELECT=*PSEUDOC*
- VARSELECT=*FAR*
- VARSELECT=*PRICE*
- VARSELECT=*PENALTY*

The default value is FAR. See the section "Integer Programming" on page 205 for further discussion.

WOBJECTIVE=*r*

specifies that PROC LP should delay examination of active nodes that cannot lead to an integer solution with objective at least as small (as large for maximizations) as *r*, until all other active nodes have been explored. The default value is $+\infty$ for minimization ($-\infty$ for maximization).

Sensitivity/Parametric/Ranging Control Options

NORANGEPRICE

is the inverse of the RANGEPRICE option.

NORANGERHS

is the inverse of the RANGERHS option.

PRICEPHI=Φ

specifies the limit for parametric programming when perturbing the price vector. See the section "Parametric Programming" on page 216 for further discussion. See Example 3.5 for an illustration of this option.

RANGEPRICE
indicates that range analysis is to be performed on the price coefficients. See the section "Range Analysis" on page 216 for further discussion.

RANGERHS
indicates that range analysis is to be performed on the right-hand-side vector. See the section "Range Analysis" on page 216 for further discussion.

RHSPHI=Φ
specifies the limit for parametric programming when perturbing the right-hand-side vector. See the section "Parametric Programming" on page 216 for further discussion.

Simplex Algorithm Control Options

DEVEX
indicates that the devex method of weighting the reduced costs be used in pricing (Harris 1975).

EPSILON=*e*
specifies a positive number close to zero. It is used in the following instances:

During phase 1, if the sum of the basic artificial variables is within *e* of zero, the current solution is considered feasible. If this sum is not exactly zero, then there are artificial variables within *e* of zero in the current solution. In this case, a note is displayed on the SAS log.

During phase 1, if all reduced costs are $\leq$ *e* for nonbasic variables at their lower bounds and $\geq$ *e* for nonbasic variables at their upper bounds and the sum of infeasibilities is greater than *e*, then the problem is considered infeasible. If the maximum reduced cost is within *e* of zero, a note is displayed on the SAS log.

During phase 2, if all reduced costs are $\leq$ *e* for nonbasic variables at their lower bounds and $\geq$ *e* for nonbasic variables at their upper bounds, then the current solution is considered optimal.

During phases 1, 2, and 3, the EPSILON= option is also used to test if the denominator is different from zero before performing the ratio test to determine which basic variable should leave the basis.

The default value is $1.0\text{E}{-8}$.

GOALPROGRAM
specifies that multiple objectives in the input data set are to be treated as sequential objectives in a goal-programming model. The value of the right-hand-side variable in the objective row gives the priority of the objective. Lower numbers have higher priority.

INFINITY=*r*
specifies the largest number PROC LP uses in computation. The INFINITY= option is used to determine when a problem has an unbounded variable value. The default value is the largest double precision number. *

*This value is system dependent.

INVFREQ=*m*
reinverts the current basis matrix after *m* major and minor iterations. The default value is 100.

INVTOL=*r*
reinverts the current basis matrix if the largest element in absolute value in the decomposed basis matrix is greater than *r*. If after reinversion this condition still holds, then the value of the INVTOL= option is increased by a factor of 10 and a note indicating this modification is displayed on the SAS log. When *r* is frequently exceeded, this may be an indication of a numerically unstable problem. The default value is 1000.

MAXIT=*n*
simultaneously sets the values of the MAXIT1=, MAXIT2=, MAXIT3=, and IMAXIT= options.

MAXIT1=*n*
performs at most $n \geq 0$ phase 1 iterations. The default value is 100.

MAXIT2=*n*
performs at most $n \geq 0$ phase 2 iterations. If MAXIT2=0, then only phase 1 is entered so that on successful termination PROC LP will have found a feasible, but not necessarily optimal, solution. The default value is 100.

MAXIT3=*n*
performs at most $n \geq 0$ phase 3 iterations. All dual pivots are counted as phase 3 pivots. The default value is 99999999.

NODEVEX
is the inverse of the DEVEX option.

PARARESTORE
indicates that following a parametric programming analysis, PROC LP should restore the basis.

PHASEMIX=*r*
specifies a number between 0 and 1. When the number is positive, PROC LP tries to improve the objective function of phase 2 during phase 1. The PHASEMIX= option is a weight factor of the phase 2 objective function in phase 1. The default value is 0.

PRICE=*m*
specifies the number of columns to subset when multiple pricing is used in selecting the column to enter the basis (Greenberg 1978). The type of suboptimization used is determined by the PRICETYPE= option. See the section "Pricing" on page 203 for a description of this process.

PRICETYPE=*pricetype*
specifies the type of multiple pricing to be performed. If this option is specified and the PRICE= option is not specified, then PRICE= is assumed to be 10. Valid values for the PRICETYPE= option are

- PRICETYPE=*COMPLETE*

- PRICETYPE=*DYNAMIC*
- PRICETYPE=*NONE*
- PRICETYPE=*PARTIAL*

The default value is PARTIAL. See the section "Pricing" on page 203 for a description of this process.

RANDOMPRICEMULT=*r*
specifies a number between 0 and 1. This option sets a limit, in phase 1, on the number of iterations when PROC LP will randomly pick the entering variables. The limit equals *r* times the number of nonbasic variables, or the number of basic variables, whichever is smaller. The default value of the RANDOMPRICEMULT= option is 0.01.

REPSILON=*e*
specifies a positive number close to zero. The REPSILON= option is used in the ratio test to determine which basic variable is to leave the basis. The default value is $1.0\text{E}-10$.

SCALE=*scale*
specifies the type of scaling to be used. Valid values for the SCALE= option are

- SCALE=*BOTH*
- SCALE=*COLUMN*
- SCALE=*NONE*
- SCALE=*ROW*

The default value is BOTH. See the section "Scaling" on page 204 for further discussion.

SMALL=*e*
specifies a positive number close to zero. Any element in a matrix with a value less than *e* is set to zero. The default value is machine dependent.

TIME=*t*
checks at each iteration to see if *t* seconds have elapsed since PROC LP began. If more than *t* seconds have elapsed, the procedure pauses and displays the current solution. The default value is 120 seconds.

U=*r*
enables PROC LP to control the choice of pivots during LU decomposition and updating the basis matrix. The variable *r* should take values between EPSILON and 1.0 because small values of *r* bias the algorithm toward maintaining sparsity at the expense of numerical stability and vice versa. The more sparse the decomposed basis is, the less time each iteration takes. The default value is 0.1.

COEF Statement

COEF *variables* **;**

For the sparse input format, the COEF statement specifies the numeric variables in the problem data set that contain the coefficients in the model. The value of the coefficient variable in a given observation is the value of the coefficient in the column and row specified in the COLUMN and ROW variables in that observation. For multiple ROW variables, the LP procedure maps the ROW variables to the COEF variables on the basis of their order in the COEF and ROW statements. There must be the same number of COEF variables as ROW variables. If the COEF statement is omitted, the procedure looks for the default variable names that have the prefix _COEF.

COL Statement

COL *variable* **;**

For the sparse input format, the COL statement specifies a character variable in the problem data set that contains the names of the columns in the model. Columns in the model are either structural variables, right-hand-side vectors, right-hand-side change vectors, or a range vector. The COL variable must be a character variable. If the COL statement is omitted, the LP procedure looks for the default variable name _COL_.

ID Statement

ID *variable(s)* **;**

For the dense input format, the ID statement specifies a character variable in the problem data set that contains a name for each constraint coefficients row, objective coefficients row, and variable definition row. If the ID statement is omitted, the LP procedure looks for the default variable name, _ID_. If this variable is not in the problem data set, the procedure assigns the default name _OBSxx_ to each row, where xx specifies the observation number in the problem data set.

For the sparse input format, the ID statement specifies the character variables in the problem data set that contain the names of the rows in the model. Rows in the model are one of the following types: constraints, objective functions, bounding rows, or variable describing rows. The ID variables must be character variables. There must be the same number of ID variables as variables specified in the COEF statement. If the ID statement is omitted, the LP procedure looks for the default variable names having the prefix _ID.

Note: The ID statement is an alias for the ROW statement.

IPIVOT Statement

IPIVOT ;

The IPIVOT statement causes the LP procedure to execute one integer branch-and-bound pivot and pause. If you use the IPIVOT statement while the PROXIMITYPAUSE= option is in effect, pause interferences may occur. To avoid such interferences, you must either reset the PROXIMITYPAUSE value or submit IPIVOT**;** RUN**;** instead of IPIVOT**;**.

PIVOT Statement

PIVOT ;

The PIVOT statement causes the LP procedure to execute one simplex pivot and pause.

PRINT Statement

PRINT *options* **;**

The PRINT statement is useful for displaying part of a solution summary, examining intermediate tableaus, performing sensitivity analysis, and using parametric programming. In the options, the colnames and rownames lists can be empty, in which case the LP procedure displays tables with all columns or rows, or both. If a column name or a row name has spaces or other special characters in it, the name must be enclosed in single or double quotes when it appears in the argument. The options that can be used with this statement are as follows.

BEST
displays a Solution, Variable, and Constraint Summary for the best integer solution found.

COLUMN(*colnames***) / SENSITIVITY**
displays a Variable Summary containing the logical and structural variables listed in the colnames list. If the / SENSITIVITY option is included, then sensitivity analysis is performed on the price coefficients for the listed colnames structural variables.

INTEGER
displays a Variable Summary containing only the integer variables.

INTEGER_NONZEROS
displays a Variable Summary containing only the integer variables with nonzero activity.

INTEGER_ZEROS
displays a Variable Summary containing only the integer variables with zero activity.

MATRIX(*rownames***,***colnames***) / PICTURE**
displays the submatrix of the matrix of constraint coefficients defined by the rownames and colnames lists. If the / PICTURE option is included, then the formatted submatrix is displayed. The format used is summarized in Table 3.2.

Table 3.2. Format Summary

Condition on the Coefficient x						Symbols Printed	
			abs(x)	=	0		“ ”
0		<	abs(x)	<	.000001	sgn(x)	“Z”
	.000001	≤	abs(x)	<	.00001	sgn(x)	“Y”
	.00001	≤	abs(x)	<	.0001	sgn(x)	“X”
	.0001	≤	abs(x)	<	.001	sgn(x)	“W”
	.001	≤	abs(x)	<	.01	sgn(x)	“V”
	.01	≤	abs(x)	<	.1	sgn(x)	“U”
	.1	≤	abs(x)	<	1	sgn(x)	“T”
			abs(x)	=	1	sgn(x)	“1”
1		<	abs(x)	<	10	sgn(x)	“A”
10		≤	abs(x)	<	100	sgn(x)	“B”
100		≤	abs(x)	<	1000	sgn(x)	“C”
1000		≤	abs(x)	<	10000	sgn(x)	“D”
10000		≤	abs(x)	<	100000	sgn(x)	“E”
100000		≤	abs(x)	<	1.0E06	sgn(x)	“F”

NONINTEGER
displays a Variable Summary containing only the continuous variables.

NONINTEGER_NONZEROS
displays a Variable Summary containing only the continuous variables with nonzero activity.

NONZEROS
displays a Variable Summary containing only the variables with nonzero activity.

PRICESEN
displays the results of parametric programming for the current value of the PRICEPHI= option, the price coefficients, and all of the price change vectors.

RANGEPRICE
performs range analysis on the price coefficients.

RANGERHS
performs range analysis on the right-hand-side vector.

RHSSEN
displays the results of parametric programming for the current value of the RHSPHI= option, the right-hand-side coefficients, and all of the right-hand-side change vectors.

ROW(*rownames***) / SENSITIVITY**
displays a constraint summary containing the rows listed in the rowname list. If the / SENSITIVITY option is included, then sensitivity analysis is performed on the right-hand-side coefficients for the listed rownames.

SOLUTION
displays the Solution Summary, including the Variable Summary and the Constraint Summary.

TABLEAU
displays the current tableau.

ZEROS
displays a Variable Summary containing only the variables with zero activity. This may be useful in the analysis of ON/OFF, ZERO/ONE, scheduling, and assignment applications.

QUIT Statement

QUIT *options* **;**

The QUIT statement causes the LP procedure to terminate processing immediately. No further displaying is performed and no output data sets are created.

The QUIT/SAVE statement causes the LP procedure to save the output data sets, defined in the PROC LP statement or in the RESET statement, and then terminate the procedure.

RANGE Statement

RANGE *variable* **;**

For the dense input format, the RANGE statement identifies the variable in the problem data set that contains the range coefficients. These coefficients enable you to specify the feasible range of a row. For example, if the ith row is

$$a^T x \leq b_i$$

and the range coefficient for this row is $r_i > 0$, then all values of x that satisfy

$$b_i - r_i \leq a^T x \leq b_i$$

are feasible for this row. Table 3.3 shows the bounds on a row as a function of the row type and the sign on a nonmissing range coefficient r.

Table 3.3. Interpretation of the Range Coefficient

		Bounds	
r	_TYPE_	Lower	Upper
$\neq 0$	LE	$b - \|r\|$	b
$\neq 0$	GE	b	$b + \|r\|$
> 0	EQ	b	$b + r$
< 0	EQ	$b + r$	b

If you include a range variable in the model and have a missing value or zero for it in a constraint row, then that constraint is treated as if no range variable had been included.

If the RANGE statement is omitted, the LP procedure assumes that the variable named _RANGE_ contains the range coefficients.

For the sparse input format, the RANGE statement gives the name of a column in the problem data set that contains the range constants. If the RANGE statement is omitted, then the LP procedure assumes that the column named _RANGE_ or the column with the 'RANGE' keyword in the problem data set contains the range constants.

RESET Statement

RESET *options* **;**

The RESET statement is used to change options after the LP procedure has started execution. All of the options that can be set in the PROC LP statement can also be reset with the RESET statement, except for the DATA=, the PRIMALIN=, and the ACTIVEIN= options. In addition to the options available with the PROC LP statement, the following two options can be used.

LOWER(*colnames***)=***n***;**
During phase 3, this sets the lower bound on all of the structural variables listed in the colnames list to an integer value *n*. This may contaminate the branch-and-bound tree. All nodes that descend from the current problem have lower bounds that may be different from those input in the problem data set.

UPPER(*colnames***)=***n***;**
During phase 3, this sets the upper bound on all of the structural variables listed in the colnames list to an integer value *n*. This may contaminate the branch-and-bound tree. All nodes that descend from the current problem have upper bounds that may be different from those input in the problem data set.

Note that the LOWER= and UPPER= options only apply to phase 3 for integer problems. Therefore, they should only be applied once the integer iterations have started; if they are applied before then, they will be ignored.

RHS Statement

RHS *variables* **;**

For the dense input format, the RHS statement identifies variables in the problem data set that contain the right-hand-side constants of the linear program. Only numeric variables can be specified. If more than one variable is included in the RHS statement, the LP procedure assumes that problems for several linear programs are defined in the problem data set. A new linear program is defined for each variable in the RHS list. If the RHS statement is omitted, the procedure assumes that a variable named _RHS_ contains the right-hand-side constants.

For the sparse input format, the RHS statement gives the names of one or more columns in the problem data set that are to be considered as right-hand-side constants. If the RHS statement is omitted, then the LP procedure assumes that the column named _RHS_ or columns with the 'RHS' keyword in the problem data set

contain the right-hand-side constants. See the section "Sparse Data Input Format" on page 196 for further discussion.

As default, the LP procedure assumes that the RHS constant is a zero vector for the dense and sparse input formats.

RHSSEN Statement

RHSSEN *variables* **;**

For the dense input format, the RHSSEN statement identifies variables in the problem data set that define change vectors for examining the sensitivity of the optimal solution to changes in the RHS constants. If the RHSSEN statement is omitted, then the LP procedure assumes that a variable named _RHSSEN_ contains a right-hand-side change vector.

For the sparse input format, the RHSSEN statement gives the names of one or more columns in the problem data set that are to be considered as change vectors. If the RHSSEN statement is omitted, then the LP procedure assumes that the column named _RHSSEN_ or columns with the 'RHSSEN' keyword in the problem data set contain the right-hand-side change vectors. For further information, see the section "Sparse Data Input Format" on page 196, the section "Right-Hand-Side Sensitivity Analysis" on page 214, and the section "Right-Hand-Side Parametric Programming" on page 216.

ROW Statement

ROW *variable(s)* **;**

For the dense input format, the ROW statement specifies a character variable in the problem data set that contains a name for each row of constraint coefficients, each row of objective coefficients and each variable describing row. If the ROW statement is omitted, the LP procedure looks for the default variable name, _ROW_. If there is no such variable in the problem data set, the procedure assigns the default name _OBSxx_ to each row, where xx specifies the observation number in the problem data set.

For the sparse input format, the ROW statement specifies the character variables in the problem data set that contain the names of the rows in the model. Rows in the model are one of the following types: constraints, objective functions, bounding rows, or variable describing rows. The ROW variables must be character variables. There must be the same number of ROW variables as variables specified in the COEF statement. If the ROW statement is omitted, the LP procedure looks for the default variable names having the prefix _ROW.

RUN Statement

RUN ;

The RUN statement causes optimization to be started or resumed. The TITLE or OPTIONS statement should not appear between PROC LP and RUN statements.

SHOW Statement

SHOW *options* **;**

The SHOW statement specifies that the LP procedure display either the *current options* or the *current solution status* on the SAS log.

OPTIONS
requests that the current options be displayed on the SAS log.

STATUS
requests that the status of the current solution be displayed on the SAS log.

TYPE Statement

TYPE *variable* **;**

The TYPE statement specifies a character variable in the problem data set that contains the type identifier for each observation. This variable has keyword values that specify how the LP procedure should interpret the observation. If the TYPE statement is omitted, the procedure assumes that a variable named _TYPE_ contains the type keywords.

For the dense input format, the type variable identifies the constraint and objective rows and rows that contain information about the variables. The type variable should have nonmissing values in all observations.

For the sparse input format, the type variable identifies a model's rows and columns. In an observation, a nonmissing type is associated with either a row or a column. If there are many columns sharing the same type, you can define a row of that type. Then, any nonmissing values in that row set the types of the corresponding columns.

The following are valid values for the TYPE variable in an observation:

MIN	contains the price coefficients of an objective row, for example, c in the problem (MIP), to be minimized.
MAX	contains the price coefficients of an objective row, for example, c, to be maximized.
EQ $(=)$	contains coefficients of an equality constrained row.
LE $(\leq)$	contains coefficients of an inequality, less than or equal to, constrained row.
GE $(\geq)$	contains coefficients of an inequality, greater than or equal to, constrained row.

SOSEQ — identifies the row as specifying a special ordered set. The variables flagged in this row are members of a set *exactly one* of which must be above its lower bound in the optimal solution. Note that variables in this type of special ordered set must be *integer*.

SOSLE — identifies the row as specifying a special ordered set. The variables flagged in this row are members of a set in which only one can be above its lower bound in the optimal solution.

UNRSTRT
UNRSTRCT — identifies those structural variables to be considered as unrestricted variables. These are variables for which $\ell_i = -\infty$ and $u_i = +\infty$. Any variable that has a 1 in this observation is considered an unrestricted variable.

LOWERBD — identifies lower bounds on the structural variables. If all structural variables are to be nonnegative, that is, $\ell_i = 0$, then you do not need to include an observation with the 'LOWERBD' keyword in a variable specified in the TYPE statement. Missing values for variables in a lower-bound row indicate that the variable has lower bound equal to zero.

Note: A variable with lower or upper bounds cannot be identified as unrestricted.

UPPERBD — identifies upper bounds u_i on the structural variables. For each structural variable that is to have an upper bound $u_i = +\infty$, the observation must contain a missing value or the current value of INFINITY. All other values are interpreted as upper bounds, including 0.

FIXED — identifies variables that have fixed values. A nonmissing value in a row with 'FIXED' type keyword gives the constant value of that variable.

INTEGER — identifies variables that are integer-constrained. In a feasible solution, these variables must have integer values. A missing value in a row with 'INTEGER' type keyword indicates that the variable is not integer-constrained. The value of variables in the 'INTEGER' row gives an ordering to the integer-constrained variables that is used when the VARSELECT= option equals PRIOR.

Note: Every integer-constrained variable must have an upper bound defined in a row with type 'UPPERBD'. See the section "Controlling the Branch-and-Bound Search" on page 208 for further discussion.

BINARY — identifies variables that are constrained to be either 0 or 1. This is equivalent to specifying that the variable is an integer variable and has a lower bound of 0 and an upper bound of 1. A missing value in a row with 'BINARY' type keyword indicates that the variable is not constrained to be 0 or 1. The value of variables in the 'BINARY' row gives an ordering to the integer-constrained variables that is used when the VARSELECT= option equals PRIOR. See the section "Controlling the Branch-and-Bound Search" on page 208 for further discussion.

BASIC	identifies variables that form an initial basic feasible solution. A missing value in a row with 'BASIC' type indicates that the variable is not basic.
PRICESEN	identifies a vector that is used to evaluate the sensitivity of the optimal solution to changes in the objective function. See the section "Price Sensitivity Analysis" on page 215 and the section "Price Parametric Programming" on page 217 for further discussion.
FREE	identifies a nonbinding constraint. Any number of FREE constraints can appear in a problem data set.
RHS	identifies a right-hand-side column in the sparse input format. This replaces the RHS statement. It is useful when converting the MPS format into the sparse format of PROC LP. See the section "Converting MPS Format" on page 198 for more information.
RHSSEN	identifies a right-hand-side sensitivity analysis vector in the sparse input format. This replaces the RHSSEN statement. It is useful when converting the MPS format into the sparse format of PROC LP. See the section "Converting MPS Format" on page 198 for more information.
RANGE	identifies a range vector in the sparse input format. This replaces the RANGE statement. It is useful when converting the MPS format into the sparse format of PROC LP. See the section "Converting MPS Format" on page 198 for more information.

VAR Statement

VAR *variables* **;**

For the dense input format, the VAR statement identifies variables in the problem data set that are to be interpreted as structural variables in the linear program. Only numeric variables can be specified. If no VAR statement is specified, the LP procedure uses all numeric variables not included in an RHS or RHSSEN statement as structural variables.

Details: LP Procedure

Missing Values

The LP procedure treats missing values as missing in all rows except those that identify either upper or lower bounds on structural variables. If the row is an upper-bound row, then the type identifier is 'UPPERBD' and the LP procedure treats missing values as $+\infty$. If the row is a lower-bound row, then the type identifier is 'LOWERBD' and the LP procedure treats missing values as 0, except for the variables that are identified as 'UNRSTRT'.

Dense Data Input Format

In the dense format, a model is expressed in a similar way as it is formulated. Each SAS variable corresponds to a model's column and each SAS observation corresponds to a model's row. A SAS variable in the input data set is one of the following:

- a type variable
- an id variable
- a structural variable
- a right-hand-side variable
- a right-hand-side sensitivity analysis variable
- a range variable

The type variable tells PROC LP how to interpret the observation as a part of the mathematical programming problem. It identifies and classifies objectives, constraints, and the rows that contain information of variables like types, bounds, and so on. PROC LP recognizes the following keywords as values for the type variable: MIN, MAX, EQ, LE, GE, SOSEQ, SOSLE, UNRSTRT, LOWERBD, UPPERBD, FIXED, INTEGER, BINARY, BASIC, PRICESEN, and FREE. The values of the id variable are the names of the rows in the model. The other variables identify and classify the columns with numerical values.

The TYPE, ID (or ROW), and RHS statements can be omitted if the input data set contains variables _TYPE_, _ID_ (or _ROW_), and _RHS_; otherwise, they must be used. The VAR statement is optional. When it is not specified, PROC LP uses as structural variables all numeric variables not explicitly or implicitly included in statement lists. The RHSSEN and RANGE statements are optional statements for sensitivity and range analyses. They can be omitted if the input data set contains the _RHSSEN_ and _RANGE_ variables.

Sparse Data Input Format

The sparse format to PROC LP is designed to enable you to specify only the nonzero coefficients in the description of linear programs, integer programs, and mixed-integer programs. The SAS data set that describes the sparse model must contain at least four SAS variables:

- a type variable
- a column variable
- a row variable
- a coefficient variable

Each observation in the data set associates a type with a row or column, and defines a coefficient or numerical value in the model. The value of the type variable is a keyword that tells PROC LP how to interpret the observation. In addition to the keywords in the dense format, PROC LP also recognizes the keywords RHS, RHSSEN, and RANGE as values of the type variable. Table 3.4 shows the keywords that are recognized by PROC LP and in which variables can appear in the problem data set.

The values of the row and column variables are the names of the rows and columns in the model. The values of the coefficient variables define basic coefficients and lower and upper bounds, and identify model variables with types BASIC, FIXED, BINARY, and INTEGER. All character values in the sparse data input format are case insensitive.

The SAS data set can contain multiple pairs of rows and coefficient variables. In this way, more information about the model can be specified in each observation in the data set. See Example 3.2 for details.

Table 3.4. Variable Keywords Used in the Problem Data Set

TYPE (_TYPE_)	COL (_COL_)
MIN	
MAX	
EQ	
LE	
GE	
SOSEQ	
SOSLE	
UNRSTRT	
LOWERBD	
UPPERBD	
FIXED	
INTEGER	
BINARY	
BASIC	
PRICESEN	
FREE	
RHS	_RHS_
RHSSEN	_RHSSEN_
RANGE	_RANGE_
∗xxxxxxx	

Follow these rules for sparse data input:

- The order of the observations is unimportant.
- Each unique column name appearing in the COL variable defines a unique column in the model.
- Each unique row name appearing in the ROW variable defines a unique row in the model.
- The type of the row is identified when an observation in which the row name appears (in a ROW variable) has type MIN, MAX, LE, GE, EQ, SOSLE, SOSEQ, LOWERBD, UPPERBD, UNRSTRT, FIXED, BINARY, INTEGER, BASIC, FREE, or PRICESEN.
- The type of each row must be identified at least once. If a row is given a type more than once, the multiple definitions must be identical.
- When there are multiple rows named in an observation (that is, when there are multiple ROW variables), the TYPE variable applies to each row named in the observation.
- The type of a column is identified when an observation in which the column name but no row name appears has the type LOWERBD, UPPERBD, UNRSTRT, FIXED, BINARY, INTEGER, BASIC, RHS, RHSSEN, or RANGE. A column type can also be identified in an observation in which both

column and row names appear and the row name has one of the preceding types.

- Each column is assumed to be a structural column in the model unless the column is identified as a right-hand-side vector, a right-hand-side change vector, or a range vector. A column can be identified as one of these types using either the keywords RHS, RHSSEN, or RANGE in the TYPE variable, the special column names _RHS_, _RHSSEN_, or _RANGE_, or the RHS, RHSSEN, or RANGE statements following the PROC LP statement.
- A TYPE variable beginning with the character * causes the observation to be interpreted as a comment.

When the column names appear in the Variable Summary in the PROC LP output, they are listed in alphabetical order. The row names appear in the order in which they appear in the problem data set.

Converting MPS Format

MPS input format was introduced by IBM. It has been a way of creating inputs for linear and integer programs. SASMPSXS is a SAS macro function that converts the standard MPS format to the sparse format of the LP procedure. The following is an example of the MPS format:

```
NAME          EXAMPLE
* THIS IS DATA FOR THE PRODUCT MIX PROBLEM.
ROWS
 N  PROFIT
 L  STAMP
 L  ASSEMB
 L  FINISH
 N  CHNROW
 N  PRICE
COLUMNS
    DESK      STAMP          3.00000   ASSEMB        10.00000
    DESK      FINISH        10.00000   PROFIT        95.00000
    DESK      PRICE        175.00000
    CHAIR     STAMP          1.50000   ASSEMB         6.00000
    CHAIR     FINISH         8.00000   PROFIT        41.00000
    CHAIR     PRICE         95.00000
    CABINET   STAMP          2.00000   ASSEMB         8.00000
    CABINET   FINISH         8.00000   PROFIT        84.00000
    CABINET   PRICE        145.00000
    BOOKCSE   STAMP          2.00000   ASSEMB         7.00000
    BOOKCSE   FINISH         7.00000   PROFIT        76.00000
    BOOKCSE   PRICE        130.00000   CHNROW         1.00000
RHS
    TIME      STAMP        800.00000   ASSEMB       1200.0000
    TIME      FINISH       800.00000
RANGES
    T1        ASSEMB       900.00000
BOUNDS
```

```
    UP        CHAIR         75.00000
    LO        BOOKCSE       50.00000
ENDATA
```

In this example, the company tries to find an optimal product mix of four items: a DESK, a CHAIR, a CABINET, and a BOOKCASE. Each item is processed in a stamping department (STAMP), an assembly department (ASSEMB), and a finishing department (FINISH). The time each item requires in each department is given in the input data. Because of resource limitations, each department has an upper limit on the time available for processing. Furthermore, because of labor constraints, the assembly department must work at least 300 hours. Finally, marketing tells you not to make more than 75 chairs, to make at least 50 bookcases, and to find the range over which the selling price of a bookcase can vary without changing the optimal product mix.

The SASMPSXS macro function uses MPSFILE='FILENAME' as an argument to read an MPS input file. It then converts the file and saves the conversion to a default SAS data set, PROB. The FILENAME should include the path.

Running the following statements on the preceding example

```
%sasmpsxs(mpsfile='filename');

proc print data=prob;
run;
```

produces the sparse input form of the LP procedure:

OBS	_TYPE_	_COL_	_ROW1_	_COEF1_	_ROW2_	_COEF2_
1	*OW			.		.
2	FREE		PROFIT	.		.
3	LE		STAMP	.		.
4	LE		ASSEMB	.		.
5	LE		FINISH	.		.
6	FREE		CHNROW	.		.
7	FREE		PRICE	.		.
8	*OL	MNS		.		.
9		DESK	STAMP	3.0	ASSEMB	10
10		DESK	FINISH	10.0	PROFIT	95
11		DESK	PRICE	175.0		.
12		CHAIR	STAMP	1.5	ASSEMB	6
13		CHAIR	FINISH	8.0	PROFIT	41
14		CHAIR	PRICE	95.0		.
15		CABINET	STAMP	2.0	ASSEMB	8
16		CABINET	FINISH	8.0	PROFIT	84
17		CABINET	PRICE	145.0		.
18		BOOKCSE	STAMP	2	ASSEMB	7
19		BOOKCSE	FINISH	7	PROFIT	76
20		BOOKCSE	PRICE	130	CHNROW	1
21	*HS			.		.

```
22    RHS         TIME       STAMP       800      ASSEMB     1200
23    RHS         TIME       FINISH      800                    .
24    *AN         ES                       .                     .
25    RANGE       T1         ASSEMB      900                    .
26    *OU         DS                       .                     .
27    UPPERBDD    CHAIR      UP           75                    .
28    LOWERBDD    BOOKCSE    LO           50                    .
```

SASMPSXS recognizes four MPS row types: E, L, G, and N. It converts them into types EQ, LE, GE, and FREE. Since objective rows, price change rows and free rows all share the same type N in the MPS format, you need a DATA step to assign proper types to the objective rows and price change rows.

```
data;
   set prob;
   if _type_='free' and _row1_='profit' then _type_='max';
   if _type_='free' and _row1_='chnrow' then _type_='pricesen';
run;

proc lp sparsedata;
run;
```

In the MPS format, the variable types include LO, UP, FX, FR, MI, and BV. The SASMPSXS macro converts them into types LOWERBD, UPPERBD, FIXED, UNRESTRICTED, -INFINITY, and BINARY, respectively. Occasionally, you may need to define your own variable types, in which case, you must add corresponding type handling entries in the SASMPSXS.SAS program and use the SAS %INCLUDE macro to include the file at the beginning of your program. The SASMPSXS macro function can be found in the SAS sample library. Information on the MPS format can be obtained from Murtagh (1981).

SASMPSXS can take no arguments, or it can take one or two arguments. If no arguments are present, SASMPSXS assumes that the MPS input file has been saved to a SAS data set named RAW. The macro then takes information from that data set and converts it into the sparse form of the LP procedure. The RAW data set should have the following six variables:

```
data RAW;
   infile ...;
   input field1 $ 2-3   field2 $ 5-12
         field3 $ 15-22 field4   25-36
         field5 $ 40-47 field6   50-61;
   ...
run;
```

If the preceding MPS input data set has a name other than RAW, you can use MPSDATA=*SAS-data-set* as an argument in the SASMPSXS macro function. If you want the converted sparse form data set to have a name other than PROB, you can use LPDATA=*SAS-data-set* as an argument. The order of the arguments in the SASMPSXS macro function is not important.

The Reduced Costs, Dual Activities, and Current Tableau

The evaluation of reduced costs and the dual activities is independent of problem structure. For a basic solution, let B be the matrix composed of the basic columns of A and let N be the matrix composed of the nonbasic columns of A. The reduced cost associated with the ith variable is

$$(c^T - c_B^T B^{-1} A)_i$$

and the dual activity of the jth row is

$$(c_B^T B^{-1})_j$$

The Current Tableau is a section displayed when you specify either the TABLEAUPRINT option in the PROC LP statement or the TABLEAU option in the PRINT statement. The output contains a row for each basic variable and a column for each nonbasic variable. In addition, there is a row for the reduced costs and a column for the product

$$B^{-1}b$$

This column is labeled INV(B)*R. The body of the tableau contains the matrix

$$B^{-1}N$$

Macro Variable _ORLP_

The LP procedure defines a macro variable named _ORLP_. This variable contains a character string that indicates the status of the procedure. It is set whenever the user gets control, at breakpoints, and at procedure termination. The form of the _ORLP_ character string is STATUS= PHASE= OBJECTIVE= P_FEAS= D_FEAS= INT_ITER= INT_FEAS= ACTIVE= INT_BEST= PHASE1_ITER= PHASE2_ITER= PHASE3_ITER=. The terms are interpreted as follows:

STATUS=	the status of the current solution
PHASE=	the phase the procedure is in (1, 2, or 3)
OBJECTIVE=	the current objective value
P_FEAS=	whether the current solution is primal feasible
D_FEAS=	whether the current solution is dual feasible
INT_ITER=	the number of integer iterations performed
INT_FEAS=	the number of integer feasible solutions found
ACTIVE=	the number of active nodes in the current branch-and-bound tree

INT_BEST=	the best integer objective value found
PHASE1_ITER=	the number of iterations performed in phase 1
PHASE2_ITER=	the number of iterations performed in phase 2
PHASE3_ITER=	the number of iterations performed in phase 3

Table 3.5 shows the possible values for the nonnumeric terms in the string.

Table 3.5. Possible Values for Nonnumeric Terms

STATUS=	P_FEAS=	D_FEAS=
SUCCESSFUL	YES	YES
UNBOUNDED	NO	NO
INFEASIBLE		
MAX_TIME		
MAX_ITER		
PIVOT		
BREAK		
INT_FEASIBLE		
INT_INFEASIBLE		
INT_MAX_ITER		
PAUSE		
FEASIBLEPAUSE		
IPAUSE		
PROXIMITYPAUSE		
ACTIVE		
RELAXED		
FATHOMED		
IPIVOT		
UNSTABLE		
SINGULAR		
MEMORY_ERROR		
IO_ERROR		
SYNTAX_ERROR		
SEMANTIC_ERROR		
BADDATA_ERROR		
UNKNOWN_ERROR		

This information can be used when PROC LP is one step in a larger program that needs to identify how the LP procedure terminated. Because _ORLP_ is a standard SAS macro variable, it can be used in the ways that all macro variables can be used (see the *SAS Guide to Macro Processing*).

Pricing

PROC LP performs multiple pricing when determining which variable will enter the basis at each pivot (Greenberg 1978). This heuristic can shorten execution time in many problems. The specifics of the multiple pricing algorithm depend on the value of the PRICETYPE= option. However, in general, when some form of multiple pricing is used, during the first iteration PROC LP places the PRICE= nonbasic columns yielding the greatest marginal improvement to the objective function in a candidate list. This list identifies a subproblem of the original. On subsequent iterations, only the reduced costs for the nonbasic variables in the candidate list are calculated. This accounts for the potential time savings. When either the candidate list is empty or the subproblem is optimal, a new candidate list must be identified and the process repeats. Because identification of the subproblem requires pricing the complete problem, an iteration in which this occurs is called a *major iteration*. A *minor iteration* is an iteration in which only the subproblem is to be priced.

The value of the PRICETYPE= option determines the type of multiple pricing that is to be used. The types of multiple pricing include partial suboptimization (PRICETYPE=PARTIAL), complete suboptimization (PRICETYPE=COMPLETE), and complete suboptimization with dynamically varying the value of the PRICE= option (PRICETYPE=DYNAMIC).

When partial suboptimization is used, in each minor iteration the nonbasic column in the subproblem yielding the greatest marginal improvement to the objective is brought into the basis and removed from the candidate list. The candidate list now has one less entry. At each subsequent iteration, another column from the subproblem is brought into the basis and removed from the candidate list. When there are either no remaining candidates or the remaining candidates do not improve the objective, the subproblem is abandoned and a major iteration is performed. If the objective cannot be improved on a major iteration, the current solution is optimal and PROC LP terminates.

Complete suboptimization is identical to partial suboptimization with one exception. When a nonbasic column from the subproblem is brought into the basis, it is replaced in the candidate list by the basic column that is leaving the basis. As a result, the candidate list does not diminish at each iteration.

When PRICETYPE=DYNAMIC, complete suboptimization is performed, but the value of the PRICE= option changes so that the ratio of minor to major iterations is within two units of the PRICE= option.

These heuristics can shorten execution time for small values of the PRICE= option. Care should be exercised in choosing a value from the PRICE= option because too large a value can use more time than if pricing were not used.

Scaling

Based on the SCALE= option specified, the procedure scales the coefficients of both the constraints and objective rows before iterating. This technique can improve the numerical stability of an ill-conditioned problem. If you want to modify the default matrix scaling used, which is SCALE=BOTH, use the SCALE=COLUMN, SCALE=ROW, or SCALE=NONE option in the PROC LP statement. If SCALE=BOTH, the matrix coefficients are scaled so that the largest element in absolute value in each row or column equals 1. They are scaled by columns first and then by rows. If SCALE=COLUMN (ROW), the matrix coefficients are scaled so that the largest element in absolute value in each column (row) equals 1. If SCALE=NONE, no scaling is performed.

Preprocessing

With the preprocessing option, you can identify redundant and infeasible constraints, improve lower and upper bounds of variables, fix variable values and improve coefficients and RHS values before solving a problem. Preprocessing can be applied to LP, IP and MIP problems. For an LP problem, it may significantly reduce the problem size. For an IP or MIP problem, it can often reduce the gap between the optimal solution and the solution of the relaxed problem, which could lead to a smaller search tree in the branch-and-bound algorithm. As a result, the CPU time may be reduced on many problems. Although there is no guarantee that preprocessing will always yield a faster solution, it does provide a highly effective approach to solving large and difficult problems.

Preprocessing is especially useful when the original problem causes numerical difficulties to PROC LP. Since preprocessing could identify redundant constraints and tighten lower and upper bounds of variables, the reformulated problem may eliminate the numerical difficulties in practice.

When a constraint is identified as redundant, its type is marked as ‘FREE’ in the Constraint Summary. If a variable is fixed, its type is marked as ‘FIXED’ in the Variables Summary. If a constraint is identified as infeasible, PROC LP stops immediately and displays the constraint name in the SAS log file. This capability sometimes gives valuable insight into the model or the formulation and helps establish if the model is reasonable and the formulation is correct.

For a large and dense problem, preprocessing may take a longer time for each iteration. To limit the number of preprocessings, use the PMAXIT= option. To stop any further preprocessings during the preprocessing stage, press the CTRL-BREAK key. PROC LP will enter phase 1 at the end of the current iteration.

Integer Programming

Formulations of mathematical programs often require that some of the decision variables take only integer values. Consider the formulation

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \{\geq, =, \leq\} b \\ & \ell \leq x \leq u \\ & x_i \text{ is integer}, i \in \mathcal{S} \end{aligned}$$

The set of indices $\mathcal{S}$ identifies those variables that must take only integer values. When $\mathcal{S}$ does not contain all of the integers between 1 and n, inclusive, this problem is called a mixed-integer program (MIP). Otherwise, it is known as an integer program. Let x^{opt}(MIP) denote an optimal solution to (MIP). An integer variable with bounds between 0 and 1 is also called a binary variable.

Specifying the Problem

An integer or mixed-integer problem can be solved with PROC LP. To solve this problem, you must identify the integer variables. You can do this with a row in the input data set that has the keyword 'INTEGER' for the type variable. Any variable that has a nonmissing and nonzero value for this row is interpreted as an integer variable. It is important to note that integer variables must have upper bounds explicitly defined using the 'UPPERBD' keyword. The values in the 'INTEGER' row not only identify those variables that must be integers, but they also give an ordering to the integer variables that can be used in the solution technique.

You can follow the same steps to identify binary variables. For the binary variables, there is no need to supply any upper bounds.

Following the rules of sparse data input format, you can also identify individual integer or binary variables.

The Branch-and-Bound Technique

The branch-and-bound approach is used to solve integer and mixed-integer problems. The following discussion outlines the approach and explains how to use several options to control the procedure.

The branch-and-bound technique solves an integer program by solving a sequence of linear programs. The sequence can be represented by a tree, with each node in the tree being identified with a linear program that is derived from the problems on the path leading to the root of the tree. The root of the tree is identified with a linear program that is identical to (MIP), except that $\mathcal{S}$ is empty. This relaxed version of (MIP), called (LP(0)), can be written as

$$\begin{aligned} x^{opt}(0) = \quad & \min c^T x \\ \text{subject to} \quad & Ax \{\geq, =, \leq\} b \\ & \ell \leq x \leq u \end{aligned}$$

The branch-and-bound approach generates linear programs along the nodes of the tree using the following scheme. Consider $x^{opt}(0)$, the optimal solution to (LP(0)). If $x^{opt}(0)_i$ is integer for all $i \in \mathcal{S}$, then $x^{opt}(0)$ is optimal in (MIP). Suppose for some $i \in \mathcal{S}$, $x^{opt}(0)_i$ is nonintegral. In that case, define two new problems (LP(1)) and (LP(2)), descendants of the parent problem (LP(0)). The problem (LP(1)) is identical to (LP(0)) except for the additional constraint

$$x_i \leq \lfloor x^{opt}(0)_i \rfloor$$

and the problem (LP(2)) is identical to (LP(0)) except for the additional constraint

$$x_i \geq \lceil x^{opt}(0)_i \rceil$$

The notation $\lceil y \rceil$ means the smallest integer greater than or equal to y, and the notation $\lfloor y \rfloor$ means the largest integer less than or equal to y. Note that the two new problems do not have $x^{opt}(0)$ as a feasible solution, but because the solution to (MIP) must satisfy one of the preceding constraints, x_i^{opt}(MIP) must satisfy one of the new constraints. The two problems thus defined are called active nodes in the branch-and-bound tree, and the variable x_i is called the branching variable.

Next, the algorithm chooses one of the problems associated with an active node and attempts to solve it using the dual simplex algorithm. The problem may be infeasible, in which case the problem is dropped. If it can be solved, and it in turn does not have an integer solution (that is, a solution for which x_i is integer for all $i \in \mathcal{S}$), then it defines two new problems. These new problems each contain all of the constraints of the parent problems plus the appropriate additional one.

Branching continues in this manner until either there are no active nodes or an integer solution is found. When an integer solution is found, its objective value provides a bound for the objective of (MIP). In particular, if z is the objective value of the current best integer solution, then any active problems whose parent problem has objective value $\geq z$ can be discarded (assuming that the problem is a minimization). This can be done because all problems that descend from this parent will also have objective value $\geq z$. This technique is known as *fathoming*. When there are no active nodes remaining to be solved, the current integer solution is optimal in (MIP). If no integer solution has been found, then (MIP) is (integer) infeasible.

It is important to realize that integer programs are NP-complete. Roughly speaking, this means that the effort required to solve them grows exponentially with the size of the problem. For example, a problem with 10 binary variables can, in the worst case, generate $2^{10} = 1024$ nodes in the branch-and-bound tree. A problem with 20 binary variables can, in the worst case, generate $2^{20} = 1048576$ nodes in the branch-and-bound tree. Although the algorithm is unlikely to have to generate every single possible node, the need to explore even a small fraction of the potential number of nodes for a large problem can be resource intensive.

The Integer Iteration Log

To help monitor the growth of the branch-and-bound tree, the LP procedure reports on the status of each problem that is solved. The report, displayed in the Integer Iteration Log, can be used to reconstruct the branch-and-bound tree. Each row in the report describes the results of the attempted solution of the linear program at a node in the tree. In the following discussion, a problem on a given line in the log is called the current problem. The following columns are displayed in the report:

Iter	identifies the number of the branch-and-bound iteration.
Problem	identifies how the current problem fits in the branch-and-bound tree.
Condition	reports the result of the attempted solution of the current problem. Values for Condition are: • ACTIVE: The current problem was solved successfully. • INFEASIBLE: The current problem is infeasible. • FATHOMED: The current problem cannot lead to an improved integer solution and therefore it is dropped. • SINGULAR: A singular basis was encountered in attempting to solve the current problem. Solution of this relaxed problem is suspended and will be attempted later if necessary. • SUBOPTIMAL: The current problem has an integer feasible solution.
Objective	reports the objective value of the current problem.
Branched	names the variable that is branched in subtrees defined by the descendants of this problem.
Value	gives the current value of the variable named in the column labeled Branched.
Sinfeas	gives the sum of the integer infeasibilities in the optimal solution to the current problem.
Active	reports the total number of nodes currently active in the branch-and-bound tree.
Proximity	reports the gap between the best integer solution and the current lower (upper for maximizations) bound of all active nodes.

To reconstruct the branch-and-bound tree from this report, consider the interpretation of iteration j. If Iter=j and Problem=k, then the problem solved on iteration j is identical to the problem solved on iteration $|k|$ with an additional constraint. If

$k > 0$, then the constraint is an upper bound on the variable named in the Branched column on iteration $|k|$. If $k < 0$, then the constraint is a lower bound on that variable. The value of the bound can be obtained from the value of Value in iteration $|k|$ as described in the previous section.

Example 3.8 in the section "Examples: LP Procedure" on page 229 shows an Integer Iteration Log in its output.

Controlling the Branch-and-Bound Search

There are several options you can use to control branching. This is accomplished by controlling the program's choice of the branching variable and of the next active node. In the discussion that follows, let

$$f_i(k) = x^{opt}(k)_i - \lfloor x^{opt}(k)_i \rfloor$$

where $x^{opt}(k)$ is the optimal solution to the problem solved in iteration k.

The CANSELECT= option directs the choice of the next active node. Valid keywords for this option include LIFO, FIFO, OBJ, PROJECT, PSEUDOC, and ERROR. The following list describes the action that each of these causes when the procedure must choose for solution a problem from the list of active nodes.

LIFO
: chooses the last problem added to the tree of active nodes. This search has the effect of a depth-first search of the branch-and-bound tree.

FIFO
: chooses the first node added to the tree of active nodes. This search has the effect of a breadth-first search of the branch-and-bound tree.

OBJ
: chooses the problem whose parent has the smallest (largest if the problem is a maximization) objective value.

PROJECT
: chooses the problem with the largest (smallest if the problem is a maximization) projected objective value. The projected objective value is evaluated using the sum of integer infeasibilities, $s(k)$, associated with an active node (LP(k)), defined by

$$s(k) = \sum_{i \in \mathcal{S}} \min\{f_i(k), 1 - f_i(k)\}$$

An empirical measure of the rate of increase (decrease) in the objective value is defined as

$$\lambda = (z^* - z(0))/s(0)$$

where

- $z(k)$ is the optimal objective value for (LP(k))
- z^* is the objective value of the current best integer solution

The projected objective value for problems (LP($k+1$)) and (LP($k+2$)) is defined as

$$z(k) + \lambda s(k)$$

PSEUDOC chooses the problem with the largest (least if the problem is a maximization) projected pseudocost. The projected pseudocost is evaluated using the weighted sum of infeasibilities $s_w(k)$ associated with an active problem (LP(k)), defined by

$$s_w(k) = \sum_{i \in \mathcal{S}} \min\{d_i(k)f_i(k), u_i(k)(1 - f_i(k))\}$$

The weights u_i and d_i are initially equal to the absolute value of the ith objective coefficient and are updated at each integer iteration. They are modified by examining the empirical marginal change in the objective as additional constraints are placed on the variables in $\mathcal{S}$ along the path from (LP(0)) to a node associated with an integer feasible solution. In particular, if the definition of problems (LP(k+1)) and (LP(k+2)) from parent (LP(k)) involve the addition of constraints $x_i \leq \lfloor x^{opt}(k)_i \rfloor$ and $x_i \geq \lceil x^{opt}(k)_i \rceil$, respectively, and one of them is on a path to an integer feasible solution, then only one of the following is true:

$$d_i(k) = (z(k+1) - z(k))/f_i(k)$$

$$u_i(k) = (z(k+2) - z(k))/(1 - f_i(k))$$

Note the similarity between $s_w(k)$ and $s(k)$. The weighted quantity $s_w(k)$ accounts to some extent for the influence of the objective function. The projected pseudocost for problems (LP($k+1$)) and (LP($k+2$)) is defined as

$$z_w(k) \equiv z(k) + s_w(k)$$

ERROR chooses the problem with the largest error. The error associated with problems (LP($k+1$)) and (LP($k+2$)) is defined as

$$(z^* - z_w(k))/(z^* - z(k))$$

The BACKTRACK= option controls the search for the next problem. This option can take the same values as the CANSELECT= option. In addition to the case outlined under the DELTAIT= option, backtracking is required as follows based on the CANSELECT= option in effect:

- If CANSELECT=LIFO and there is no active node in the portion of the active tree currently under exploration with a bound better than the value of WOBJECTIVE=, then the procedure must backtrack.
- If CANSELECT=FIFO, PROJECT, PSEUDOC, or ERROR and the bound corresponding to the node under consideration is not better than the value of WOBJECTIVE=, then the procedure must backtrack.

The default value is OBJ.

The VARSELECT= option directs the choice of branching variable. Valid keywords for this option include CLOSE, FAR, PRIOR, PSEUDOC, PRICE, and PENALTY. The following list describes the action that each of these causes when $x^{opt}(k)$, an optimal solution of problem (LP(k)), is used to define active problems (LP($k+1$)) and (LP($k+2$)).

CLOSE chooses as branching variable the variable x_i such that i minimizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and}$$

$$\text{IEPSILON} \leq f_i(k) \leq 1 - \text{IEPSILON}\}$$

FAR chooses as branching variable the variable x_i such that i maximizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and}$$

$$\text{IEPSILON} \leq f_i(k) \leq 1 - \text{IEPSILON}\}$$

PRIOR chooses as branching variable x_i such that $i \in \mathcal{S}$, $x^{opt}(k)_i$ is non-integral, and variable x_i has the minimum value in the INTEGER row in the input data set. This choice for the VARSELECT= option is recommended when you have enough insight into the model to identify those integer variables that have the most significant effect on the objective value.

PENALTY chooses as branching variable x_i such that $i \in \mathcal{S}$ and a bound on the increase in the objective of (LP(k)) (penalty) resulting from adding the constraint

$$x_i \leq \lfloor x^{opt}(k)_i \rfloor \quad \text{or} \quad x_i \geq \lceil x^{opt}(k)_i \rceil$$

is maximized. The bound is calculated without pivoting using techniques of sensitivity analysis (Garfinkel and Nemhauser 1972). Because the cost of calculating the maximum penalty can be large if $\mathcal{S}$ is large, you may want to limit the number of variables in $\mathcal{S}$ for which the penalty is calculated. The penalty is calculated for PENALTYDEPTH= variables in $\mathcal{S}$.

PRICE chooses as branching variable x_i such that $i \in \mathcal{S}$, $x^{opt}(k)_i$ is non-integral, and variable x_i has the minimum price coefficient (maximum for maximization).

PSEUDOC chooses as branching variable the variable x_i such that i maximizes

$$\{\min\{d_i f_i(k), u_i(1 - f_i(k))\} \mid i \in \mathcal{S} \text{ and}$$

$$\text{IEPSILON} \leq f_i(k) \leq 1 - \text{IEPSILON}\}$$

The weights u_i and d_i are initially equal to the absolute value of the ith objective coefficient and are updated whenever an integer feasible solution is encountered. See the discussion on the CANSELECT= option for details on the method of updating the weights.

Customizing Search Heuristics

Often a good heuristic for searching the branch-and-bound tree of a problem can be found. You are tempted to continue using this heuristic when the problem data changes but the problem structure remains constant. The ability to reset procedure options interactively enables you to experiment with search techniques in an attempt to identify approaches that perform well. Then you can easily reapply these techniques to subsequent problems.

For example, the PIP branch-and-bound strategy (Crowder, Johnson, and Padberg 1983) describes one such heuristic. The following program uses a similar strategy. Here, the OBJ rule (choose the active node with least parent objective function in the case of a minimization problem) is used for selecting the next active node to be solved until an integer feasible solution is found. Once such a solution is found, the search procedure is changed to the LIFO rule: choose the problem most recently placed in the list of active nodes.

```
proc lp canselect=obj ifeasiblepause=1;
run;
   reset canselect=lifo ifeasiblepause=9999999;
run;
```

Further Discussion on AUTO and CONTROL= options

Consider a minimization problem. At each integer iteration, PROC LP will select a node to solve from a pool of active nodes. The best bound strategy (CANSELECT=OBJ) will pick the node with the smallest projected objective value. This strategy improves the lower bound of the integer program and usually takes fewer integer iterations. One disadvantage is that PROC LP must recalculate the inverse of the basis matrix at almost every integer iteration; such recalculation is relatively expensive. Another disadvantage is that this strategy does not pay attention to improving the upper bound of the integer program. Thus the number of active nodes tends to grow rapidly if PROC LP cannot quickly find an optimal integer solution.

On the other hand, the LIFO strategy is very efficient and does not need to calculate the inverse of the basis matrix unless the previous node is fathomed. It is a depth-first strategy so it tends to find an integer feasible solution quickly. However, this strategy will pick nodes locally and usually will take longer integer iterations than the best bound strategy.

There is another strategy that is often overlooked. Here it is called the **best upper bound** strategy. With this strategy, each time you select an active node, instead of picking the node with the smallest projected objective value, you select the one with the largest projected objective value. This strategy is as efficient as the LIFO strategy. Moreover, it selects active nodes globally. This strategy tries to improve the upper bound of the integer program by searching for new integer feasible solutions. It also fathoms active nodes quickly and keeps the total number of active nodes below the current level. A disadvantage is that this strategy may evaluate more nodes that do not have any potential in finding an optimal integer solution.

The best bound strategy has the advantage of improving the lower bound. The LIFO strategy has the advantages of efficiency and finding a local integer feasible solution. The best upper bound strategy has the advantages of keeping the size of active nodes under control and at the same time trying to identify any potential integer feasible solution globally.

Although the best bound strategy is generally preferred, in some instances other strategies may be more effective. For example, if you have found an integer optimal solution but you do not know it, you still have to enumerate all possible active nodes. Then the three strategies will basically take the same number of integer iterations after an optimal solution is found but not yet identified. Since the LIFO and best upper bound strategies are very efficient per integer iteration, both will outperform the best bound strategy.

Since no one strategy suits all situations, a hybrid strategy has been developed to increase applicability. The CONTROL= option combines the above three strategies naturally and provides a simple control parameter in [0, 1] dealing with different integer programming problems and different solution situations. The AUTO option automatically sets and adjusts the CONTROL= parameter so that you do not need to know any problem structure or decide a node selection strategy in advance.

Since the LIFO strategy is less costly, you should use it as much as possible in the combinations. The following process is called a **diving process**. Starting from an active node, apply the LIFO strategy as much as you can until the current node becomes feasible or is fathomed, or exceeds a preset limit. During this process, there is no inverse matrix calculation involved except for the first node. When the diving process is over, apply one of the three strategies to select the next starting node. One set of combinations is called a cycle.

The control parameter r controls the frequency of the three strategies being applied and the depth of the diving process in a cycle. It starts with a pure best bound strategy at $r = 0$, and then gradually increases the frequency of the diving processes and their depths as r increases. At $r = 0.5$, one cycle contains a best bound strategy plus a full diving process. After $r = 0.5$, the number of the diving processes will gradually

increase in a cycle. In addition, the best upper bound strategy will join the cycle. As r continues to increase, the frequency of the best upper bound strategy will increase. At $r = 1$, it becomes a pure best upper bound strategy.

The AUTO option will automatically adjust the value of the CONTROL= option. At the start, it sets CONTROL=0.7, which emphasizes finding an upper bound. After an integer feasible solution is found, it sets CONTROL=0.5, which emphasizes efficiency and lower bound improvement. When the number of active nodes grows over the default or user defined limit m, the number indicates that a better upper bound is needed. The AUTO option will start to increase the value of CONTROL= from 0.5. If the size of the active nodes continues to grow, so will the value of the CONTROL= option. When the size of active nodes reaches to the default or user-defined limit n, CONTROL= will be set to 1. At this moment, the growth of active nodes is stopped. When the size of active nodes reduces, AUTO will decrease the value of CONTROL= option.

You can use other strategies to improve the lower bound by setting CANSELECT= to other options.

Saving and Restoring the List of Active Nodes

The list of active nodes can be saved in a SAS data set for use at a subsequent invocation of PROC LP. The ACTIVEOUT= option in the PROC LP statement names the data set into which the current list of active nodes is saved when the procedure terminates due to an error termination condition. Examples of such conditions are time limit exceeded, integer iterations exceeded, and phase 3 iterations exceeded. The ACTIVEIN= option in the PROC LP statement names a data set that can be used to initialize the list of active nodes. To achieve the greatest benefit when restarting PROC LP, use the PRIMALOUT= and PRIMALIN= options in conjunction with the ACTIVEOUT= and ACTIVEIN= options. See Example 3.10 in the section "Examples: LP Procedure" on page 229 for an illustration.

Sensitivity Analysis

Sensitivity analysis is a technique for examining the effects of changes in model parameters on the optimal solution. The analysis enables you to examine the size of a perturbation to the right-hand-side or objective vector by an arbitrary change vector for which the basis of the current optimal solution remains optimal.

Note: When sensitivity analysis is performed on integer-constrained problems, the integer variables are fixed at the value they obtained in the integer optimal solution. Therefore, care must be used when interpreting the results of such analyses. Care must also be taken when preprocessing is enabled, because preprocessing usually alters the original formulation.

Right-Hand-Side Sensitivity Analysis

Consider the problem $(lpr(\phi))$:

$$\begin{aligned} x^{opt}(\phi) = \quad & \min c^T x \\ \text{subject to} \quad & Ax \{\geq, =, \leq\} b + \phi r \\ & \ell \leq x \leq u \end{aligned}$$

where r is a right-hand-side change vector.

Let $x^{opt}(\phi)$ denote an optimal basic feasible solution to $(lpr(\phi))$. PROC LP can be used to examine the effects of changes in ϕ on the solution $x^{opt}(0)$ of problem $(lpr(0))$. For the basic solution $x^{opt}(0)$, let B be the matrix composed of the basic columns of A and let N be the matrix composed of the nonbasic columns of A. For the basis matrix B, the basic components of $x^{opt}(0)$, written as $x^{opt}(0)_B$, can be expressed as

$$x^{opt}(0)_B = B^{-1}(b - Nx^{opt}(0)_N)$$

Furthermore, because $x^{opt}(0)$ is feasible,

$$\ell_B \leq B^{-1}(b - Nx^{opt}(0)_N) \leq u_B$$

where ℓ_B is a column vector of the lower bounds on the structural basic variables, and u_B is a column vector of the upper bounds on the structural basic variables. For each right-hand-side change vector r identified in the RHSSEN statement, PROC LP finds an interval $[\phi_{min}, \phi_{max}]$ such that

$$\ell_B \leq B^{-1}(b + \phi r - Nx^{opt}(0)_N) \leq u_B$$

for $\phi \in [\phi_{min}, \phi_{max}]$. Furthermore, because changes in the right-hand side do not affect the reduced costs, for $\phi \in [\phi_{min}, \phi_{max}]$,

$$x^{opt}(\phi)^T = ((B^{-1}(b + \phi r - Nx^{opt}(0)_N))^T, x^{opt}(0)_N^T)$$

is optimal in $(lpr(\phi))$.

For $\phi = \phi_{min}$ and $\phi = \phi_{max}$, PROC LP reports the following:

- the names of the leaving variables
- the value of the optimal objective in the modified problems
- the RHS values in the modified problems
- the solution status, reduced costs and activities in the modified problems

The leaving variable identifies the basic variable x_i that first reaches either the lower bound ℓ_i or the upper bound u_i as ϕ reaches ϕ_{min} or ϕ_{max}. This is the basic variable that would leave the basis to maintain primal feasibility. Multiple RHSSEN variables can appear in a problem data set.

Price Sensitivity Analysis

Consider the problem $(lpp(\phi))$:

$$\begin{aligned} x^{opt}(\phi) = \quad & \min(c+\phi r)^T x \\ \text{subject to} \quad & Ax \{\geq, =, \leq\} b \\ & \ell \leq x \leq u \end{aligned}$$

where r is a price change vector.

Let $x^{opt}(\phi)$ denote an optimal basic feasible solution to $(lpp(\phi))$. PROC LP can be used to examine the effects of changes in ϕ on the solution $x^{opt}(0)$ of problem $(lpp(0))$. For the basic solution $x^{opt}(0)$, let B be the matrix composed of the basic columns of A and let N be the matrix composed of the nonbasic columns of A. For basis matrix B, the reduced cost associated with the ith variable can be written as

$$rc_i(\phi) = ((c+\phi r)_N^T - (c+\phi r)_B^T B^{-1} N)_i$$

where $(c+\phi r)_N$ and $(c+\phi r)_B$ is a partition of the vector of price coefficients into nonbasic and basic components. Because $x^{opt}(0)$ is optimal in $(lpp(0))$, the reduced costs satisfy

$$rc_i(\phi) \geq 0$$

if the nonbasic variable in column i is at its lower bound, and

$$rc_i(\phi) \leq 0$$

if the nonbasic variable in column i is at its upper bound.

For each price coefficient change vector r identified with the keyword PRICESEN in the TYPE variable, PROC LP finds an interval $[\phi_{min}, \phi_{max}]$ such that for $\phi \in [\phi_{min}, \phi_{max}]$,

$$rc_i(\phi) \geq 0$$

if the nonbasic variable in column i is at its lower bound, and

$$rc_i(\phi) \leq 0$$

if the nonbasic variable in column i is at its upper bound. Because changes in the price coefficients do not affect feasibility, for $\phi \in [\phi_{min}, \phi_{max}]$, $x^{opt}(\phi)$ is optimal in $(lpp(\phi))$. For $\phi = \phi_{min}$ and $\phi = \phi_{max}$, PROC LP reports the following:

- the names of entering variables
- the value of the optimal objective in the modified problems

- the price coefficients in the modified problems
- the solution status, reduced costs, and activities in the modified problems

The entering variable identifies the variable whose reduced cost first goes to zero as ϕ reaches ϕ_{min} or ϕ_{max}. This is the nonbasic variable that would enter the basis to maintain optimality (dual feasibility). Multiple PRICESEN variables may appear in a problem data set.

Range Analysis

Range analysis is sensitivity analysis for specific change vectors. As with the sensitivity analysis case, care must be used in interpreting the results of range analysis when the problem has integers or the preprocessing option is enabled.

Right-Hand-Side Range Analysis

The effects on the optimal solution of changes in each right-hand-side value can be studied using the RANGERHS option in the PROC LP or RESET statement. This option results in sensitivity analysis for the m right-hand-side change vectors specified by the columns of the $m \times m$ identity matrix.

Price Range Analysis

The effects on the optimal solution of changes in each price coefficient can be studied using the RANGEPRICE option in the PROC LP or RESET statement. This option results in sensitivity analysis for the n price change vectors specified by the rows of the $n \times n$ identity matrix.

Parametric Programming

Sensitivity analysis and range analysis examine how the optimal solution behaves with respect to perturbations of model parameter values. These approaches assume that the basis at optimality is not allowed to change. When greater flexibility is desired and a change of basis is acceptable, parametric programming can be used.

As with the sensitivity analysis case, care must be used in interpreting the results of parametric programming when the problem has integers or the preprocessing option is enabled.

Right-Hand-Side Parametric Programming

As discussed in the section "Right-Hand-Side Sensitivity Analysis" on page 214, for each right-hand-side change vector r, PROC LP finds an interval $[\phi_{min}, \phi_{max}]$ such that for $\phi \in [\phi_{min}, \phi_{max}]$,

$$x^{opt}(\phi)^T = ((B^{-1}(b + \phi r - N x^{opt}(0)_N))^T, x^{opt}(0)_N^T)$$

is optimal in $(lpr(\phi))$ for the fixed basis B. Leaving variables that inhibit further changes in ϕ without a change in the basis B are associated with the quantities ϕ_{min} and ϕ_{max}. By specifying RHSPHI=Φ in either the PROC LP statement or the RESET

statement, you can examine the solution $x^{opt}(\phi)$ as ϕ increases or decreases from 0 to Φ.

When RHSPHI=Φ is specified, the procedure first finds the interval $[\phi_{min}, \phi_{max}]$ as described previously. Then, if $\Phi \in [\phi_{min}, \phi_{max}]$, no further investigation is needed. However, if $\Phi > \phi_{max}$ or $\Phi < \phi_{min}$, then the procedure attempts to solve the new problem $(lpr(\Phi))$. To accomplish this, it pivots the leaving variable out of the basis while maintaining dual feasibility. If this new solution is primal feasible in $(lpr(\Phi))$, no further investigation is needed; otherwise, the procedure identifies the new leaving variable and pivots it out of the basis, again maintaining dual feasibility. Dual pivoting continues in this manner until a solution that is primal feasible in $(lpr(\Phi))$ is identified. Because dual feasibility is maintained at each pivot, the $(lpr(\Phi))$ primal feasible solution is optimal.

At each pivot, the procedure reports on the variables that enter and leave the basis, the current range of ϕ , and the objective value. When $x^{opt}(\Phi)$ is found, it is displayed. If you want the solution $x^{opt}(\phi)$ at each pivot, then specify the PARAPRINT option in either the PROC LP or the RESET statement.

Price Parametric Programming

As discussed in the section "Price Sensitivity Analysis" on page 215, for each price change vector r, PROC LP finds an interval $[\phi_{min}, \phi_{max}]$ such that for each $\phi \in [\phi_{min}, \phi_{max}]$,

$$rc_i(\phi) = ((c + \phi r)_N^T - (c + \phi r)_B^T B^{-1} N)_i$$

satisfies the conditions for optimality in $(lpp(\phi))$ for the fixed basis B. Entering variables that inhibit further changes in ϕ without a change in the basis B are associated with the quantities ϕ_{min} and ϕ_{max}. By specifying PRICEPHI=Φ in either the PROC LP statement or the RESET statement, you can examine the solution $x^{opt}(\phi)$ as ϕ increases or decreases from 0 to Φ.

When PRICEPHI=Φ is specified, the procedure first finds the interval $[\phi_{min}, \phi_{max}]$, as described previously. Then, if $\Phi \in [\phi_{min}, \phi_{max}]$, no further investigation is needed. However, if $\Phi > \phi_{max}$ or $\Phi < \phi_{min}$, the procedure attempts to solve the new problem $(lpp(\Phi))$. To accomplish this, it pivots the entering variable into the basis while maintaining primal feasibility. If this new solution is dual feasible in $(lpp(\Phi))$, no further investigation is needed; otherwise, the procedure identifies the new entering variable and pivots it into the basis, again maintaining primal feasibility. Pivoting continues in this manner until a solution that is dual feasible in $(lpp(\Phi))$ is identified. Because primal feasibility is maintained at each pivot, the $(lpp(\Phi))$ dual feasible solution is optimal.

At each pivot, the procedure reports on the variables that enter and leave the basis, the current range of ϕ , and the objective value. When $x^{opt}(\Phi)$ is found, it is displayed. If you want the solution $x^{opt}(\phi)$ at each pivot, then specify the PARAPRINT option in either the PROC LP or the RESET statement.

Interactive Facilities

The interactive features of the LP procedure enable you to examine intermediate results, perform sensitivity analysis, parametric programming, and range analysis, and control the solution process.

Controlling Interactive Features

You can gain control of the LP procedure for interactive processing by setting a breakpoint or pressing the CTRL-BREAK key combination, or when certain error conditions are encountered:

- when a feasible solution is found
- at each pivot of the simplex algorithm
- when an integer feasible solution is found
- at each integer pivot of the branch-and-bound algorithm
- after the data are read but before iteration begins
- after at least one integer feasible solution has been found which is within desirable proximity of optimality
- after the problem has been solved but before results are displayed

When the LP procedure pauses, you can enter any of the interactive statements RESET, PIVOT, IPIVOT, PRINT, SHOW, QUIT, and RUN.

Breakpoints are set using the FEASIBLEPAUSE, PAUSE=, IFEASIBLEPAUSE=, IPAUSE=, PROXIMITYPAUSE=, READPAUSE, and ENDPAUSE options. The LP procedure displays a message on the SAS log when it gives you control because of encountering one of these breakpoints.

During phase 1, 2, or 3, the CTRL-BREAK key pauses the LP procedure and releases the control at the beginning of the next iteration.

The error conditions, which usually cause the LP procedure to pause, include time limit exceeded, phase 1 iterations exceeded, phase 2 iterations exceeded, phase 3 iterations exceeded, and integer iterations exceeded. You can use the RESET statement to reset the option that caused the error condition.

The PIVOT and IPIVOT statements result in control being returned to you after a single simplex algorithm pivot and an integer pivot. The PRINT and SHOW statements display current solution information and return control to you. On the other hand, the QUIT statement requests that you leave the LP procedure immediately. If you want to quit but save output data sets, then type QUIT/SAVE. The RUN statement requests the LP procedure to continue its execution immediately.

Displaying Intermediate Results

Once you have control of the procedure, you can examine the current values of the options and the status of the problem being solved using the SHOW statement. All displaying done by the SHOW statement goes to the SAS log.

Details about the current status of the solution are obtained using the PRINT statement. The various display options enable you to examine parts of the variable and constraint summaries, display the current tableau, perform sensitivity analysis on the current solution, and perform range analysis.

Interactive Facilities in Batch Mode

All of the interactive statements can be used when processing in batch mode. This is particularly convenient when the interactive facilities are used to combine different search strategies in solving integer problems.

Sensitivity Analysis

Two features that enhance the ability to perform sensitivity analysis need further explanation. When you specify /SENSITIVITY in a PRINT COLUMN(*colnames*) statement, the LP procedure defines a new change row to use in sensitivity analysis and parametric programming. This new change row has a +1 entry for each variable listed in the PRINT statement. This enables you to define new change rows interactively.

When you specify /SENSITIVITY in a PRINT ROW (*rownames*) statement, the LP procedure defines a new change column to use in sensitivity analysis and parametric programming. This new change column has a +1 entry for each right-hand-side coefficient listed in the PRINT statement. This enables you to define new change columns interactively.

In addition, you can interactively change the RHSPHI= and PRICEPHI= options using the RESET statement. This enables you to perform parametric programming interactively.

Memory Management

There are no restrictions on the problem size in the LP procedure. The number of constraints and variables in a problem that PROC LP can solve depends on the host platform, the available memory, and the available disk space for utility data sets.

Memory usage is affected by a great many factors including the density of the technological coefficient matrix, the model structure, and the density of the decomposed basis matrix. The algorithm requires that the decomposed basis fit completely in memory. Any additional memory is used for nonbasic columns. The partition between the decomposed basis and the nonbasic columns is dynamic so that as the inverse grows, which typically happens as iterations proceed, more memory is available to it and less is available for the nonbasic columns.

The LP procedure determines the initial size of the decomposed basis matrix. If the area used is too small, PROC LP must spend time compressing this matrix, which degrades performance. If PROC LP must compress the decomposed basis matrix on the

average more than 15 times per iteration, then the size of the memory devoted to the basis is increased. If the work area cannot be made large enough to invert the basis, an error return occurs. On the other hand, if PROC LP compresses the decomposed basis matrix on the average once every other iteration, then memory devoted to the decomposed basis is decreased, freeing memory for the nonbasic columns.

For many models, memory constraints are not a problem because both the decomposed basis and all the nonbasic columns will have no problem fitting. However, when the models become large relative to the available memory, the algorithm tries to adjust memory distribution in order to solve the problem. In the worst cases, only one nonbasic column fits in memory with the decomposed basis matrix.

Problems involving memory use can occur when solving mixed-integer problems. Data associated with each node in the branch-and-bound tree must be kept in memory. As the tree grows, competition for memory by the decomposed basis, the nonbasic columns, and the branch-and-bound tree may become critical. If the situation becomes critical, the procedure automatically switches to branching strategies that use less memory. However, it is possible to reach a point where no further processing is possible. In this case, PROC LP terminates on a memory error.

Output Data Sets

The LP procedure can optionally produce four output data sets. These are the ACTIVEOUT=, PRIMALOUT=, DUALOUT=, and TABLEAUOUT= data sets. Each contains two variables that identify the particular problem in the input data set. These variables are

_OBJ_ID_ identifies the objective function ID.

_RHS_ID_ identifies the right-hand-side variable.

Additionally, each data set contains other variables, which are discussed below.

ACTIVEOUT= Data Set

The ACTIVEOUT= data set contains a representation of the current active branch-and-bound tree. You can use this data set to initialize the branch-and-bound tree to continue iterations on an incompletely solved problem. Each active node in the tree generates two observations in this data set. The first is a 'LOWERBD' observation that is used to reconstruct the lower-bound constraints on the currently described active node. The second is an 'UPPERBD' observation that is used to reconstruct the upper-bound constraints on the currently described active node. In addition to these, an observation that describes the current best integer solution is included. The data set contains the following variables:

STATUS contains the keywords LOWERBD, UPPERBD, and INTBEST for identifying the type of observation.

PROB contains the problem number for the current observation.

OBJECT contains the objective value of the parent problem that generated the current observation's problem.

SINFEA contains the sum of the integer infeasibilities of the current observation's problem.

PROJEC contains the data needed for CANSELECT=PROJECT when the branch-and-bound tree is read using the ACTIVEIN= option.

PSEUDO contains the data needed for CANSELECT=PSEUDOC when the branch-and-bound tree is read using the ACTIVEIN= option.

INTEGER VARIABLES Integer-constrained structural variables are also included in the ACTIVEOUT= data set. For each observation, these variables contain values for defining the active node in the branch-and-bound tree.

PRIMALOUT= Data Set

The PRIMALOUT= data set contains the current primal solution. If the problem has integer-constrained variables, the PRIMALOUT= data set contains the current best integer feasible solution. If none have been found, the PRIMALOUT= data set contains the relaxed solution. In addition to _OBJ_ID_ and _RHS_ID_, the data set contains the following variables:

VAR identifies the variable name.

TYPE identifies the type of the variable as specified in the input data set. Artificial variables are labeled as type 'ARTIFCL'.

STATUS identifies whether the variable is basic, nonbasic, or at an upper bound in the current solution.

LBOUND contains the input lower bound on the variable unless the variable is integer-constrained and an integer solution is given. In this case, _LBOUND_ contains the lower bound on the variable needed to realize the integer solution on subsequent calls to PROC LP when using the PRIMALIN= option.

VALUE identifies the value of the variable in the current solution or the current best integer feasible solution.

UBOUND contains the input upper bound on the variable unless the variable is integer-constrained and an integer solution is given. In this case, _UBOUND_ contains the upper bound on the variable needed to realize the integer solution on subsequent calls to PROC LP when using the PRIMALIN= option.

PRICE contains the input price coefficient of the variable.

_R_COST_ identifies the value of the reduced cost in the current solution. Example 3.3 in the section "Examples: LP Procedure" on page 229 shows a typical PRIMALOUT= data set. Note that it is necessary to include the information on objective function and right-hand side in order to distinguish problems in multiple problem data sets.

DUALOUT= Data Set

The DUALOUT= data set contains the dual solution for the current solution. If the problem has integer-constrained variables, the DUALOUT= data set contains the dual for the current best integer solution, if any. Otherwise it contains the dual for the relaxed solution. In addition to _OBJ_ID_ and _RHS_ID_, it contains the following variables:

_ROW_ID_	identifies the row or constraint name.
TYPE	identifies the type of the row as specified in the input data set.
RHS	gives the value of the right-hand side on input.
_L_RHS_	gives the lower bound for the row evaluated from the input right-hand-side value, the TYPE of the row, and the value of the RANGE variable for the row.
VALUE	gives the value of the row, at optimality, excluding logical variables.
_U_RHS_	gives the upper bound for the row evaluated from the input right-hand-side value, the TYPE of the row, and the value of the RANGE variable for the row.
DUAL	gives the value of the dual variable associated with the row.

TABLEAUOUT= Data Set

The TABLEAUOUT= data set contains the current tableau. Each observation, except for the first, corresponds to a basic variable in the solution. The observation labeled R_COSTS contains the reduced costs $c_N^T - c_B^T B^{-1} N$. In addition to _OBJ_ID_ and _RHS_ID_, it contains the following variables:

BASIC	gives the names of the basic variables in the solution.
INVB_R	gives the values of $B^{-1}r$, where r is the right-hand-side vector.
STRUCTURAL VARIABLES	give the values in the tableau, namely $B^{-1}A$.

Input Data Sets

In addition to the DATA= input data set, PROC LP recognizes the ACTIVEIN= and the PRIMALIN= data sets.

ACTIVEIN= Data Set

The ACTIVEIN= data set contains a representation of the current active tree. The format is identical to that of the ACTIVEOUT= data set.

PRIMALIN= Data Set

The format of the PRIMALIN= data set is identical to the PRIMALOUT= data set. PROC LP uses the PRIMALIN= data set to identify variables at their upper bounds in the current solution and variables that are basic in the current solution.

You can add observations to the end of the problem data set if they define cost (right-hand-side) sensitivity change vectors and have PRICESEN (RHSSEN) types. This enables you to solve a problem, save the solution in a SAS data set, and perform sensitivity analysis later. You can also use the PRIMALIN= data set to restart problems that have not been completely solved or to which new variables have been added.

Displayed Output

The output from the LP procedure is discussed in the following six sections:

- Problem Summary
- Solution Summary including a Variable Summary and a Constraint Summary
- Infeasible Information Summary
- RHS Sensitivity Analysis Summary (the RHS Range Analysis Summary is not discussed)
- Price Sensitivity Analysis Summary (the Price Range Analysis Summary is not discussed)
- Iteration Log

For integer-constrained problems, the procedure also displays an Integer Iteration Log. The description of this Log can be found in the section "Integer Programming" on page 205. When you request that the tableau be displayed, the procedure displays the Current Tableau. The description of this can be found in the section "The Reduced Costs, Dual Activities, and Current Tableau" on page 201.

A problem data set can contain a set of constraints with several right-hand sides and several objective functions. PROC LP considers each combination of right-hand side and objective function as defining a new linear programming problem and solves each, performing all specified sensitivity analysis on each problem. For each problem defined, PROC LP displays a new sequence of output sections. Example 3.1 in the section "Examples: LP Procedure" on page 229 discusses each of these elements.

The LP procedure produces the following displayed output by default.

The Problem Summary

The problem summary includes the

- type of optimization and the name of the objective row (as identified by the ID or ROW variable)
- name of the SAS variable that contains the right-hand-side constants
- name of the SAS variable that contains the type keywords

- density of the coefficient matrix (the ratio of the number of nonzero elements to the number of total elements) after the slack and surplus variables have been appended
- number of each type of variable in the mathematical program
- number of each type of constraint in the mathematical program

The Solution Summary

The solution summary includes the

- termination status of the procedure
- objective value of the current solution
- number of phase 1 iterations that were completed
- number of phase 2 iterations that were completed
- number of phase 3 iterations that were completed
- number of integer iterations that were completed
- number of integer feasible solutions that were found
- number of initial basic feasible variables identified
- time used in solving the problem excluding reading the data and displaying the solution
- number of inversions of the basis matrix
- current value of several of the options

The Variable Summary

The variable summary includes the

- column number associated with each structural or logical variable in the problem
- name of each structural or logical variable in the problem. (PROC LP gives the logical variables the name of the constraint ID. If no ID variable is specified, the procedure names the logical variable _OBSn_, where n is the observation that describes the constraint.)
- variable's status in the current solution. The status can be BASIC, DEGEN, ALTER, blank, LOWBD, or UPPBD, depending upon whether the variable is a basic variable, a degenerate variable (that is, a basic variable whose activity is at its input lower bound), a nonbasic variable that can be brought into the basis to define an alternate optimal solution, a nonbasic variable at its default lower bound 0, a nonbasic variable at its lower bound, or a nonbasic variable at its upper bound.
- type of variable (whether it is logical or structural, and, if structural, its bound type, or other value restriction). See Example 3.1 for a list of possible types in the variable summary.

- value of the objective coefficient associated with each variable
- activity of the variable in the current solution
- variable's reduced cost in the current solution

The Constraint Summary

The constraint summary includes the

- constraint row number and its ID
- kind of constraint (whether it is an OBJECTIVE, LE, EQ, GE, RANGELE, RANGEEQ, RANGEGE, or FREE row)
- number of the slack or surplus variable associated with the constraint row
- value of the right-hand-side constant associated with the constraint row
- current activity of the row (excluding logical variables)
- current activity of the dual variable (shadow price) associated with the constraint row

The Infeasible Information Summary

The infeasible information summary includes the

- name of the infeasible row or variable
- current activity for the row or variable
- type of the row or variable
- value of right-hand-side constant
- name of each nonzero and nonmissing variable in the row
- activity and upper and lower bounds for the variable

The RHS Sensitivity Analysis Summary

The RHS sensitivity analysis summary includes the

- value of ϕ_{min}
- leaving variable when $\phi = \phi_{min}$
- objective value when $\phi = \phi_{min}$
- value of ϕ_{max}
- leaving variable when $\phi = \phi_{max}$
- objective value when $\phi = \phi_{max}$
- column number and name of each logical and structural variable
- variable's status when $\phi \in [\phi_{min}, \phi_{max}]$
- variable's reduced cost when $\phi \in [\phi_{min}, \phi_{max}]$
- value of right-hand-side constant when $\phi = \phi_{min}$
- activity of the variable when $\phi = \phi_{min}$
- value of right-hand-side constant when $\phi = \phi_{max}$
- activity of the variable when $\phi = \phi_{max}$

The Price Sensitivity Analysis Summary

The price sensitivity analysis summary includes the

- value of ϕ_{min}
- entering variable when $\phi = \phi_{min}$
- objective value when $\phi = \phi_{min}$
- value of ϕ_{max}
- entering variable when $\phi = \phi_{max}$
- objective value when $\phi = \phi_{max}$
- column number and name of each logical and structural variable
- variable's status when $\phi \in [\phi_{min}, \phi_{max}]$
- activity of the variable when $\phi \in [\phi_{min}, \phi_{max}]$
- price of the variable when $\phi = \phi_{min}$
- variable's reduced cost when $\phi = \phi_{min}$
- price of the variable when $\phi = \phi_{max}$
- variable's reduced cost when $\phi = \phi_{max}$

The Iteration Log

The iteration log includes the

- phase number
- iteration number in each phase
- name of the leaving variable
- name of the entering variable
- variable's reduced cost
- objective value

ODS Table and Variable Names

PROC LP assigns a name to each table it creates. You can use these names to select output tables when using the Output Delivery System (ODS).

Table 3.6. ODS Tables Produced in PROC LP

Table Name	Description	Statement/Option
ProblemSummary	Problem summary	default
SolutionSummary	Solution summary	default
VariableSummary	Variable summary	default
ConstraintSummary	Constraint summary	default
IterationLog	Iteration log	FLOW
IntegerIterationLog	Integer iteration log	default
PriceSensitivitySummary	Price sensitivity analysis summary	default, PRINT PRICESEN, or PRINT COLUMN/SENSITIVITY
PriceActivities	Price activities at ϕ_{min} and ϕ_{max}	default, PRINT PRICESEN, or PRINT COLUMN/SENSITIVITY
PriceActivity	Price activity at ϕ_{min} or ϕ_{max}	PRICEPHI= and PARAPRINT
PriceParametricLog	Price parametric programming log	PRICEPHI=
PriceRangeSummary	Price range analysis	RANGEPRICE or PRINT RANGEPRICE
RhsSensitivitySummary	RHS sensitivity analysis summary	default, PRINT RHSSEN, or PRINT ROW/SENSITIVITY
RhsActivities	RHS activities at ϕ_{min} and ϕ_{max}	default, PRINT RHSSEN, or PRINT ROW/SENSITIVITY
RhsActivity	RHS activity at ϕ_{min} or ϕ_{max}	RHSPHI= and PARAPRINT
RhsParametricLog	RHS parametric programming log	RHSPHI=
RhsRangeSummary	RHS range analysis	RANGERHS or PRINT RANGERHS
InfeasibilitySummary	Infeasible row or variable summary	default
InfeasibilityActivity	Variable activity in an infeasible row	default
CurrentTableau	Current tableau	TABLEAUPRINT or PRINT TABLEAU
Matrix	Technological matrix	PRINT MATRIX
MatrixPicture	Technological matrix picture	PRINT MATRIX/PICTURE
MatrixPictureLegend	Technological matrix picture legend	PRINT MATRIX/PICTURE

The following table lists the variable names of the preceding tables used in the ODS template of the LP procedure.

Table 3.7. Variable Names for the ODS Tables Produced in PROC LP

Table Name	Variables
VariableSummary	VarName, Status, Type, Price, Activity, ReducedCost
ConstraintSummary	Row, RowName, Type, SSCol, Rhs, Activity, Dual
IterationLog	Phase, Iteration, EnterVar, EnterCol, LeaveVar, LeaveCol, ReducedCost, ObjValue
IntegerIterationLog	Iteration, Problem, Condition, Objective, Branch, Value, SumOfInfeas, Active, Proximity
PriceActivities	Col, VarName, Status, Activity, MinPrice, MinReducedCost, MaxPrice, MaxReducedCost
PriceActivity	Col, VarName, Status, Activity, Price, ReducedCost
PriceParametricLog	LeaveVar, LeaveCol, EnterVar, EnterCol, ObjValue, CurrentPhi
PriceRangeSummary	Col, VarName, MinPrice, MinEnterVar, MinObj, MaxPrice, MaxEnterVar, MaxObj
RhsActivities	Col, VarName, Status, ReducedCost, MinRhs, MinActivity, MaxRhs, MaxActivity
RhsActivity	Col, VarName, Status, ReducedCost, Rhs, Activity,
RhsParametricLog	LeaveVar, LeaveCol, EnterVar, EnterCol, ObjValue, CurrentPhi
RhsRangeSummary	RowName, MinRhs, MinLeaveVar, MinObj, MaxRhs, MaxLeaveVar, MaxObj
InfeasibilityActivity	VarName, Coefficient, Activity, Lower, Upper

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the LP procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to allow the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the *SAS Companion* book for your operating system.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the *SAS Companion* book for your operating system.

Examples: LP Procedure

The following fourteen examples illustrate some of the capabilities of PROC LP. These examples, together with the other SAS/OR examples, can be found in the SAS sample library. A description of the features of PROC LP as shown in the examples are

Example 3.1	dense input format
Example 3.2	sparse input format
Example 3.3	the RANGEPRICE option to show you the range over which each objective coefficient can vary without changing the variables in the basis
Example 3.4	more sensitivity analysis and restarting a problem
Example 3.5	parametric programming
Example 3.6	special ordered sets
Example 3.7	goal programming
Example 3.8	integer programming
Example 3.9	an infeasible problem
Example 3.10	restarting integer programs
Example 3.11	controlling the search of the branch-and-bound tree
Example 3.12	matrix generation and report writing for an assignment problem
Example 3.13	matrix generation and report writing for a scheduling problem
Example 3.14	a multicommodity transshipment problem

Example 3.1. An Oil Blending Problem

The blending problem presented in the introduction is a good example for demonstrating some of the features of the LP procedure. Recall that a step in refining crude oil into finished oil products involves a distillation process that splits crude into various streams. Suppose that there are three types of crude available: Arabian light, Arabian heavy, and Brega. These are distilled into light naphtha, intermediate naphtha, and heating oil. Using one of two recipes, these in turn are blended into jet fuel.

Assume that you can sell as much fuel as is produced. What production strategy maximizes the profit from jet fuel sales? The following SAS code demonstrates a way of answering this question using linear programming. The SAS data set is a representation of the formulation for this model given in the introductory section.

```
data;
   input _row_ $17.
         a_light a_heavy brega naphthal naphthai heatingo jet_1
         jet_2  _type_ $ _rhs_;
   datalines;
profit             -175 -165 -205    0   0   0   300   300  max      .
naphtha_l_conv     .035 .030 .045   -1   0   0     0     0  eq       0
naphtha_i_conv     .100 .075 .135    0  -1   0     0     0  eq       0
heating_o_conv     .390 .300 .430    0   0  -1     0     0  eq       0
recipe_1              0    0    0    0  .3  .7    -1     0  eq       0
recipe_2              0    0    0   .2   0  .8     0    -1  eq       0
available           110  165   80    .   .   .     .     .  upperbd .
;

proc lp;
run;
```

The _ROW_ variable contains the names of the rows in the model; the variables A_LIGHT to JET_2 are the names of the structural variables in the model; the _TYPE_ variable contains the keywords that tell the LP procedure how to interpret each row in the model; and the _RHS_ variable gives the value of the right-hand-side constants.

The structural variables are interpreted as the quantity of each type of constituent or finished product. For example, the value of A_HEAVY in the solution is the amount of Arabian heavy crude to buy while the value of JET_1 in the solution is the amount of recipe 1 jet fuel that is produced. As discussed previously, the values given in the model data set are the technological coefficients whose interpretation depends on the model. In this example, the coefficient -175 in the PROFIT row for the variable A_LIGHT gives a cost coefficient (because the row with _ROW_=PROFIT has _TYPE_=MAX) for the structural variable A_LIGHT. This means that for each unit of Arabian heavy crude purchased, a cost of 175 units is incurred.

The coefficients 0.035, 0.100, and 0.390 for the A_LIGHT variable give the percentages of each unit of Arabian light crude that is distilled into the light naphtha, intermediate naphtha, and heating oil components. The 110 value in the row _ROW_=AVAILABLE gives the quantity of Arabian light that is available.

PROC LP produces the following Problem Summary output. Included in the summary is an identification of the objective, defined by the first observation of the problem data set; the right-hand-side variable, defined by the variable _RHS_; and the type identifier, defined by the variable _TYPE_. See Output 3.1.1.

Output 3.1.1. Problem Summary for the Oil Blending Problem

The LP Procedure

Problem Summary

Objective Function	Max profit
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	45.00
Variables	**Number**
Non-negative	5
Upper Bounded	3
Total	8
Constraints	**Number**
EQ	5
Objective	1
Total	6

The next section of output (Output 3.1.2) contains the Solution Summary, which indicates whether or not an optimal solution was found. In this example, the procedure terminates successfully (with an optimal solution), with 1544 as the value of the objective function. Also included in this section of output is the number of phase 1 and phase 2 iterations, the number of variables used in the initial basic feasible solution, and the time used to solve the problem. For several options specified in the PROC LP statement, the current option values are also displayed.

Output 3.1.2. Solution Summary for the Oil Blending Problem

The LP Procedure

Solution Summary

Terminated Successfully

Objective Value	1544
Phase 1 Iterations	0
Phase 2 Iterations	4
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	5
Time Used (seconds)	0
Number of Inversions	3
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The next section of output (Output 3.1.3) contains the Variable Summary. A line is displayed for each variable in the mathematical program with the variable name, the status of the variable in the solution, the type of variable, the variable's price coefficient, the activity of the variable in the solution, and the reduced cost for the variable. The status of a variable can be

BASIC	if the variable is a basic variable in the solution.
DEGEN	if the variable is a basic variable whose activity is at its input lower bound.
ALTER	if the variable is nonbasic and is basic in an alternate optimal solution.
LOWBD	if the variable is nonbasic and is at its lower bound.
UPPBD	if the variable is nonbasic and is at its upper bound.

The TYPE column shows how PROC LP interprets the variable in the problem data set. Types include the following:

NON-NEG	if the variable is a nonnegative variable with lower bound 0 and upper bound $+\infty$.
LOWERBD	if the variable has a lower bound specified in a LOWERBD observation and upper bound $+\infty$.
UPPERBD	if the variable has an upper bound that is less than $+\infty$ and lower bound 0. This upper bound is specified in an UPPERBD observation.
UPLOWBD	if the variable has a lower bound specified in a LOWERBD observation and an upper bound specified in an UPPERBD observation.
INTEGER	if the variable is constrained to take integer values. If this is the case, then it must also be upper and lower bounded.
BINARY	if the variable is constrained to take value 0 or 1.
UNRSTRT	if the variable is an unrestricted variable having bounds of $-\infty$ and $+\infty$.
SLACK	if the variable is a slack variable that PROC LP has appended to a LE constraint. For variables of this type, the variable name is the same as the name of the constraint (given in the ROW variable) for which this variable is the slack. A nonzero slack variable indicates that the constraint is not tight. The slack is the amount by which the right-hand side of the constraint exceeds the left-hand side.

SURPLUS — if the variable is a surplus variable that PROC LP has appended to a GE constraint. For variables of this type, the variable name is the same as the name of the constraint (given in the ROW variable) for which this variable is the surplus. A nonzero surplus variable indicates that the constraint is not tight. The surplus is the amount by which the left-hand side of the constraint exceeds the right-hand side.

The Variable Summary gives the value of the structural variables at optimality. In this example, it tells you how to produce the jet fuel to maximize your profit. You should buy 110 units of A_LIGHT and 80 units of BREGA. These are used to make 7.45 units of NAPHTHAL, 21.8 units of NAPHTHAI, and 77.3 units of HEATINGO. These in turn are used to make 60.65 units of JET_1 using recipe 1 and 63.33 units of JET_2 using recipe 2.

Output 3.1.3. Variable Summary for the Oil Blending Problem

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	a_light	UPPBD	UPPERBD	-175	110	11.6
2	a_heavy		UPPERBD	-165	0	-21.45
3	brega	UPPBD	UPPERBD	-205	80	3.35
4	naphthal	BASIC	NON-NEG	0	7.45	0
5	naphthai	BASIC	NON-NEG	0	21.8	0
6	heatingo	BASIC	NON-NEG	0	77.3	0
7	jet_1	BASIC	NON-NEG	300	60.65	0
8	jet_2	BASIC	NON-NEG	300	63.33	0

The reduced cost associated with each nonbasic variable is the marginal value of that variable if it is brought into the basis. In other words, the objective function value would (assuming no constraints were violated) increase by the reduced cost of a nonbasic variable if that variable's value increased by one. Similarly, the objective function value would (assuming no constraints were violated) decrease by the reduced cost of a nonbasic variable if that variable's value were decreased by one. Basic variables always have a zero reduced cost. At optimality, for a maximization problem, nonbasic variables that are not at an upper bound have nonpositive reduced costs (for example, A_HEAVY has a reduced cost of -21.45). The objective would decrease if they were to increase beyond their optimal values. Nonbasic variables at upper bounds have nonnegative reduced costs, showing that increasing the upper bound (if the reduced cost is not zero) does not decrease the objective. For a nonbasic variable at its upper bound, the reduced cost is the marginal value of increasing its upper bound, often called its shadow price.

For minimization problems, the definition of reduced costs remains the same but the conditions for optimality change. For example, at optimality the reduced costs of all non-upper-bounded variables are nonnegative, and the reduced costs of upper-bounded variables at their upper bound are nonpositive.

The next section of output (Output 3.1.4) contains the Constraint Summary. For each constraint row, free row, and objective row, a line is displayed in the Constraint Summary. Included on the line are the constraint name, the row type, the slack or surplus variable associated with the row, the right-hand-side constant associated with the row, the activity of the row (not including the activity of the slack and surplus variables), and the dual activity (shadow prices).

A dual variable is associated with each constraint row. At optimality, the value of this variable, the dual activity, tells you the marginal value of the right-hand-side constant. For each unit increase in the right-hand-side constant, the objective changes by this amount. This quantity is also known as the shadow price. For example, the marginal value for the right-hand-side constant of constraint HEATING_O_CONV is -450.

Output 3.1.4. Constraint Summary for the Oil Blending Problem

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	profit	OBJECTVE	.	0	1544	.
2	naphtha_l_conv	EQ	.	0	0	-60
3	naphtha_i_conv	EQ	.	0	0	-90
4	heating_o_conv	EQ	.	0	0	-450
5	recipe_1	EQ	.	0	0	-300
6	recipe_2	EQ	.	0	0	-300

Example 3.2. A Sparse View of the Oil Blending Problem

Typically, mathematical programming models are very sparse. This means that only a small percentage of the coefficients are nonzero. The sparse problem input is ideal for these models. The oil blending problem in the section "An Introductory Example" on page 164 has a sparse form. This example shows the same problem in a sparse form with the data given in a different order. In addition to representing the problem in a concise form, the sparse format

- allows long column names
- enables easy matrix generation (see Example 3.12, Example 3.13, and Example 3.14)
- is compatible with MPS sparse format

The model in the sparse format is solved by invoking PROC LP with the SPARSEDATA option as follows.

```
data oil;
   format _type_ $8. _col_ $14. _row_ $16. ;
   input _type_ $ _col_ $ _row_ $ _coef_ ;
   datalines;
```

```
max      .             profit              .
.        arabian_light profit           -175
.        arabian_heavy profit           -165
.        brega         profit           -205
.        jet_1         profit            300
.        jet_2         profit            300
eq       .             napha_l_conv        .
.        arabian_light napha_l_conv     .035
.        arabian_heavy napha_l_conv     .030
.        brega         napha_l_conv     .045
.        naphtha_light napha_l_conv       -1
eq       .             napha_i_conv        .
.        arabian_light napha_i_conv     .100
.        arabian_heavy napha_i_conv     .075
.        brega         napha_i_conv     .135
.        naphtha_inter napha_i_conv       -1
eq       .             heating_oil_conv    .
.        arabian_light heating_oil_conv .390
.        arabian_heavy heating_oil_conv .300
.        brega         heating_oil_conv .430
.        heating_oil   heating_oil_conv   -1
eq       .             recipe_1            .
.        naphtha_inter recipe_1           .3
.        heating_oil   recipe_1           .7
eq       .             recipe_2            .
.        jet_1         recipe_1           -1
.        naphtha_light recipe_2           .2
.        heating_oil   recipe_2           .8
.        jet_2         recipe_2           -1
upperbd .              available           .
.        arabian_light available          110
.        arabian_heavy available          165
.        brega         available           80
;

proc lp data=oil sparsedata;
run;
```

The output from PROC LP follows.

Output 3.2.1. Output for the Sparse Oil Blending Problem

The LP Procedure

Problem Summary

Objective Function	Max profit
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	45.00

Variables	Number
Non-negative	5
Upper Bounded	3
Total	8

Constraints	Number
EQ	5
Objective	1
Total	6

The LP Procedure

Solution Summary

Terminated Successfully

Objective Value	1544
Phase 1 Iterations	0
Phase 2 Iterations	5
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	5
Time Used (seconds)	0
Number of Inversions	3
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	arabian_heavy		UPPERBD	-165	0	-21.45
2	arabian_light	UPPBD	UPPERBD	-175	110	11.6
3	brega	UPPBD	UPPERBD	-205	80	3.35
4	heating_oil	BASIC	NON-NEG	0	77.3	0
5	jet_1	BASIC	NON-NEG	300	60.65	0
6	jet_2	BASIC	NON-NEG	300	63.33	0
7	naphtha_inter	BASIC	NON-NEG	0	21.8	0
8	naphtha_light	BASIC	NON-NEG	0	7.45	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	profit	OBJECTVE	.	0	1544	.
2	napha_l_conv	EQ	.	0	0	-60
3	napha_i_conv	EQ	.	0	0	-90
4	heating_oil_conv	EQ	.	0	0	-450
5	recipe_1	EQ	.	0	0	-300
6	recipe_2	EQ	.	0	0	-300

Example 3.3. Sensitivity Analysis: Changes in Objective Coefficients

Simple solution of a linear program is often not enough. A manager needs to evaluate how sensitive the solution is to changing assumptions. The LP procedure provides several tools that are useful for "what if," or sensitivity, analysis. One tool studies the effects of changes in the objective coefficients.

For example, in the oil blending problem, the cost of crude and the selling price of jet fuel can be highly variable. If you want to know the range over which each objective coefficient can vary without changing the variables in the basis, you can use the RANGEPRICE option in the PROC LP statement.

```
proc lp data=oil sparsedata
        rangeprice primalout=solution;
run;
```

In addition to the Problem and Solution summaries, the LP procedure produces a Price Range Summary, shown in Output 3.3.1.

For each structural variable, the upper and lower ranges of the price (objective function coefficient) and the objective value are shown. The blocking variables, those variables that would enter the basis if the objective coefficient were perturbed further, are also given. For example, the output shows that if the cost of ARABIAN_LIGHT

crude were to increase from 175 to 186.6 per unit (remember that you are maximizing profit so the ARABIAN_LIGHT objective coefficient would decrease from -175 to -186.6), then it would become optimal to use less of this crude for any fractional increase in its cost. Increasing the unit cost to 186.6 would drive its reduced cost to zero. Any additional increase would drive its reduced cost negative and would destroy the optimality conditions; thus, you would want to use less of it in your processing. The output shows that, at the point where the reduced cost is zero, you would only be realizing a profit of 268 = 1544 - (110 × 11.6) and that ARABIAN_LIGHT enters the basis, that is, leaves its upper bound. On the other hand, if the cost of ARABIAN_HEAVY were to decrease to 143.55, you would want to stop using the formulation of 110 units of ARABIAN_LIGHT and 80 units of BREGA and switch to a production scheme that included ARABIAN_HEAVY, in which case the profit would increase from the 1544 level.

Output 3.3.1. Price Range Summary for the Oil Blending Problem

The LP Procedure

Price Range Analysis

Col	Variable Name	Minimum Phi Price Entering		Objective
1	arabian_heavy	-INFINITY	.	1544
2	arabian_light	-186.6	arabian_light	268
3	brega	-208.35	brega	1276
4	heating_oil	-7.790698	brega	941.77907
5	jet_1	290.19034	brega	949.04392
6	jet_2	290.50992	brega	942.99292
7	naphtha_inter	-24.81481	brega	1003.037
8	naphtha_light	-74.44444	brega	989.38889

Price Range Analysis

Col	Maximum Phi Price Entering		Objective
1	-143.55	arabian_heavy	1544
2	INFINITY	.	INFINITY
3	INFINITY	.	INFINITY
4	71.5	arabian_heavy	7070.95
5	392.25806	arabian_heavy	7139.4516
6	387.19512	arabian_heavy	7066.0671
7	286	arabian_heavy	7778.8
8	715	arabian_heavy	6870.75

Note that in the PROC LP statement, the PRIMALOUT=SOLUTION option was given. This caused the procedure to save the optimal solution in a SAS data set named SOLUTION. This data set can be used to perform further analysis on the problem without having to restart the solution process. Example 3.4 shows how this is done. A display of the data follows in Output 3.3.2.

Output 3.3.2. The PRIMALOUT= Data Set for the Oil Blending Problem

Obs	_OBJ_ID_	_RHS_	VAR	_TYPE_	_STATUS_	_LB_	_VALUE_	_UB_	_PRICE_	_RC_
1	profit	_rhs_	arabian_heavy	UPPERBD		0	0.00	165	-165	-21.45
2	profit	_rhs_	arabian_light	UPPERBD	_UPPER_	0	110.00	110	-175	11.60
3	profit	_rhs_	brega	UPPERBD	_UPPER_	0	80.00	80	-205	3.35
4	profit	_rhs_	heating_oil	NON-NEG	_BASIC_	0	77.30	1.7977E308	0	0.00
5	profit	_rhs_	jet_1	NON-NEG	_BASIC_	0	60.65	1.7977E308	300	0.00
6	profit	_rhs_	jet_2	NON-NEG	_BASIC_	0	63.33	1.7977E308	300	0.00
7	profit	_rhs_	naphtha_inter	NON-NEG	_BASIC_	0	21.80	1.7977E308	0	-0.00
8	profit	_rhs_	naphtha_light	NON-NEG	_BASIC_	0	7.45	1.7977E308	0	0.00
9	profit	_rhs_	PHASE_1_OBJECTIVE	OBJECT	_DEGEN_	0	0.00	0	0	0.00
10	profit	_rhs_	profit	OBJECT	_BASIC_	0	1544.00	1.7977E308	0	0.00

Example 3.4. Additional Sensitivity Analysis

The objective coefficient ranging analysis, discussed in the last example, is useful for assessing the effects of changing costs and returns on the optimal solution if each objective function coefficient is modified in isolation. However, this is often not the case.

Suppose you anticipate that the cost of crude will be increasing and you want to examine how that will affect your optimal production plans. Furthermore, you estimate that if the price of ARABIAN_LIGHT goes up by 1 unit, then the price of ARABIAN_HEAVY will rise by 1.2 units and the price of BREGA will increase by 1.5 units. However, you plan on passing some of your increased overhead on to your jet fuel customers, and you decide to increase the price of jet fuel 1 unit for each unit of increased cost of ARABIAN_LIGHT.

An examination of the solution sensitivity to changes in the cost of crude is a two-step process. First, add the information on the proportional rates of change in the crude costs and the jet fuel price to the problem data set. Then, invoke the LP procedure. The following program accomplishes this. First, it adds a new row, named CHANGE, to the model. It gives this row a type of PRICESEN. That tells PROC LP to perform objective function coefficient sensitivity analysis using the given rates of change. The program then invokes PROC LP to perform the analysis. Notice that the PRIMALIN=SOLUTION option is used in the PROC LP statement. This tells the LP procedure to use the saved solution. Although it is not necessary to do this, it will eliminate the need for PROC LP to re-solve the problem and can save computing time.

```
data sen;
   format _type_ $8. _col_ $14. _row_ $6.;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
pricesen .             change   .
.        arabian_light change   1
.        arabian_heavy change 1.2
.        brega         change 1.5
.        jet_1         change  -1
.        jet_2         change  -1
;

data;
   set oil sen;
run;

proc lp sparsedata primalin=solution;
run;
```

Output 3.4.1 shows the range over which the current basic solution remains optimal so that the current production plan need not change. The objective coefficients are modified by adding ϕ times the change vector given in the SEN data set, where ϕ ranges from a minimum of -4.15891 to a maximum of 29.72973. At the minimum value of ϕ, the profit decreases to 1103.073. This value of ϕ corresponds to an increase in the cost of ARABIAN_HEAVY to 169.99 (namely, $-175 + \phi \times 1.2$), ARABIAN_LIGHT to 179.16 ($= -175 + \phi \times 1$), and BREGA to 211.24 ($= -205 + \phi \times 1.5$), and corresponds to an increase in the price of JET_1 and JET_2 to 304.16 (= 300 + $\phi \times$ (-1)). These values can be found in the Price column under the section labeled Minimum Phi.

Output 3.4.1. The Price Sensitivity Analysis Summary for the Oil Blending Problem

The LP Procedure

Price Sensitivity Analysis Summary
Sensitivity Vector change

Minimum Phi	-4.158907511
Entering Variable	brega
Optimal Objective	1103.0726257
Maximum Phi	29.72972973
Entering Variable	arabian_heavy
Optimal Objective	4695.9459459

				----Minimum Phi----		----Maximum Phi----	
Col	Variable Name	Status	Activity	Price	Reduced Cost	Price	Reduced Cost
1	arabian_heavy		0	-169.9907	-24.45065	-129.3243	0
2	arabian_light	UPPBD	110	-179.1589	10.027933	-145.2703	22.837838
3	brega	UPPBD	80	-211.2384	0	-160.4054	27.297297
4	heating_oil	BASIC	77.3	0	0	0	0
5	jet_1	BASIC	60.65	304.15891	0	270.27027	0
6	jet_2	BASIC	63.33	304.15891	0	270.27027	0
7	naphtha_inter	BASIC	21.8	0	0	0	0
8	naphtha_light	BASIC	7.45	0	0	0	0

The Price Sensitivity Analysis Summary also shows the effects of lowering the cost of crude and lowering the price of jet fuel. In particular, at the maximum ϕ of 29.72973, the current optimal production plan yields a profit of 4695.95. Any increase or decrease in ϕ beyond the limits given results in a change in the production plan. More precisely, the columns that constitute the basis change.

Example 3.5. Price Parametric Programming for the Oil Blending Problem

This example continues to examine the effects of a change in the cost of crude and the selling price of jet fuel. Suppose that you know the cost of ARABIAN_LIGHT crude is likely to increase 30 units, with the effects on oil and fuel prices as described in Example 3.4. The analysis in the last example only accounted for an increase of a little over 4 units (because the minimum ϕ was -4.15891). Because an increase in the cost of ARABIAN_LIGHT beyond 4.15891 units requires a change in the optimal basis, it may require a change in the optimal production strategy as well. This type of analysis, where you want to find how the solution changes with changes in the objective function coefficients or right-hand-side vector, is called parametric programming.

You can answer this question by using the PRICEPHI= option in the PROC LP statement. The following program instructs PROC LP to continually increase the cost of the crudes and the return from jet fuel using the ratios given previously, until the cost of ARABIAN_LIGHT increases at least 30 units.

```
proc lp sparsedata primalin=solution pricephi=-30;
run;
```

The PRICEPHI= option in the PROC LP statement tells PROC LP to perform parametric programming on any price change vectors specified in the problem data set. The value of the PRICEPHI= option tells PROC LP how far to change the value of ϕ and in what direction. A specification of PRICEPHI=-30 tells PROC LP to continue pivoting until the problem has objective function equal to (original objective function value) $-$ 30 $\times$ (change vector).

Output 3.5.1 shows the result of this analysis. The first page is the Price Sensitivity Analysis Summary, as discussed in Example 3.4. The next page is an accounting for the change in basis as a result of decreasing ϕ beyond -4.1589. It shows that BREGA left the basis at an upper bound and entered the basis at a lower bound. The interpretation of these basis changes can be difficult (Hadley 1962; Dantzig 1963).

The last page of output shows the optimal solution at the displayed value of ϕ, namely -30.6878. At an increase of 30.6878 units in the cost of ARABIAN_LIGHT and the related changes to the other crudes and the jet fuel, it is optimal to modify the production of jet fuel as shown in the activity column. Although this plan is optimal, it results in a profit of 0. This may suggest that the ratio of a unit increase in the price of jet fuel per unit increase in the cost of ARABIAN_LIGHT is lower than desirable.

Output 3.5.1. Price Parametric Programming for the Oil Blending Problem

The LP Procedure

Price Sensitivity Analysis Summary
Sensitivity Vector change

Minimum Phi	-4.158907511
Entering Variable	brega
Optimal Objective	1103.0726257
Maximum Phi	29.72972973
Entering Variable	arabian_heavy
Optimal Objective	4695.9459459

					Minimum Phi		Maximum Phi	
Col	Variable Name	Status	Activity	Price	Reduced Cost	Price	Reduced Cost
1	arabian_heavy		0	-169.9907	-24.45065	-129.3243	0
2	arabian_light	UPPBD	110	-179.1589	10.027933	-145.2703	22.837838
3	brega	UPPBD	80	-211.2384	0	-160.4054	27.297297
4	heating_oil	BASIC	77.3	0	0	0	0
5	jet_1	BASIC	60.65	304.15891	0	270.27027	0
6	jet_2	BASIC	63.33	304.15891	0	270.27027	0
7	naphtha_inter	BASIC	21.8	0	0	0	0
8	naphtha_light	BASIC	7.45	0	0	0	0

The LP Procedure

Price Parametric Programming Log
Sensitivity Vector change

Leaving Variable	Entering Variable	Objective	Current Phi
brega	brega	1103.0726	-4.158908

The LP Procedure

Price Sensitivity Analysis Summary
Sensitivity Vector change

Minimum Phi	-30.68783069
Entering Variable	arabian_light
Optimal Objective	0

				----Minimum Phi----	
Col	Variable Name	Status	Activity	Price	Reduced Cost
1	arabian_heavy		0	-201.8254	-43.59127
2	arabian_light	ALTER	110	-205.6878	0
3	brega		0	-251.0317	-21.36905
4	heating_oil	BASIC	42.9	0	0
5	jet_1	BASIC	33.33	330.68783	0
6	jet_2	BASIC	35.09	330.68783	0
7	naphtha_inter	BASIC	11	0	0
8	naphtha_light	BASIC	3.85	0	0

What is the optimal return if ϕ is exactly -30? Because the change in the objective is linear as a function of ϕ, you can calculate the objective for any value of ϕ between those given by linear interpolation. For example, for any ϕ between -4.1589 and -30.6878, the optimal objective value is

$$\phi \times (1103.0726 - 0)/(-4.1589 - 30.6878) + b$$

where

$$b = 30.6878 \times (1103.0726 - 0)/(-4.1589 - 30.6878)$$

For ϕ =-30, this is 28.5988.

Example 3.6. Special Ordered Sets and the Oil Blending Problem

Often managers want to evaluate the cost of making a choice among alternatives. In particular, they want to make the most profitable choice. Suppose that only one oil crude can be used in the production process. This identifies a set of variables of which only one can be above its lower bound. This additional restriction could be included in the model by adding a binary integer variable for each of the three crudes. Constraints would be needed that would drive the appropriate binary variable

to 1 whenever the corresponding crude is used in the production process. Then a constraint limiting the total of these variables to only one would be added. A similar formulation for a fixed charge problem is shown in Example 3.8.

The SOSLE type implicitly does this. The following DATA step adds a row to the model that identifies which variables are in the set. The SOSLE type tells the LP procedure that only one of the variables in this set can be above its lower bound. If you use the SOSEQ type, it tells PROC LP that exactly one of the variables in the set must be above its lower bound. Only integer variables can be in an SOSEQ set.

```
data special;
   format _type_ $6. _col_ $14. _row_ $8. ;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
SOSLE .             special .
.     arabian_light special 1
.     arabian_heavy special 1
.     brega         special 1
;

data;
   set oil special;
run;

proc lp sparsedata;
run;
```

Output 3.6.1 includes an Integer Iteration Log. This log shows the progress that PROC LP is making in solving the problem. This is discussed in some detail in Example 3.8.

Output 3.6.1. The Oil Blending Problem with a Special Ordered Set

```
                          The LP Procedure

                        Problem Summary

Objective Function              Max profit
Rhs Variable                        _rhs_
Type Variable                       _type_
Problem Density (%)                  45.00

Variables                          Number

Non-negative                            5
Upper Bounded                           3

Total                                   8

Constraints                        Number

EQ                                      5
Objective                               1

Total                                   6
```

The LP Procedure

Integer Iteration Log

Iter	Problem	Condition	Objective	Branched	Value	Sinfeas	Active	Proximity
1	0	ACTIVE	1544	arabian_light	110	0	2	.
2	-1	SUBOPTIMAL	1276	.	.	.	1	268
3	1	FATHOMED	268	.	.	.	0	.

The LP Procedure

Solution Summary

Integer Optimal Solution

Objective Value	1276
Phase 1 Iterations	0
Phase 2 Iterations	5
Phase 3 Iterations	0
Integer Iterations	3
Integer Solutions	1
Initial Basic Feasible Variables	5
Time Used (seconds)	0
Number of Inversions	5
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	arabian_heavy		UPPERBD	-165	0	-21.45
2	arabian_light	UPPBD	UPPERBD	-175	110	11.6
3	brega		UPPERBD	-205	0	3.35
4	heating_oil	BASIC	NON-NEG	0	42.9	0
5	jet_1	BASIC	NON-NEG	300	33.33	0
6	jet_2	BASIC	NON-NEG	300	35.09	0
7	naphtha_inter	BASIC	NON-NEG	0	11	0
8	naphtha_light	BASIC	NON-NEG	0	3.85	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	profit	OBJECTVE	.	0	1276	.
2	napha_l_conv	EQ	.	0	0	-60
3	napha_i_conv	EQ	.	0	0	-90
4	heating_oil_conv	EQ	.	0	0	-450
5	recipe_1	EQ	.	0	0	-300
6	recipe_2	EQ	.	0	0	-300

The solution shows that only the ARABIAN_LIGHT crude is purchased. The requirement that only one crude be used in the production is met, and the profit is 1276. This tells you that the value of purchasing crude from an additional source, namely BREGA, is worth $1544 - 1276 = 268$.

Example 3.7. Goal-Programming a Product Mix Problem

This example shows how to use PROC LP to solve a linear goal-programming problem. PROC LP has the ability to solve a series of linear programs, each with a new objective function. These objective functions are ordered by priority. The first step is to solve a linear program with the highest priority objective function constrained only by the formal constraints in the model. Then, the problem with the next highest priority objective function is solved, constrained by the formal constraints in the model and by the value that the highest priority objective function realized. That is, the second problem optimizes the second highest priority objective function among the alternate optimal solutions to the first optimization problem. The process continues until a linear program is solved for each of the objectives.

This technique is useful for differentiating among alternate optimal solutions to a linear program. It also fits into the formal paradigm presented in goal programming. In goal programming, the objective functions typically take on the role of driving a linear function of the structural variables to meet a target level as closely as possible. The details of this can be found in many books on the subject, including Ignizio (1976).

Consider the following problem taken from Ignizio (1976). A small paint company manufactures two types of paint, latex and enamel. In production, the company uses 10 hours of labor to produce 100 gallons of latex and 15 hours of labor to produce 100 gallons of enamel. Without hiring outside help or requiring overtime, the company has 40 hours of labor available each week. Furthermore, each paint generates a profit at the rate of $1.00 per gallon. The company has the following objectives listed in decreasing priority:

- avoid the use of overtime
- achieve a weekly profit of $1000
- produce at least 700 gallons of enamel paint each week

The program to solve this problem follows.

```
data object;
   input _row_ $ latex enamel n1 n2 n3 p1 p2 p3 _type_ $ _rhs_;
   datalines;
overtime   .   .   .   .   .  1   .   . min   1
profit     .   .   .  1   .   .   .   . min   2
enamel     .   .   .   .  1   .   .   . min   3
overtime  10  15  1   .   . -1   .   . eq   40
profit   100 100   .  1   .   . -1   . eq 1000
enamel     .   1   .   .  1   .   . -1 eq    7
;

proc lp data=object goalprogram;
run;
```

The data set called OBJECT contains the model. Its first three observations are the objective rows, and the next three observations are the constraints. The values in the right-hand-side variable _RHS_ in the objective rows give the priority of the objectives. The objective in the first observation with _ROW_='OVERTIME' has the highest priority, the objective named PROFIT has the next highest, and the objective named ENAMEL has the lowest. Note that the value of the right-hand-side variable determines the priority, not the order, in the data set.

Because this example is set in the formal goal-programming scheme, the model has structural variables representing negative (n1, n2, and n3) and positive (p1, p2, and p3) deviations from target levels. For example, n1+p1 is the deviation from the objective of avoiding the use of overtime and underusing the normal work time, namely using exactly 40 work hours. The other objectives are handled similarly.

Notice that the PROC LP statement includes the GOALPROGRAM option. Without this option, the procedure would solve three separate problems: one for each of the three objective functions. In that case, however, the procedure would not constrain the second and third programs using the results of the preceding programs; also, the values 1, 2, and 3 for _RHS_ in the objective rows would have no effect.

Output 3.7.1 shows the solution of the goal program, apparently as three linear program outputs. However, examination of the constraint summaries in the second and third problems shows that the constraints labeled by the objectives OVERTIME and PROFIT have type FIXEDOBJ. This indicates that these objective rows have become constraints in the subsequent problems.

Output 3.7.1. Goal Programming

The LP Procedure

Problem Summary

Objective Function	Min overtime
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	45.83

Variables	Number
Non-negative	8
Total	8

Constraints	Number
EQ	3
Objective	3
Total	6

The LP Procedure

Solution Summary

Terminated Successfully

Objective Value	0
Phase 1 Iterations	2
Phase 2 Iterations	0
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	7
Time Used (seconds)	0
Number of Inversions	2
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	latex	ALTER	NON-NEG	0	0	0
2	enamel	ALTER	NON-NEG	0	0	0
3	n1	BASIC	NON-NEG	0	40	0
4	n2	BASIC	NON-NEG	0	1000	0
5	n3	BASIC	NON-NEG	0	7	0
6	p1		NON-NEG	1	0	1
7	p2	ALTER	NON-NEG	0	0	0
8	p3	ALTER	NON-NEG	0	0	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	overtime	OBJECTVE	.	0	0	.
2	profit	FREE_OBJ	.	0	1000	.
3	enamel	FREE_OBJ	.	0	7	.
4	overtime	EQ	.	40	40	0
5	profit	EQ	.	1000	1000	0
6	enamel	EQ	.	7	7	0

The LP Procedure

Problem Summary

Objective Function	Min profit
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	45.83

Variables	Number
Non-negative	8
Total	8

Constraints	Number
EQ	3
Objective	3
Total	6

The LP Procedure

Solution Summary

Terminated Successfully

Objective Value	600
Phase 1 Iterations	0
Phase 2 Iterations	3
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	7
Time Used (seconds)	0
Number of Inversions	5
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	latex	BASIC	NON-NEG	0	4	0
2	enamel		NON-NEG	0	0	50
3	n1		NON-NEG	0	0	10
4	n2	BASIC	NON-NEG	1	600	0
5	n3	BASIC	NON-NEG	0	7	0
6	p1	DEGEN	NON-NEG	0	0	0
7	p2		NON-NEG	0	0	1
8	p3	ALTER	NON-NEG	0	0	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	overtime	FIXEDOBJ	.	0	0	.
2	profit	OBJECTVE	.	0	600	.
3	enamel	FREE_OBJ	.	0	7	.
4	overtime	EQ	.	40	40	-10
5	profit	EQ	.	1000	1000	1
6	enamel	EQ	.	7	7	0

The LP Procedure

Problem Summary

Objective Function	Min enamel
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	45.83

Variables	Number
Non-negative	8
Total	8

Constraints	Number
EQ	3
Objective	3
Total	6

The LP Procedure

Solution Summary

Terminated Successfully

Objective Value	7
Phase 1 Iterations	0
Phase 2 Iterations	1
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	7
Time Used (seconds)	0
Number of Inversions	8
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	latex	BASIC	NON-NEG	0	4	0
2	enamel	DEGEN	NON-NEG	0	0	0
3	n1		NON-NEG	0	0	0.2
4	n2	BASIC	NON-NEG	0	600	0
5	n3	BASIC	NON-NEG	1	7	0
6	p1	DEGEN	NON-NEG	0	0	0
7	p2		NON-NEG	0	0	0.02
8	p3		NON-NEG	0	0	1

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	overtime	FIXEDOBJ	.	0	0	.
2	profit	FIXEDOBJ	.	0	600	.
3	enamel	OBJECTVE	.	0	7	.
4	overtime	EQ	.	40	40	-0.2
5	profit	EQ	.	1000	1000	0.02
6	enamel	EQ	.	7	7	1

The solution to the last linear program shows a value of 4 for the variable LATEX and a value of 0 for the variable ENAMEL. This tells you that the solution to the linear goal program is to produce 400 gallons of latex and no enamel paint.

The values of the objective functions in the three linear programs tell you whether you can achieve the three objectives. The activities of the constraints labeled OVERTIME, PROFIT, and ENAMEL tell you values of the three linear program objectives. Because the first linear programming objective OVERTIME is 0, the highest priority objective, which is to avoid using additional labor, is accomplished. However, because the second and third objectives are nonzero, the second and third priority objectives are not satisfied completely. The PROFIT objective is 600. Because the PROFIT objective is to minimize the negative deviation from the profit constraint, this means that a profit of only $400 = 1000 - 600$ is realized. Similarly, the ENAMEL objective is 7, indicating that there is a negative deviation from the ENAMEL target of 7 units.

Example 3.8. A Simple Integer Program

Recall the linear programming problem presented in Chapter 1, "Introduction to Optimization." In that problem, a firm produces two products, chocolates and gumdrops, that are processed by four processes: cooking, color/flavor, condiments, and packaging. The objective is to determine the product mix that maximizes the profit to the firm while not exceeding manufacturing capacities. The problem is extended to demonstrate a use of integer-constrained variables.

Suppose that you must manufacture only one of the two products. In addition, there is a setup cost of 100 if you make the chocolates and 75 if you make the gumdrops. To identify which product will maximize profit, you define two zero-one integer variables, ICHOCO and IGUMDR, and you also define two new constraints, CHOCOLATE and GUM. The constraint labeled CHOCOLATE forces ICHOCO to equal one when chocolates are manufactured. Similarly, the constraint labeled GUM forces IGUMDR to equal 1 when gumdrops are manufactured. Also, you should include a constraint labeled ONLY_ONE that requires the sum of ICHOCO and IGUMDR to equal 1. (Note that this could be accomplished more simply by including ICHOCO and IGUMDR in a SOSEQ set.) Since ICHOCO and IGUMDR are integer variables, this constraint eliminates the possibility of both products being manufactured. Notice the coefficients -10000, which are used to force ICHOCO or IGUMDR to 1 whenever CHOCO and GUMDR are nonzero. This technique, which is often used in integer programming, can cause severe numerical problems. If this driving coefficient is too large, then arithmetic overflows and underflow may result. If the driving coefficient is too small, then the integer variable may not be driven to 1 as desired by the modeler.

The objective coefficients of the integer variables ICHOCO and IGUMDR are the negatives of the setup costs for the two products. The following is the data set that describes this problem and the call to PROC LP to solve it:

```
data;
   format _row_ $10. ;
   input _row_ $ choco gumdr ichoco igumdr _type_ $ _rhs_;
   datalines;
object         .25    .75   -100    -75 max          .
cooking         15     40      0      0 le       27000
color            0  56.25      0      0 le       27000
package      18.75      0      0      0 le       27000
condiments      12     50      0      0 le       27000
chocolate        1      0 -10000      0 le           0
gum              0      1      0 -10000 le           0
only_one         0      0      1      1 eq           1
binary           .      .      1      2 binary       .
;

proc lp;
run;
```

The solution shows that gumdrops are produced. See Output 3.8.1.

Output 3.8.1. Summaries and an Integer Programming Iteration Log

The LP Procedure

Problem Summary

Objective Function	Max object
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	25.71

Variables	Number
Non-negative	2
Binary	2
Slack	6
Total	10

Constraints	Number
LE	6
EQ	1
Objective	1
Total	8

The LP Procedure

Integer Iteration Log

Iter	Problem	Condition	Objective	Branched	Value	Sinfeas	Active	Proximity
1	0	ACTIVE	397.5	ichoco	0.1	0.2	2	.
2	-1	SUBOPTIMAL	260	.	.	.	1	70
3	1	SUBOPTIMAL	285	.	.	.	0	.

The LP Procedure

Solution Summary

Integer Optimal Solution

Objective Value	285
Phase 1 Iterations	0
Phase 2 Iterations	5
Phase 3 Iterations	5
Integer Iterations	3
Integer Solutions	2
Initial Basic Feasible Variables	9
Time Used (seconds)	0
Number of Inversions	5
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	choco	DEGEN	NON-NEG	0.25	0	0
2	gumdr	BASIC	NON-NEG	0.75	480	0
3	ichoco		BINARY	-100	0	2475
4	igumdr	BASIC	BINARY	-75	1	0
5	cooking	BASIC	SLACK	0	7800	0
6	color		SLACK	0	0	-0.013333
7	package	BASIC	SLACK	0	27000	0
8	condiments	BASIC	SLACK	0	3000	0
9	chocolate		SLACK	0	0	-0.25
10	gum	BASIC	SLACK	0	9520	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	object	OBJECTVE	.	0	285	.
2	cooking	LE	5	27000	19200	0
3	color	LE	6	27000	27000	0.0133333
4	package	LE	7	27000	0	0
5	condiments	LE	8	27000	24000	0
6	chocolate	LE	9	0	0	0.25
7	gum	LE	10	0	-9520	0
8	only_one	EQ	.	1	1	-75

The branch-and-bound tree can be reconstructed from the information contained in the integer iteration log. The column labeled Iter numbers the integer iterations. The column labeled Problem identifies the Iter number of the parent problem from which the current problem is defined. For example, Iter=2 has Problem=-1. This means that problem 2 is a direct descendant of problem 1. Furthermore, because problem 1 branched on ICHOCO, you know that problem 2 is identical to problem 1 with an additional constraint on variable ICHOCO. The minus sign in the Problem=-1 in Iter=2 tells you that the new constraint on variable ICHOCO is a lower bound. Moreover, because Value=0.1 in Iter=1, you know that ICHOCO=0.1 in Iter=1 so that the added constraint in Iter=2 is ICHOCO $\geq \lceil 0.1 \rceil$. In this way, the information in the log can be used to reconstruct the branch-and-bound tree. In fact, when you save an ACTIVEOUT= data set, it contains information in this format that is used to reconstruct the tree when you restart a problem using the ACTIVEIN= data set. See Example 3.10.

Note that if you defined a SOSEQ special ordered set containing the variables CHOCO and GUMDR, the integer variables ICHOCO and IGUMDR and the three associated constraints would not have been needed.

Example 3.9. An Infeasible Problem

This is an example of the Infeasible Information Summary that is displayed when an infeasible problem is encountered. Consider the following problem:

$$\begin{array}{ll} \max & x + y + z + w \\ \text{subject to} & x + 3y + 2z + 4w \leq 5 \\ & 3x + y + 2z + w \leq 4 \\ & 5x + 3y + 3z + 3w = 9 \\ & x,\ y,\ z,\ w \geq 0 \end{array}$$

Examination of this problem reveals that it is unsolvable. Consequently, PROC LP identifies it as infeasible. The following program attempts to solve it.

```
data infeas;
   format _id_ $6.;
   input _id_ $ x1-x4 _type_ $ _rhs_;
   datalines;
profit   1    1    1    1    max   .
const1   1    3    2    4    le    5
const2   3    1    2    1    le    4
const3   5    3    3    3    eq    9
;

proc lp;
run;
```

The results are shown in Output 3.9.1.

Output 3.9.1. The Solution of an Infeasible Problem

```
                        The LP Procedure

                         Problem Summary

Objective Function      Max profit
Rhs Variable                 _rhs_
Type Variable               _type_
Problem Density (%)          77.78

Variables                   Number

Non-negative                     4
Slack                            2

Total                            6

Constraints                 Number

LE                               2
EQ                               1
Objective                        1

Total                            4
```

ERROR: Infeasible problem. Note the constraints in the constraint summary that are identified as infeasible. If none of the constraints are flagged then check the implicit bounds on the variables.

The LP Procedure

Solution Summary

Infeasible Problem

Objective Value	2.5
Phase 1 Iterations	2
Phase 2 Iterations	0
Phase 3 Iterations	0
Integer Iterations	0
Integer Solutions	0
Initial Basic Feasible Variables	5
Time Used (seconds)	0
Number of Inversions	2
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	x1	BASIC	NON-NEG	1	0.75	0
2	x2	BASIC	NON-NEG	1	1.75	0
3	x3		NON-NEG	1	0	0.5
4	x4		NON-NEG	1	0	0
INF	const1	BASIC	SLACK	0	-1	0
6	const2		SLACK	0	0	0.5

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	profit	OBJECTVE	.	0	2.5	.
INF	const1	LE	5	5	6	0
3	const2	LE	6	4	4	-0.5
4	const3	EQ	.	9	9	0.5

```
                          The LP Procedure

                    Infeasible Information Summary

                    Infeasible Row          const1
                    Constraint Activity          6
                    Row Type                    LE
                    Rhs Data                     5


                                                  Lower      Upper
          Variable Coefficient  Activity          Bound      Bound

          x1                1      0.75              0   INFINITY
          x2                3      1.75              0   INFINITY
          x3                2         0              0   INFINITY
          x4                4         0              0   INFINITY
```

Note the information given in the Infeasible Information Summary for the infeasible row CONST1. It shows that the inequality row CONST1 with right-hand side 5 was found to be infeasible with activity 6. The summary also shows each variable that has a nonzero coefficient in that row and its activity level at the infeasibility. Examination of these model parameters might give you a clue as to the cause of infeasibility, such as an incorrectly entered coefficient or right-hand-side value.

Example 3.10. Restarting an Integer Program

The following example is attributed to Haldi (Garfinkel and Nemhauser 1972) and is used in the literature as a test problem. Notice that the ACTIVEOUT= and the PRIMALOUT= options are used when invoking PROC LP. These cause the LP procedure to save the primal solution in the data set named P and the active tree in the data set named A. If the procedure fails to find an optimal integer solution on the initial call, it can be called later using the A and P data sets as starting information.

```
data haldi10;
   input x1-x12 _type_ $ _rhs_;
   datalines;
  0   0   0   0   0   0   1   1   1   1   1   1  MAX    .
  9   7  16   8  24   5   3   7   8   4   6   5  LE   110
 12   6   6   2  20   8   4   6   3   1   5   8  LE    95
 15   5  12   4   4   5   5   5   6   2   1   5  LE    80
 18   4   4  18  28   1   6   4   2   9   7   1  LE   100
-12   0   0   0   0   0   1   0   0   0   0   0  LE     0
  0 -15   0   0   0   0   0   1   0   0   0   0  LE     0
  0   0 -12   0   0   0   0   0   1   0   0   0  LE     0
  0   0   0 -10   0   0   0   0   0   1   0   0  LE     0
  0   0   0   0 -11   0   0   0   0   0   1   0  LE     0
  0   0   0   0   0 -11   0   0   0   0   0   1  LE     0
  1   1   1   1   1   1 1000 1000 1000 1000 1000 1000 UPPERBD .
  1   2   3   4   5   6   7   8   9  10  11  12  INTEGER .
  ;

proc lp data=haldi10 activeout=a primalout=p;
run;
```

The ACTIVEOUT= data set contains a representation of the current active problems in the branch-and-bound tree. The PRIMALOUT= data set contains a representation of the solution to the current problem. These two can be used to restore the procedure to an equivalent state to the one it was in when it stopped.

The results from the call to PROC LP is shown in Output 3.10.1. Notice that the procedure performed 100 iterations and then terminated on maximum integer iterations. This is because, by default, IMAXIT=100. The procedure reports the current best integer solution.

Output 3.10.1. Output from the HALDI10 Problem

```
                    The LP Procedure

                     Problem Summary

Objective Function      Max _OBS1_
Rhs Variable                 _rhs_
Type Variable               _type_
Problem Density (%)          31.82

Variables                   Number

Integer                          6
Binary                           6
Slack                           10

Total                            22

Constraints                 Number

LE                               10
Objective                         1

Total                            11
```

The LP Procedure

Integer Iteration Log

Iter	Problem	Condition	Objective	Branched	Value	Sinfeas	Active	Proximity
1	0	ACTIVE	18.709524	x9	1.543	1.11905	2	.
2	1	ACTIVE	18.467723	x12	9.371	0.88948	3	.
3	2	ACTIVE	18.460133	x8	0.539	1.04883	4	.
4	-3	ACTIVE	18.453638	x12	8.683	1.12993	5	.
5	4	ACTIVE	18.439678	x10	7.448	1.20125	6	.
6	5	ACTIVE	18.403728	x6	0.645	1.3643	7	.
7	-6	ACTIVE	18.048289	x4	0.7	1.18395	8	.
8	-7	ACTIVE	17.679087	x8	1.833	0.52644	9	.
9	8	ACTIVE	17.52	x10	6.667	0.70111	10	.
10	9	ACTIVE	17.190085	x12	7.551	1.37615	11	.
11	-10	ACTIVE	17.02	x1	0.085	0.255	12	.
12	11	ACTIVE	16.748	x11	0.748	0.47	13	.
13	-12	ACTIVE	16.509091	x9	0.509	0.69091	14	.
14	13	ACTIVE	16.261333	x11	1.261	0.44267	15	.
15	14	ACTIVE	16	x3	0.297	0.45455	16	.
16	15	ACTIVE	16	x5	0.091	0.15758	16	.
17	-16	INFEASIBLE	-0.4	.	.	.	15	.
18	-15	ACTIVE	11.781818	x10	1.782	0.37576	15	.
19	18	ACTIVE	11	x5	0.091	0.15758	15	.
20	-19	INFEASIBLE	-6.4	.	.	.	14	.
21	-14	ACTIVE	11.963636	x5	0.182	0.28485	14	.
22	-21	INFEASIBLE	-4.4	.	.	.	13	.
23	-13	ACTIVE	15.281818	x10	4.282	0.52273	13	.
24	23	ACTIVE	15.041333	x5	0.095	0.286	14	.
25	-24	INFEASIBLE	-2.9	.	.	.	13	.
26	24	INFEASIBLE	14	.	.	.	12	.
27	12	ACTIVE	16	x3	0.083	0.15	13	.
28	-27	ACTIVE	15.277778	x9	0.278	0.34444	14	.
29	-28	ACTIVE	13.833333	x10	3.833	0.23333	14	.
30	29	ACTIVE	13	x2	0.4	0.4	15	.
31	30	INFEASIBLE	12	.	.	.	14	.
32	-30	SUBOPTIMAL	10	.	.	.	13	8
33	28	ACTIVE	15	x2	0.067	0.06667	13	8
34	-33	SUBOPTIMAL	12	.	.	.	12	6
35	27	ACTIVE	15	x2	0.067	0.06667	12	6
36	-35	SUBOPTIMAL	15	.	.	.	11	3
37	-11	FATHOMED	14.275	.	.	.	10	3
38	10	ACTIVE	16.804848	x1	0.158	0.50313	11	3
39	-38	FATHOMED	14.784	.	.	.	10	3
40	38	ACTIVE	16.40381	x11	1.404	0.68143	11	3
41	-40	ACTIVE	16.367677	x10	5.368	0.69949	12	3
42	41	ACTIVE	16.113203	x11	2.374	1.00059	12	3
43	42	ACTIVE	16	x5	0.182	0.33182	12	3
44	-43	FATHOMED	13.822222	.	.	.	11	3
45	-41	FATHOMED	12.642424	.	.	.	10	3
46	40	ACTIVE	16	x5	0.229	0.37857	10	3
47	46	FATHOMED	15	.	.	.	9	3
48	-9	ACTIVE	17.453333	x7	0.453	0.64111	10	3
49	48	ACTIVE	17.35619	x11	0.356	0.53857	11	3
50	49	ACTIVE	17	x5	0.121	0.27143	12	3

51	50	ACTIVE	17	x3	0.083	0.15	13	3
52	-51	FATHOMED	15.933333	.	.	.	12	3
53	51	ACTIVE	16	x2	0.067	0.06667	12	3
54	-53	SUBOPTIMAL	16	.	.	.	8	2
55	-8	ACTIVE	17.655399	x12	7.721	0.56127	9	2
56	55	ACTIVE	17.519375	x10	6.56	0.76125	10	2
57	56	ACTIVE	17.256874	x2	0.265	0.67388	11	2
58	57	INFEASIBLE	17.167622	.	.	.	10	2
59	-57	FATHOMED	16.521755	.	.	.	9	2
60	-56	FATHOMED	17.03125	.	.	.	8	2
61	-55	ACTIVE	17.342857	x9	0.343	0.50476	8	2
62	61	ACTIVE	17.2225	x7	0.16	0.37333	9	2
63	62	ACTIVE	17.1875	x8	2.188	0.33333	9	2
64	63	ACTIVE	17.153651	x11	0.154	0.30095	10	2
65	-64	FATHOMED	12.381818	.	.	.	9	2
66	64	ACTIVE	17	x2	0.133	0.18571	9	2
67	-66	FATHOMED	13	.	.	.	8	2
68	-62	FATHOMED	14.2	.	.	.	7	2
69	7	FATHOMED	15.428583	.	.	.	6	2
70	6	FATHOMED	16.75599	.	.	.	5	2
71	-5	ACTIVE	17.25974	x6	0.727	0.82078	5	2
72	-71	FATHOMED	17.142857	.	.	.	4	2
73	-4	ACTIVE	18.078095	x4	0.792	0.70511	5	2
74	-73	ACTIVE	17.662338	x10	7.505	0.91299	5	2
75	74	ACTIVE	17.301299	x9	0.301	0.57489	5	2
76	75	ACTIVE	17.210909	x7	0.211	0.47697	5	2
77	76	FATHOMED	17.164773	.	.	.	4	2
78	73	FATHOMED	12.872727	.	.	.	3	2
79	3	ACTIVE	18.368316	x10	7.602	1.20052	4	2
80	79	ACTIVE	18.198323	x7	1.506	1.85351	5	2
81	80	ACTIVE	18.069847	x12	8.517	1.67277	6	2
82	-81	ACTIVE	17.910909	x4	0.7	0.73015	7	2
83	-82	ACTIVE	17.790909	x7	0.791	0.54015	8	2
84	-83	ACTIVE	17.701299	x9	0.701	0.62229	8	2
85	84	ACTIVE	17.17619	x6	0.818	0.45736	8	2
86	-85	ACTIVE	17.146667	x11	0.147	0.24333	8	2
87	86	ACTIVE	17	x1	0.167	0.16667	8	2
88	87	INFEASIBLE	16	.	.	.	7	2
89	83	ACTIVE	17.58	x11	0.58	0.73788	8	2
90	-89	FATHOMED	17.114286	.	.	.	7	2
91	-80	ACTIVE	18.044048	x12	8.542	1.71158	8	2
92	91	ACTIVE	17.954536	x11	0.477	1.90457	9	2
93	92	ACTIVE	17.875084	x4	0.678	1.16624	10	2
94	93	FATHOMED	13.818182	.	.	.	9	2
95	-93	ACTIVE	17.231221	x6	0.727	0.76182	9	2
96	-95	FATHOMED	17.085714	.	.	.	8	2
97	-92	FATHOMED	17.723058	.	.	.	7	2
98	-91	FATHOMED	16.378788	.	.	.	6	2
99	89	ACTIVE	17	x6	0.818	0.26515	6	2
100	-99	ACTIVE	17	x3	0.083	0.08333	6	2

WARNING: The maximum number of integer iterations has been exceeded. Increase this limit with the 'IMAXIT=' option on the RESET statement.

The LP Procedure

Solution Summary

Terminated on Maximum Integer Iterations
Integer Feasible Solution

Objective Value	16
Phase 1 Iterations	0
Phase 2 Iterations	13
Phase 3 Iterations	161
Integer Iterations	100
Integer Solutions	4
Initial Basic Feasible Variables	12
Time Used (seconds)	0
Number of Inversions	37
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	x1	DEGEN	BINARY	0	0	0
2	x2	ALTER	BINARY	0	1	0
3	x3		BINARY	0	0	12
4	x4	ALTER	BINARY	0	1	0
5	x5	ALTER	BINARY	0	0	0
6	x6	ALTER	BINARY	0	1	0
7	x7		INTEGER	1	0	1
8	x8		INTEGER	1	1	1
9	x9	DEGEN	INTEGER	1	0	0
10	x10		INTEGER	1	7	1
11	x11		INTEGER	1	0	1
12	x12		INTEGER	1	8	1
13	_OBS2_	BASIC	SLACK	0	15	0
14	_OBS3_	BASIC	SLACK	0	2	0
15	_OBS4_	BASIC	SLACK	0	7	0
16	_OBS5_	BASIC	SLACK	0	2	0
17	_OBS6_	ALTER	SLACK	0	0	0
18	_OBS7_	BASIC	SLACK	0	14	0
19	_OBS8_		SLACK	0	0	-1
20	_OBS9_	BASIC	SLACK	0	3	0
21	_OBS10_	DEGEN	SLACK	0	0	0
22	_OBS11_	BASIC	SLACK	0	3	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	_OBS1_	OBJECTVE	.	0	16	.
2	_OBS2_	LE	13	110	95	0
3	_OBS3_	LE	14	95	93	0
4	_OBS4_	LE	15	80	73	0
5	_OBS5_	LE	16	100	98	0
6	_OBS6_	LE	17	0	0	0
7	_OBS7_	LE	18	0	-14	0
8	_OBS8_	LE	19	0	0	1
9	_OBS9_	LE	20	0	-3	0
10	_OBS10_	LE	21	0	0	0
11	_OBS11_	LE	22	0	-3	0

To continue with the solution of this problem, invoke PROC LP with the ACTIVEIN= and PRIMALIN= options and reset the IMAXIT= option. This restores the branch-and-bound tree and simplifies calculating a basic feasible solution from which to start processing.

```
proc lp data=haldi10 activein=a primalin=p imaxit=250;
run;
```

The procedure picks up iterating from a equivalent state to where it left off. The problem will still not be solved when IMAXIT=250 occurs.

Example 3.11. Alternative Search of the Branch-and-Bound Tree

In this example, the HALDI10 problem from Example 3.10 is solved. However, here the default strategy for searching the branch-and-bound tree is modified. By default, the search strategy has VARSELECT=FAR. This means that when searching for an integer variable on which to branch, the procedure uses the one that has a value farthest from an integer value. An alternative strategy has VARSELECT=PENALTY. This strategy causes PROC LP to look at the cost, in terms of the objective function, of branching on an integer variable. The procedure looks at PENALTYDEPTH= integer variables before choosing the one with the largest cost. This is a much more expensive strategy (in terms of execution time) than the VARSELECT=FAR strategy, but it can be beneficial if fewer integer iterations must be done to find an optimal solution.

```
proc lp data=haldi10 varselect=penalty;
run;
```

Compare the number of integer iterations needed to solve the problem using this heuristic with the default strategy used in Example 3.10. In this example, the difference is profound; in general, solution times can vary significantly with the search technique. See Output 3.11.1.

Output 3.11.1. Summaries and an Integer Programming Iteration Log: Using VARSELECT=PENALTY

The LP Procedure

Problem Summary

Objective Function	Max _OBS1_
Rhs Variable	_rhs_
Type Variable	_type_
Problem Density (%)	31.82

Variables	Number
Integer	6
Binary	6
Slack	10
Total	22

Constraints	Number
LE	10
Objective	1
Total	11

The LP Procedure

Integer Iteration Log

Iter	Problem	Condition	Objective	Branched	Value	Sinfeas	Active	Proximity
1	0	ACTIVE	18.709524	x4	0.8	1.11905	2	.
2	1	ACTIVE	16.585187	x1	0.447	2.33824	3	.
3	2	ACTIVE	14.86157	x5	0.221	2.09584	4	.
4	3	ACTIVE	14.807195	x2	0.897	1.31729	5	.
5	-4	ACTIVE	14.753205	x8	14.58	0.61538	6	.
6	5	ACTIVE	14.730078	x6	0.043	0.79446	7	.
7	-6	ACTIVE	13.755102	x3	0.051	0.58163	8	.
8	-7	ACTIVE	11.6	x8	11.6	0.4	9	.
9	8	ACTIVE	11.6	x12	0.6	0.4	10	.
10	-9	ACTIVE	11.6	x8	10.6	0.4	11	.
11	10	ACTIVE	11.6	x12	1.6	0.4	12	.
12	-11	ACTIVE	11.6	x8	9.6	0.4	13	.
13	12	ACTIVE	11.6	x12	2.6	0.4	14	.
14	-13	ACTIVE	11.571429	x9	0.143	0.57143	15	.
15	14	ACTIVE	11.5	x8	8.5	0.5	16	.
16	-15	INFEASIBLE	9	.	.	.	15	.
17	15	ACTIVE	11.375	x12	3.375	0.375	16	.
18	-17	ACTIVE	11.166667	x8	7.167	0.16667	17	.
19	18	ACTIVE	11.125	x12	4.125	0.125	18	.
20	19	SUBOPTIMAL	11	.	.	.	7	7
21	7	ACTIVE	13.5	x8	13.5	0.5	8	7
22	-21	INFEASIBLE	11	.	.	.	7	7
23	21	ACTIVE	13.375	x12	0.375	0.375	8	7
24	-23	ACTIVE	13.166667	x8	12.17	0.16667	9	7
25	24	ACTIVE	13.125	x12	1.125	0.125	10	7
26	25	SUBOPTIMAL	13	.	.	.	4	5
27	6	ACTIVE	14.535714	x3	0.045	0.50893	5	5
28	-27	FATHOMED	12.625	.	.	.	4	5
29	27	SUBOPTIMAL	14	.	.	.	1	4
30	-1	ACTIVE	18.309524	x3	0.129	1.31905	2	4
31	30	ACTIVE	17.67723	x6	0.886	0.43662	3	4
32	31	ACTIVE	15.485156	x2	0.777	1.50833	4	4
33	-32	ACTIVE	15.2625	x1	0.121	1.38333	4	4
34	33	ACTIVE	15.085106	x10	3.532	0.91489	4	4
35	34	FATHOMED	14.857143	.	.	.	3	4
36	32	FATHOMED	11.212121	.	.	.	2	4
37	-31	ACTIVE	17.56338	x10	7.93	0.43662	3	4
38	37	ACTIVE	17.225962	x8	2.38	0.69231	4	4
39	38	ACTIVE	17.221818	x1	0.016	0.37111	5	4
40	-39	FATHOMED	14.43662	.	.	.	4	4
41	39	ACTIVE	17.172375	x2	0.133	0.31948	5	4
42	41	ACTIVE	16.890196	x5	0.086	0.19608	6	4
43	42	ACTIVE	16.75	x12	9.75	0.25	7	4
44	-43	SUBOPTIMAL	15	.	.	.	6	3
45	43	SUBOPTIMAL	16	.	.	.	3	2
46	-38	FATHOMED	17.138028	.	.	.	2	2
47	-37	SUBOPTIMAL	17	.	.	.	1	1
48	-30	FATHOMED	16.566667	.	.	.	0	.

The LP Procedure

Solution Summary

Integer Optimal Solution

Objective Value	17
Phase 1 Iterations	0
Phase 2 Iterations	13
Phase 3 Iterations	79
Integer Iterations	48
Integer Solutions	6
Initial Basic Feasible Variables	12
Time Used (seconds)	0
Number of Inversions	17
Epsilon	1E-8
Infinity	1.797693E308
Maximum Phase 1 Iterations	100
Maximum Phase 2 Iterations	100
Maximum Phase 3 Iterations	99999999
Maximum Integer Iterations	100
Time Limit (seconds)	120

The LP Procedure

Variable Summary

Col	Variable Name	Status	Type	Price	Activity	Reduced Cost
1	x1	DEGEN	BINARY	0	0	0
2	x2		BINARY	0	0	-4
3	x3		BINARY	0	0	-4
4	x4		BINARY	0	1	-18
5	x5	DEGEN	BINARY	0	0	0
6	x6		BINARY	0	1	-1
7	x7		INTEGER	1	0	-6.5
8	x8		INTEGER	1	0	-3
9	x9		INTEGER	1	0	-1
10	x10		INTEGER	1	8	-8
11	x11		INTEGER	1	0	-8.545455
12	x12	BASIC	INTEGER	1	9	0
13	_OBS2_	BASIC	SLACK	0	20	0
14	_OBS3_	BASIC	SLACK	0	5	0
15	_OBS4_	BASIC	SLACK	0	10	0
16	_OBS5_		SLACK	0	0	-1
17	_OBS6_		SLACK	0	0	-1.5
18	_OBS7_	DEGEN	SLACK	0	0	0
19	_OBS8_	DEGEN	SLACK	0	0	0
20	_OBS9_	BASIC	SLACK	0	2	0
21	_OBS10_		SLACK	0	0	-2.545455
22	_OBS11_	BASIC	SLACK	0	2	0

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	_OBS1_	OBJECTVE	.	0	17	.
2	_OBS2_	LE	13	110	90	0
3	_OBS3_	LE	14	95	90	0
4	_OBS4_	LE	15	80	70	0
5	_OBS5_	LE	16	100	100	1
6	_OBS6_	LE	17	0	0	1.5
7	_OBS7_	LE	18	0	0	0
8	_OBS8_	LE	19	0	0	0
9	_OBS9_	LE	20	0	-2	0
10	_OBS10_	LE	21	0	0	2.5454545
11	_OBS11_	LE	22	0	-2	0

Although the VARSELECT=PENALTY strategy works well in this example, there is no guarantee that it will work well with your model. Experimentation with various strategies is necessary to find the one that works well with your model and data, particularly if a model is solved repeatedly with few changes to either the structure or the data.

Example 3.12. An Assignment Problem

This example departs somewhat from the emphasis of previous ones. Typically, linear programming models are large, have considerable structure, and are solved with some regularity. Some form of automatic model building, or matrix generation as it is commonly called, is a useful aid. The sparse input format provides a great deal of flexibility in model specification so that, in many cases, the DATA step can be used to generate the matrix.

The following assignment problem illustrates some techniques in matrix generation. In this example, you have four machines that can produce any of six grades of cloth, and you have five customers that demand various amounts of each grade of cloth. The return from supplying a customer with a demanded grade depends on the machine on which the cloth was made. In addition, the machine capacity depends both upon the specific machine used and the grade of cloth made.

To formulate this problem, let i denote customer, j denote grade, and k denote machine. Then let x_{ijk} denote the amount of cloth of grade j made on machine k for customer i; let r_{ijk} denote the return from selling one unit of grade j cloth made on machine k to customer i; let d_{ij} denote the demand for grade j cloth by customer i; let c_{jk} denote the number of units of machine k required to produce one unit of grade j cloth; and let a_k denote the number of units of machine k available. Then, you get

$$
\begin{array}{lll}
\max & \sum_{ijk} r_{ijk} x_{ijk} & \\
\text{subject to} & \sum_{k} x_{ijk} = d_{ij} & \text{for all } i \text{ and } j \\
 & \sum_{ij} c_{jk} x_{ijk} \leq a_k & \text{for all } k \\
 & x_{ijk} \geq 0 & \text{for all } i, j \text{ and } k
\end{array}
$$

The data are saved in three data sets. The OBJECT data set contains the returns for satisfying demand, the DEMAND data set contains the amounts demanded, and the RESOURCE data set contains the conversion factors for each grade and the total amounts of machine resources available.

```
title 'An Assignment Problem';

data object;
   input machine customer
         grade1 grade2 grade3 grade4 grade5 grade6;
   datalines;
1 1 102 140 105  105  125  148
1 2 115 133 118  118  143  166
1 3  70 108  83   83   88   86
1 4  79 117  87   87  107  105
1 5  77 115  90   90  105  148
2 1 123 150 125  124  154   .
2 2 130 157 132  131  166   .
2 3 103 130 115  114  129   .
2 4 101 128 108  107  137   .
2 5 118 145 130  129  154   .
3 1  83  .   .    97  122  147
3 2 119  .   .   133  163  180
3 3  67  .   .    91  101  101
3 4  85  .   .   104  129  129
3 5  90  .   .   114  134  179
4 1 108 121  79   .   112  132
4 2 121 132  92   .   130  150
4 3  78  91  59   .    77   72
4 4 100 113  76   .   109  104
4 5  96 109  77   .   105  145
;

data demand;
   input customer
         grade1 grade2 grade3 grade4 grade5 grade6;
   datalines;
1 100 100 150  150  175  250
2 300 125 300  275  310  325
3 400   0 400  500  340    0
4 250   0 750  750    0    0
5   0 600 300    0  210  360
;

data resource;
   input machine
         grade1 grade2 grade3 grade4 grade5 grade6 avail;
   datalines;
1 .250 .275 .300  .350  .310  .295  744
2 .300 .300 .305  .315  .320  .     244
3 .350 .    .     .320  .315  .300  790
4 .280 .275 .260  .     .250  .295  672
;
```

The linear program is built using the DATA step. The model is saved in a SAS data set in the sparse input format for PROC LP. Each section of the following DATA step generates a piece of the linear program. The first section generates the objective function; the next section generates the demand constraints; and the last section generates the machine resource availability constraints.

```
/*  build the linear programming model */

data model;
   array grade{6} grade1-grade6;
   length _type_ $ 8 _row_ $ 8 _col_ $ 8;
   keep _type_ _row_ _col_ _coef_;

   ncust=5;
   nmach=4;
   ngrade=6;

   /* generate the objective function */

   _type_='MAX';
   _row_='OBJ';
   do k=1 to nmach;
      do i=1 to ncust;
         link readobj;     /* read the objective coefficient data */
         do j=1 to ngrade;
            if grade{j}^=. then do;
               _col_='X'||put(i,1.)||put(j,1.)||put(k,1.);
               _coef_=grade{j};
               output;
            end;
         end;
      end;
   end;


   /* generate the demand constraints */

   do i=1 to ncust;
      link readdmd;        /* read the demand data */
      do j=1 to ngrade;
         if grade{j}^=. then do;
            _type_='EQ';
            _row_='DEMAND'||put(i,1.)||put(j,1.);
            _col_='_RHS_';
            _coef_=grade{j};
            output;
            _type_=' ';
            do k=1 to nmach;
               _col_='X'||put(i,1.)||put(j,1.)||put(k,1.);
               _coef_=1.0;
               output;
            end;
         end;
```

```
      end;
   end;

   /* generate the machine constraints */

   do k=1 to nmach;
      link readres;       /* read the machine data */
      _type_='LE';
      _row_='MACHINE'||put(k,1.);
      _col_='_RHS_';
      _coef_=avail;
      output;
      _type_=' ';
      do i=1 to ncust;
         do j=1 to ngrade;
            if grade{j}^=. then do;
               _col_='X'||put(i,1.)||put(j,1.)||put(k,1.);
               _coef_=grade{j};
               output;
               end;
            end;
         end;
      end;

readobj: set object;
return;
readdmd: set demand;
return;
readres: set resource;
return;
run;
```

With the model built and saved in a data set, it is ready for solution using PROC LP. The following program solves the model and saves the solution in the data set called PRIMAL:

```
/* solve the linear program */

proc lp data=model sparsedata noprint primalout=primal;
run;
```

The following output is produced by PROC LP.

Output 3.12.1. An Assignment Problem

```
                          An Assignment Problem

                            The LP Procedure

                             Problem Summary

                    Objective Function      Max OBJ
                    Rhs Variable              _RHS_
                    Type Variable            _type_
                    Problem Density (%)        5.31

                    Variables                Number

                    Non-negative                120
                    Slack                         4

                    Total                       124

                    Constraints              Number

                    LE                            4
                    EQ                           30
                    Objective                     1

                    Total                        35
```

```
                            The LP Procedure

                            Solution Summary

                         Terminated Successfully

                Objective Value                    871426.03763

                Phase 1 Iterations                            0
                Phase 2 Iterations                           40
                Phase 3 Iterations                            0
                Integer Iterations                            0
                Integer Solutions                             0
                Initial Basic Feasible Variables             36
                Time Used (seconds)                           0
                Number of Inversions                          3

                Epsilon                                    1E-8
                Infinity                           1.797693E308
                Maximum Phase 1 Iterations                  100
                Maximum Phase 2 Iterations                  100
                Maximum Phase 3 Iterations             99999999
                Maximum Integer Iterations                  100
                Time Limit (seconds)                        120
```

The solution is prepared for reporting using the DATA step, and a report is written using PROC TABULATE.

```
/* report the solution */

data solution;
   set primal;
   keep customer grade machine amount;
   if substr(_var_,1,1)='X' then do;
     if _value_^=0 then do;
       customer = substr(_var_,2,1);
       grade    = substr(_var_,3,1);
       machine  = substr(_var_,4,1);
       amount   = _value_;
       output;
     end;
  end;
run;

proc tabulate data=solution;
   class customer grade machine;
   var   amount;
   table (machine*customer), (grade*amount);
run;
```

The report shown in Output 3.12.2 gives the assignment of customer, grade of cloth, and machine that maximizes the return and does not violate the machine resource availability.

Output 3.12.2. An Assignment Problem

		grade			
		1	2	3	4
		amount	amount	amount	amount
		Sum	Sum	Sum	Sum
machine	customer				
1	1	.	100.00	150.00	150.00
	2	.	.	300.00	.
	3	.	.	256.72	210.31
	4	.	.	750.00	.
	5	.	92.27	.	.
2	3	.	.	143.28	.
	5	.	.	300.00	.
3	2	.	.	.	275.00
	3	.	.	.	289.69
	4	.	.	.	750.00
	5	.	.	.	.
4	1	100.00	.	.	.
	2	300.00	125.00	.	.
	3	400.00	.	.	.
	4	250.00	.	.	.
	5	.	507.73	.	.

(Continued)

		grade	
		5	6
		amount	amount
		Sum	Sum
machine	customer		
1	1	175.00	250.00
	2	.	.
	3	.	.
	4	.	.
	5	.	.
2	3	340.00	.
	5	.	.
3	2	310.00	325.00
	3	.	.
	4	.	.
	5	210.00	360.00
4	1	.	.
	2	.	.
	3	.	.
	4	.	.
	5	.	.

Example 3.13. A Scheduling Problem

Scheduling is an application area where techniques in model generation can be valuable. Problems involving scheduling are often solved with integer programming and are similar to assignment problems. In this example, you have eight one-hour time slots in each of five days. You have to assign four people to these time slots so that each slot is covered on every day. You allow the people to specify preference data for each slot on each day. In addition, there are constraints that must be satisfied:

- Each person has some slots for which they are unavailable.
- Each person must have either slot 4 or 5 off for lunch.
- Each person can work only two time slots in a row.
- Each person can work only a specified number of hours in the week.

To formulate this problem, let i denote person, j denote time slot, and k denote day. Then, let $x_{ijk} = 1$ if person i is assigned to time slot j on day k, and 0 otherwise;

let p_{ijk} denote the preference of person i for slot j on day k; and let h_i denote the number of hours in a week that person i will work. Then, you get

$$
\begin{array}{lll}
\max & \sum_{ijk} p_{ijk} x_{ijk} & \\
\text{subject to} & \sum_i x_{ijk} = 1 & \text{for all } j \text{ and } k \\
& x_{i4k} + x_{i5k} \leq 1 & \text{for all } i \text{ and } k \\
& x_{i,\ell,k} + x_{i,\ell+1,k} + x_{i,\ell+2,k} \leq 2 & \text{for all } i \text{ and } k, \text{ and } \ell = 1, \ldots, 6 \\
& \sum_{jk} x_{ijk} \leq h_i & \text{for all } i \\
& x_{ijk} = 0 \text{ or } 1 & \text{for all } i \text{ and } k \text{ such that } p_{ijk} > 0, \\
& & \quad \text{otherwise } x_{ijk} = 0
\end{array}
$$

To solve this problem, create a data set that has the hours and preference data for each individual, time slot, and day. A 10 represents the most desirable time slot, and a 1 represents the least desirable time slot. In addition, a 0 indicates that the time slot is not available.

```
data raw;
   input name $ hour slot mon tue wed thu fri;
   datalines;
marc  20   1    10 10 10 10 10
marc  20   2     9  9  9  9  9
marc  20   3     8  8  8  8  8
marc  20   4     1  1  1  1  1
marc  20   5     1  1  1  1  1
marc  20   6     1  1  1  1  1
marc  20   7     1  1  1  1  1
marc  20   8     1  1  1  1  1
mike  20   1    10  9  8  7  6
mike  20   2    10  9  8  7  6
mike  20   3    10  9  8  7  6
mike  20   4    10  3  3  3  3
mike  20   5     1  1  1  1  1
mike  20   6     1  2  3  4  5
mike  20   7     1  2  3  4  5
mike  20   8     1  2  3  4  5
bill  20   1    10 10 10 10 10
bill  20   2     9  9  9  9  9
bill  20   3     8  8  8  8  8
bill  20   4     0  0  0  0  0
bill  20   5     1  1  1  1  1
bill  20   6     1  1  1  1  1
bill  20   7     1  1  1  1  1
bill  20   8     1  1  1  1  1
bob   20   1    10  9  8  7  6
bob   20   2    10  9  8  7  6
bob   20   3    10  9  8  7  6
bob   20   4    10  3  3  3  3
bob   20   5     1  1  1  1  1
bob   20   6     1  2  3  4  5
bob   20   7     1  2  3  4  5
bob   20   8     1  2  3  4  5
;
```

These data are read by the following DATA step, and an integer program is built to solve the problem. The model is saved in the data set named MODEL. First, the objective function is built using the data saved in the RAW data set. Then, the constraints requiring a person to be working in each time slot are built. Next, the constraints allowing each person time for lunch are added. Then, the constraints restricting people to only two consecutive hours are added. Next, the constraints limiting the time that any one person works in a week are added. Finally, the constraints allowing a person to be assigned only to a time slot for which he is available are added. The code to build each of these constraints follows the formulation closely.

```
data model;
   array workweek{5} mon tue wed thu fri;
   array hours{4} hours1 hours2 hours3 hours4;
   retain hours1-hours4;

   set raw end=eof;

   length _row_ $ 8 _col_ $ 8 _type_ $ 8;
   keep _type_ _col_ _row_ _coef_;

   if      name='marc' then i=1;
   else if name='mike' then i=2;
   else if name='bill' then i=3;
   else if name='bob'  then i=4;

   hours{i}=hour;


   /* build the objective function */

   do k=1 to 5;
      _col_='x'||put(i,1.)||put(slot,1.)||put(k,1.);

      _row_='object';
      _coef_=workweek{k} * 1000;
      output;
      _row_='upper';
      if workweek{k}^=0 then _coef_=1;
      output;
      _row_='integer';
      _coef_=1;
      output;
   end;


   /* build the rest of the model */

   if eof then do;
      _coef_=.;
      _col_=' ';
      _type_='upper';
      _row_='upper';
      output;
```

```
_type_='max';
_row_='object';
output;
_type_='int';
_row_='integer';
output;

/* every hour 1 person working */

do j=1 to 8;
   do k=1 to 5;
      _row_='work'||put(j,1.)||put(k,1.);
      _type_='eq';
      _col_='_RHS_';
      _coef_=1;
      output;
      _coef_=1;
      _type_=' ';
      do i=1 to 4;
         _col_='x'||put(i,1.)||put(j,1.)||put(k,1.);
         output;
      end;
   end;
end;


/* each person has a lunch */

do i=1 to 4;
   do k=1 to 5;
      _row_='lunch'||put(i,1.)||put(k,1.);
      _type_='le';
      _col_='_RHS_';
      _coef_=1;
      output;
      _coef_=1;
      _type_=' ';
      _col_='x'||put(i,1.)||'4'||put(k,1.);
      output;
      _col_='x'||put(i,1.)||'5'||put(k,1.);
      output;
   end;
end;


/* work at most 2 slots in a row */

do i=1 to 4;
   do k=1 to 5;
      do l=1 to 6;
      _row_='seq'||put(i,1.)||put(k,1.)||put(l,1.);
      _type_='le';
      _col_='_RHS_';
      _coef_=2;
       output;
```

```
            _coef_=1;
            _type_=' ';
               do j=0 to 2;
                  _col_='x'||put(i,1.)||put(l+j,1.)||put(k,1.);
                  output;
               end;
            end;
         end;
      end;

      /* work at most n hours in a week */

      do i=1 to 4;
         _row_='capacit'||put(i,1.);
         _type_='le';
         _col_='_RHS_';
         _coef_=hours{i};
         output;
         _coef_=1;
         _type_=' ';
         do j=1 to 8;
            do k=1 to 5;
               _col_='x'||put(i,1.)||put(j,1.)||put(k,1.);
               output;
            end;
         end;
      end;
   end;
run;
```

The model saved in the data set named MODEL is in the sparse format. The constraint that requires one person to work in time slot 1 on day 2 is named WORK12; it is $\sum_i x_{i12} = 1$.

The following model is saved in the MODEL data set (which has 1387 observations).

TYPE	_COL_	_ROW_	_COEF_
eq	_RHS_	work12	1
	x112	work12	1
	x212	work12	1
	x312	work12	1
	x412	work12	1

The model is solved using the LP procedure. The option PRIMALOUT=SOLUTION causes PROC LP to save the primal solution in the data set named SOLUTION.

```
/* solve the linear program */

proc lp sparsedata noprint primalout=solution
   time=1000 maxit1=1000 maxit2=1000;
run;
```

The following DATA step below takes the solution data set SOLUTION and generates a report data set named REPORT. It translates the variable names x_{ijk} so that a more meaningful report can be written. Then, the PROC TABULATE procedure is used to display a schedule showing how the eight time slots are covered for the week.

```
/* report the solution */
title 'Reported Solution';

data report;
   set solution;
   keep name slot mon tue wed thu fri;
   if substr(_var_,1,1)='x' then do;
     if _value_>0 then do;
        n=substr(_var_,2,1);
        slot=substr(_var_,3,1);
        d=substr(_var_,4,1);
        if      n='1' then name='marc';
        else if n='2' then name='mike';
        else if n='3' then name='bill';
        else               name='bob';
        if      d='1' then mon=1;
        else if d='2' then tue=1;
        else if d='3' then wed=1;
        else if d='4' then thu=1;
        else               fri=1;
        output;
     end;
  end;
run;

proc format;
   value xfmt 1='  xxx   ';
run;

proc tabulate data=report;
   class name slot;
   var mon--fri;
   table (slot * name), (mon tue wed thu fri)*sum=' '*f=xfmt.
         /misstext=' ';
run;
```

Output 3.13.1 from PROC TABULATE summarizes the schedule. Notice that the constraint requiring that a person be assigned to each possible time slot on each day is satisfied.

Output 3.13.1. A Scheduling Problem

Reported Solution

		mon	tue	wed	thu	fri
slot	name					
1	bill	xxx	xxx	xxx	xxx	xxx
2	bob	xxx				
	marc		xxx	xxx	xxx	xxx
3	marc			xxx	xxx	xxx
	mike	xxx	xxx			
4	mike	xxx	xxx	xxx	xxx	xxx
5	bob	xxx	xxx	xxx	xxx	xxx
6	bob		xxx			xxx
	marc	xxx				
	mike			xxx	xxx	
7	bill	xxx				
	bob			xxx	xxx	
	mike		xxx			xxx
8	bill	xxx				
	mike		xxx	xxx	xxx	xxx

Recall that PROC LP puts a character string in the macro variable _ORLP_ that describes the characteristics of the solution on termination. This string can be parsed using macro functions and the information obtained can be used in report writing. The variable can be written to the log with the command

```
%put &_orlp_;
```

which produces Output 3.13.2.

Output 3.13.2. _ORLP_ Macro Variable

```
STATUS=SUCCESSFUL PHASE=3 OBJECTIVE=211000 P_FEAS=YES D_FEAS=YES
      INT_ITER=0 INT_FEAS=1 ACTIVE=0 INT_BEST=211000 PHASE1_ITER=34
      PHASE2_ITER=49 PHASE3_ITER=0
```

From this you learn, for example, that at termination the solution is integer optimal and has an objective value of 211000.

Example 3.14. A Multicommodity Transshipment Problem with Fixed Charges

The following example illustrates a DATA step program for generating a linear program to solve a multicommodity network flow model that has fixed charges. Consider a network consisting of the following nodes: farm-a, farm-b, farm-c, Chicago, St. Louis, and New York. You can ship four commodities from each farm to Chicago or St. Louis and from Chicago or St. Louis to New York. The following table shows the unit shipping cost for each of the four commodities across each of the arcs. The table also shows the supply (positive numbers) at each of the from nodes and the demand (negative numbers) at each of the to nodes. The fixed charge is a fixed cost for shipping any nonzero amount across an arc. For example, if any amount of any of the four commodities is sent from farm-c to St. Louis, then a fixed charge of 75 units is added to the shipping cost.

Table 3.8. Farms to Cities Network Problem

		Unit Shipping Cost				Supply and Demand				Fixed Charge
From Node	To Node	1	2	3	4	1	2	3	4	
farm-a	Chicago	20	15	17	22	100	100	40	.	100
farm-b	Chicago	15	15	15	30	100	200	50	50	75
farm-c	Chicago	30	30	10	10	40	100	75	100	100
farm-a	StLouis	30	25	27	22	.	.	.	.	150
farm-c	StLouis	10	9	11	10	.	.	.	.	75
Chicago	NY	75	75	75	75	-150	-200	-50	-75	200
StLouis	NY	80	80	80	80	.	.	.	.	200

The following program is designed to take the data in the form given in the preceding table. It builds the node arc incidence matrix for a network given in this form and adds integer variables to capture the fixed charge using the type of constraints discussed in Example 3.8. The program solves the model using PROC LP, saves the solution in the PRIMALOUT= data set named SOLUTION, and displays the solution. The DATA step can be easily modified to handle larger problems with similar structure.

```
title 'Multi-commodity Transshipment Problem with Fixed Charges';

data network;
   retain M 1.0e6;
   length _col_ $ 22 _row_ $ 22;
   keep _type_ _col_ _row_ _coef_;
   array sd sd1-sd4;
   array c c1-c4;
   format arc $10.;
   input arc $ from $ to $ c1 c2 c3 c4 sd1 sd2 sd3 sd4 fx;

   /* for the first observation define some of the rows */

   if _n_=1 then do;
      _type_='upperbd';
```

```
   _row_='upper';
   output;
   _type_='lowerbd';
   _row_='lower';
   output;
   _type_='min';
   _row_='obj';
   output;
   _type_='integer';
   _row_='int';
   output;
   end;
_col_='_rhs_';
_type_='le';

do over sd;                          /* loop for each commodity     */
   _coef_=sd;
   if sd>0 then do;                  /*  the node is a supply node  */
      _row_=from||' commodity'||put(_i_,2.);
      if from^=' ' then output;
   end;
   else if sd<0 then do;             /*  the node is a demand node  */
      _row_=to||' commodity'||put(_i_,2.);
      if to^=' ' then output;
   end;
   else if from^=' ' & to^=' ' then do;  /* a transshipment node  */
      _coef_=0;
      _row_=from||' commodity'||put(_i_,2.);
      output;
      _row_=to  ||' commodity'||put(_i_,2.);
      output;
   end;
end;

do over c;                           /* loop for each commodity     */
_col_=arc||' commodity'||put(_i_,2.);
if from^=' ' & to^=' ' then do;
                                  /* add node arc incidence matrix*/
  _type_='le';
  _row_=from||' commodity'||put(_i_,2.);
  _coef_=1;
  output;
  _row_=to     ||' commodity'||put(_i_,2.);
  _coef_=-1;
  output;
  _type_='  ';
  _row_='obj';
  _coef_=c;
  output;
                                  /* add fixed charge variables      */
  _type_='le';
  _row_=arc;
  _coef_=1;output;
```

```
      _col_='_rhs_';
      _type_='  ';
      _coef_=0;
      output;
      _col_=arc||'fx';
      _coef_=-M;
      output;
      _row_='int';
      _coef_=1;
      output;
      _row_='obj';
      _coef_=fx;
      output;
      _row_='upper';
      _coef_=1;
      output;

   end;
   end;

   datalines;
a-Chicago  farm-a  Chicago 20 15 17 22  100  100  40   . 100
b-Chicago  farm-b  Chicago 15 15 15 30  100  200  50  50  75
c-Chicago  farm-c  Chicago 30 30 10 10   40  100  75 100 100
a-StLouis  farm-a  StLouis 30 25 27 22    .    .   .   . 150
c-StLouis  farm-c  StLouis 10  9 11 10    .    .   .   .  75
Chicago-NY Chicago NY      75 75 75 75 -150 -200 -50 -75 200
StLous-NY  StLouis NY      80 80 80 80    .    .   .   . 200
;

/* solve the model */

proc lp sparsedata pout=solution  noprint;
run;

/* print the solution */

data;
   set solution;
   rename _var_=arc _value_=amount;
   if _value_^=0 & _type_='NON-NEG';
run;

proc print;
   id arc;
   var amount;
run;
```

The results from this example are shown in Output 3.14.1. The NOPRINT option in the PROC LP statement suppresses the Variable and Constraint Summary sections. This is useful when solving large models for which a report program is available. Here, the solution is saved in data set SOLUTION and reported using PROC PRINT. The solution shows the amount that is shipped over each arc.

Output 3.14.1. Multicommodity Transshipment Problem with Fixed Charges

Multi-commodity Transshipment Problem with Fixed Charges

arc		amount
a-Chicago	commodity 1	10
b-Chicago	commodity 1	100
b-Chicago	commodity 2	100
c-Chicago	commodity 3	50
c-Chicago	commodity 4	75
c-StLouis	commodity 1	40
c-StLouis	commodity 2	100
Chicago-NY	commodity 1	110
Chicago-NY	commodity 2	100
Chicago-NY	commodity 3	50
Chicago-NY	commodity 4	75
StLous-NY	commodity 1	40
StLous-NY	commodity 2	100

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Chapter 4
The NLP Procedure

Chapter Contents

Chapter 4
The NLP Procedure

Overview: NLP Procedure

The NLP procedure (**N**on**L**inear **P**rogramming) offers a set of optimization techniques for minimizing or maximizing a continuous nonlinear function $f(x)$ of n decision variables, $x = (x_1, \ldots, x_n)^T$ with lower and upper bound, linear and nonlinear, equality and inequality constraints. This can be expressed as solving

$$\begin{array}{lll} \min_{x \in \mathcal{R}^n} & f(x) & \\ \text{subject to} & c_i(x) = 0, & i = 1, \ldots, m_e \\ & c_i(x) \geq 0, & i = m_e + 1, \ldots, m \\ & l_i \leq x_i \leq u_i, & i = 1, \ldots, n \end{array}$$

where f is the objective function, the c_i's are the nonlinear functions, and the l_i's and u_i's are the lower and upper bounds. Problems of this type are found in many settings ranging from optimal control to maximum likelihood estimation.

The NLP procedure provides a number of algorithms for solving this problem that take advantage of special structure on the objective function and constraints. One example is the quadratic programming problem:

$$\begin{array}{ll} \min\ (\max) & f(x) = \frac{1}{2} x^T G x + g^T x + b \\ \text{subject to} & c_i(x) = 0, \quad i = 1, \ldots, m_e \end{array}$$

where G is an $n \times n$ symmetric matrix, $g = (g_1, \ldots, g_n)^T$ is a vector, b is a scalar, and the $c_i(x)$'s are linear functions.

Another example is the least-squares problem:

$$\begin{array}{ll} \min & f(x) = \frac{1}{2}\{f_1^2(x) + \cdots + f_l^2(x)\} \\ \text{subject to} & c_i(x) = 0, \quad i = 1, \ldots, m_e \end{array}$$

where the $c_i(x)$'s are linear functions, and $f_1(x)$, ..., $f_l(x)$ are nonlinear functions of x.

The following problems are handled by PROC NLP:

- quadratic programming with an option for sparse problems
- unconstrained minimization/maximization
- constrained minimization/maximization
- linear complementarity problem

The following optimization techniques are supported in PROC NLP:

- Quadratic Active Set Technique
- Trust Region Method
- Newton-Raphson Method with Line Search
- Newton-Raphson Method with Ridging
- Quasi-Newton Methods
- Double Dogleg Method
- Conjugate Gradient Methods
- Nelder-Mead Simplex Method
- Levenberg-Marquardt Method
- Hybrid Quasi-Newton Methods

These optimization techniques require a continuous objective function f, and all but one (NMSIMP) require continuous first-order derivatives of the objective function f. Some of the techniques also require continuous second-order derivatives. There are three ways to compute derivatives in PROC NLP:

- analytically (using a special derivative compiler), the default method
- via finite-difference approximations
- via user-supplied exact or approximate numerical functions

Nonlinear programs can be input into the procedure in various ways. The objective, constraint, and derivative functions are specified using the programming statements of PROC NLP. In addition, information in SAS data sets can be used to define the structure of objectives and constraints as well as specify constants used in objectives, constraints and derivatives.

PROC NLP uses data sets to input various pieces of information:

- The DATA= data set enables you to specify data shared by all functions involved in a least-squares problem.
- The INQUAD= data set contains the arrays appearing in a quadratic programming problem.
- The INEST= data set specifies initial values for the decision variables, the values of constants that are referred to in the program statements, and simple boundary and general linear constraints.
- The MODEL= data set specifies a model (functions, constraints, derivatives) saved at a previous execution of the NLP procedure.

PROC NLP uses data sets to output various results:

- The OUTEST= data set saves the values of the decision variables, the derivatives, the solution, and the covariance matrix at the solution.

- The OUT= output data set contains variables generated in the program statements defining the objective function as well as selected variables of the DATA= input data set, if available.
- The OUTMODEL= data set saves the programming statements. It can be used to input a model in the MODEL= input data set.

Getting Started: NLP Procedure

The NLP procedure solves general nonlinear programs. It has several optimizers that are tuned to best perform on a particular class of problems. Guidelines for choosing a particular optimizer for a problem can be found in the section "Optimization Algorithms" on page 346.

Regardless of the selected optimizer, it is necessary to specify an objective function and constraints that the optimal solution must satisfy. In PROC NLP, the objective function and the constraints are specified using SAS programming statements that are similar to those used in the SAS DATA step. Some of the differences are discussed in the section "Program Statements" on page 338 and in the section "ARRAY Statement" on page 325. As with any programming language, there are many different ways to specify the same problem. Some are more economical than others.

Introductory Examples

The following introductory examples illustrate how to get started using the NLP procedure.

An Unconstrained Problem

Consider the simple example of minimizing the Rosenbrock function (Rosenbrock 1960):

$$\begin{aligned} f(x) &= \frac{1}{2}\{100(x_2 - x_1^2)^2 + (1 - x_1)^2\} \\ &= \frac{1}{2}\{f_1^2(x) + f_2^2(x)\}, \quad x = (x_1, x_2) \end{aligned}$$

The minimum function value is $f(x^*) = 0$ at $x^* = (1, 1)$. This problem does not have any constraints.

The following statements can be used to solve this problem:

```
proc nlp;
   min f;
   decvar x1 x2;
   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;
   f  = .5 * (f1 * f1 + f2 * f2);
run;
```

The MIN statement identifies the symbol **f** that characterizes the objective function in terms of **f1** and **f2**, and the DECVAR statement names the decision variables **x1** and **x2**. Because there is no explicit optimizing algorithm option specified (TECH=), PROC NLP uses the Newton-Raphson method with ridging, the default algorithm when there are no constraints.

A better way to solve this problem is to take advantage of the fact that f is a sum of squares of f_1 and f_2 and to treat it as a least-squares problem. Using the LSQ statement instead of the MIN statement tells the procedure that this is a least-squares problem, which results in the use of one of the specialized algorithms for solving least-squares problems (for example, Levenberg-Marquardt).

```
proc nlp;
   lsq f1 f2;
   decvar x1 x2;
   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;
run;
```

The LSQ statement results in the minimization of a function that is the sum of squares of functions that appear in the LSQ statement. The least-squares specification is preferred because it enables the procedure to exploit the structure in the problem for numerical stability and performance.

PROC NLP displays the iteration history and the solution to this least-squares problem as shown in Figure 4.1. It shows that the solution has $x_1 = 1$ and $x_2 = 1$. As expected in an unconstrained problem, the gradient at the solution is very close to 0.

```
                    PROC NLP: Least Squares Minimization

                      Levenberg-Marquardt Optimization

                       Scaling Update of More (1978)

                   Parameter Estimates                 2
                   Functions (Observations)            2

                             Optimization Start

Active Constraints                    0  Objective Function          47.40627029
Max Abs Gradient Element   96.895634353  Radius                     969.01382828


                                                                        Actual
                                                      Max Abs            Over
         Rest     Func      Act     Objective  Obj Fun Gradient            Pred
 Iter    arts    Calls      Con      Function   Change  Element  Lambda   Change

    1       0        2        0     42.77209   4.6342    185.0       0   0.0978
    2       0        3        0   1.7971E-29  42.7721    6E-15       0    1.000

                           Optimization Results

Iterations                            2  Function Calls                        4
Jacobian Calls                        3  Active Constraints                    0
Objective Function          1.797124E-29  Max Abs Gradient Element   5.995204E-15
Lambda                                0  Actual Over Pred Change               1
Radius                         18.49802027

ABSGCONV convergence criterion satisfied.


                    PROC NLP: Least Squares Minimization

                            Optimization Results
                             Parameter Estimates
                                                    Gradient
                                                   Objective
                  N Parameter         Estimate        Function

                  1 x1                1.000000    5.995204E-15
                  2 x2                1.000000               0

                  Value of Objective Function = 1.797124E-29
```

Figure 4.1. Least-Squares Minimization

Boundary Constraints on the Decision Variables

Bounds on the decision variables can be used. Suppose, for example, that it is necessary to constrain the decision variables in the previous example to be less than 0.5. That can be done by adding a BOUNDS statement.

```
proc nlp;
   lsq f1 f2;
   decvar x1 x2;
   bounds x1-x2 <= .5;
   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;
run;
```

The solution in Figure 4.2 shows that the decision variables meet the constraint bounds.

```
                PROC NLP: Least Squares Minimization

                       Optimization Results
                        Parameter Estimates
                                          Gradient     Active
                                         Objective     Bound
      N Parameter         Estimate        Function     Constraint

      1 x1                0.500000       -0.500000     Upper BC
      2 x2                0.250000               0

                Value of Objective Function = 0.125
```

Figure 4.2. Least-Squares with Bounds Solution

Linear Constraints on the Decision Variables

More general linear equality or inequality constraints of the form

$$\sum_{j=1}^{n} a_{ij} x_j \left\{ \leq \mid = \mid \geq \right\} b_i \quad \text{for } i = 1, \ldots, m$$

can be specified in a LINCON statement. For example, suppose that in addition to the bounds constraints on the decision variables it is necessary to guarantee that the sum $x_1 + x_2$ is less than or equal to 0.6. That can be achieved by adding a LINCON statement:

```
proc nlp;
   lsq f1 f2;
   decvar x1 x2;
   bounds x1-x2 <= .5;
   lincon x1 + x2 <= .6;
   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;
run;
```

The output in Figure 4.3 displays the iteration history and the convergence criterion.

```
                    PROC NLP: Least Squares Minimization

                      Levenberg-Marquardt Optimization

                       Scaling Update of More (1978)

                    Parameter Estimates                 2
                    Functions (Observations)            2
                    Lower Bounds                        0
                    Upper Bounds                        2
                    Linear Constraints                  1


                                                                                Actual
                                                          Max Abs                 Over
          Rest     Func      Act     Objective  Obj Fun Gradient                 Pred
 Iter     arts    Calls      Con      Function   Change  Element   Lambda      Change

    1        0        3        0       8.19877  21.0512  39.5420   0.0170      0.729
    2        0        4        0       1.05752   7.1412  13.6170   0.0105      0.885
    3        0        5        1       1.04396   0.0136  18.6337        0      0.0128
    4        0        6        1       0.16747   0.8765   0.5552        0      0.997
    5        0        7        1       0.16658 0.000895 0.000324        0      0.998
    6        0        8        1       0.16658 3.06E-10 5.911E-7        0      0.998

                             Optimization Results

Iterations                                6  Function Calls                          9
Jacobian Calls                            7  Active Constraints                      1
Objective Function           0.1665792899  Max Abs Gradient Element     5.9108825E-7
Lambda                                    0  Actual Over Pred Change      0.9981769215
Radius                       0.0000532357

GCONV convergence criterion satisfied.
```

Figure 4.3. Least-Squares with Bounds and Linear Constraints Iteration History

Figure 4.4 shows that the solution satisfies the linear constraint. Note that the procedure displays the active constraints (the constraints that are tight) at optimality.

```
                    PROC NLP: Least Squares Minimization

                             Optimization Results
                              Parameter Estimates
                                                          Gradient
                                                         Objective
                    N Parameter         Estimate          Function

                    1 x1                0.423645         -0.312000
                    2 x2                0.176355         -0.312001

                    Value of Objective Function = 0.1665792899


                     Linear Constraints Evaluated at Solution

           1 ACT          0  =     0.6000  -     1.0000 * x1  -     1.0000 * x2
```

Figure 4.4. Least-Squares with Bounds and Linear Constraints Solution

Nonlinear Constraints on the Decision Variables

More general nonlinear equality or inequality constraints can be specified using an NLINCON statement. Consider the least-squares problem with the additional constraint

$$x_1^2 - 2x_2 \geq 0$$

This constraint is specified by a new function **c1** constrained to be greater than or equal to 0 in the NLINCON statement. The function **c1** is defined in the programming statements.

```
proc nlp tech=QUANEW;
   min f;
   decvar x1 x2;
   bounds x1-x2 <= .5;
   lincon x1 + x2 <= .6;
   nlincon c1 >= 0;

   c1 = x1 * x1 - 2 * x2;

   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;

   f = .5 * (f1 * f1 + f2 * f2);
run;
```

Figure 4.5 shows the iteration history, and Figure 4.6 shows the solution to this problem.

PROC NLP: Nonlinear Minimization

Dual Quasi-Newton Optimization

Modified VMCWD Algorithm of Powell (1978, 1982)

Dual Broyden - Fletcher - Goldfarb - Shanno Update (DBFGS)
Lagrange Multiplier Update of Powell(1982)

Parameter Estimates	2
Lower Bounds	0
Upper Bounds	2
Linear Constraints	1
Nonlinear Constraints	1

Optimization Start

Objective Function	29.25	Maximum Constraint Violation	0
Maximum Gradient of the Lagran Func	76.5		

Iter	Restarts	Function Calls	Objective Function	Maximum Constraint Violation	Predicted Function Reduction	Step Size	Maximum Gradient Element of the Lagrange Function
1	0	4	2.88501	0	2.9362	1.000	20.961
2	0	5	0.91110	0	0.5601	1.000	6.777
3	0	6	0.61803	0	0.00743	1.000	1.148
4′	0	7	0.61090	0	0.0709	1.000	1.194
5′	0	8	0.54427	0	0.6015	1.000	0.988
6	0	10	0.49223	0	0.3369	0.100	0.970
7	0	12	0.45729	0	0.1848	0.114	1.332
8	0	14	0.40786	0	0.0749	0.355	2.390
9	0	15	0.36176	0	0.0556	1.000	1.129
10	0	16	0.33086	0	0.00178	1.000	0.139
11	0	17	0.33017	0	0.000290	1.000	0.0521
12	0	18	0.33004	0	0.000012	1.000	0.00222
13	0	19	0.33003	0	2.963E-8	1.000	0.00004

Optimization Results

Iterations	13	Function Calls	20
Gradient Calls	16	Active Constraints	1
Objective Function	0.3300307304	Maximum Constraint Violation	0
Maximum Projected Gradient	0.0000142688	Value Lagrange Function	0.3300307155
Maximum Gradient of the Lagran Func	0.0000138527	Slope of Search Direction	-2.962973E-8

Figure 4.5. Least-Squares with Bounds, Linear and Nonlinear Constraints, Iteration History

PROC NLP: Nonlinear Minimization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Gradient Objective Function	Gradient Lagrange Function
1	x1	0.246953	0.753018	-0.000013853
2	x2	0.030493	-3.049292	-0.000003421

Value of Objective Function = 0.3300307304

Value of Lagrange Function = 0.3300307155

Linear Constraints Evaluated at Solution

1 0.32255 = 0.6000 - 1.0000 * x1 - 1.0000 * x2

Values of Nonlinear Constraints

	Constraint	Value	Residual	Lagrange Multiplier		
[2]	c1_G	9.71E-9	9.71E-9	1.5246	Active	NLIC

Figure 4.6. Least-Squares with Bounds, Linear and Nonlinear Constraints, Solution

Not all of the optimization methods support nonlinear constraints. In particular the Levenberg-Marquardt method, the default for LSQ, does not support nonlinear constraints. (For more information about the particular algorithms, see the section "Optimization Algorithms" on page 346.) The Quasi-Newton method is the prime choice for solving nonlinear programs with nonlinear constraints. The option TECH=QUANEW in the PROC NLP statement causes the Quasi-Newton method to be used.

A Simple Maximum Likelihood Example

The following is a very simple example of a maximum likelihood estimation problem with the log likelihood function:

$$l(\mu, \sigma) = -\log(\sigma) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

The maximum likelihood estimates of the parameters μ and σ form the solution to

$$\max_{\mu,\sigma>0} \sum_i l_i(\mu, \sigma)$$

where

$$l_i(\mu,\sigma) = -\log(\sigma) - \frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

In the following DATA step, values for x are input into SAS data set X; this data set provides the values of x_i.

```
data x;
  input x @@;
datalines;
1 3 4 5 7
;
```

In the following statements, the DATA=X specification drives the building of the objective function. When each observation in the DATA=X data set is read, a new term $l_i(\mu,\sigma)$ using the value of x_i is added to the objective function LOGLIK specified in the MAX statement.

```
proc nlp data=x vardef=n covariance=h pcov phes;
   profile mean sigma / alpha=.5 .1 .05 .01;
   max loglik;
   parms mean=0, sigma=1;
   bounds sigma > 1e-12;
   loglik=-0.5*((x-mean)/sigma)**2-log(sigma);
run;
```

After a few iterations of the default Newton-Raphson optimization algorithm, PROC NLP produces the results shown in Figure 4.7.

```
                    PROC NLP: Nonlinear Maximization

                         Optimization Results
                          Parameter Estimates
                                      Approx                          Approx
N Parameter          Estimate         Std Err         t Value        Pr > |t|

1 mean               4.000000        0.894427        4.472136        0.006566
2 sigma              2.000000        0.632456        3.162278        0.025031

                         Optimization Results
                          Parameter Estimates
                                 Gradient
                                Objective
                                 Function

                              -1.33149E-10
                               5.6064147E-9

                Value of Objective Function = -5.965735903
```

Figure 4.7. Maximum Likelihood Estimates

In unconstrained maximization, the gradient (that is, the vector of first derivatives) at the solution must be very close to zero and the Hessian matrix at the solution (that is, the matrix of second derivatives) must have nonpositive eigenvalues. The Hessian matrix is displayed in Figure 4.8.

Hessian Matrix

	mean	sigma
mean	−1.250000003	1.331489E−10
sigma	1.331489E−10	−2.500000014

Determinant = 3.1250000245

Matrix has Only Negative Eigenvalues

Figure 4.8. Hessian Matrix

Under reasonable assumptions, the approximate standard errors of the estimates are the square roots of the diagonal elements of the covariance matrix of the parameter estimates, which (because of the COV=H specification) is the same as the inverse of the Hessian matrix. The covariance matrix is shown in Figure 4.9.

Covariance Matrix 2: H = (NOBS/d) inv(G)

	mean	sigma
mean	0.7999999982	4.260766E−11
sigma	4.260766E−11	0.3999999978

Factor sigm = 1

Determinant = 0.3199999975

Matrix has 2 Positive Eigenvalue(s)

Figure 4.9. Covariance Matrix

The PROFILE statement computes the values of the profile likelihood confidence limits on SIGMA and MEAN, as shown in Figure 4.10.

PROC NLP: Nonlinear Maximization

Wald and PL Confidence Limits

N	Parameter	Estimate	Alpha	Profile Likelihood Confidence Limits	
1	mean	4.000000	0.500000	3.384431	4.615569
1	mean	.	0.100000	2.305716	5.694284
1	mean	.	0.050000	1.849538	6.150462
1	mean	.	0.010000	0.670351	7.329649
2	sigma	2.000000	0.500000	1.638972	2.516078
2	sigma	.	0.100000	1.283506	3.748633
2	sigma	.	0.050000	1.195936	4.358321
2	sigma	.	0.010000	1.052584	6.064107

Wald and PL Confidence Limits

Wald Confidence Limits

3.396718	4.603282
2.528798	5.471202
2.246955	5.753045
1.696108	6.303892
1.573415	2.426585
0.959703	3.040297
0.760410	3.239590
0.370903	3.629097

Figure 4.10. Confidence Limits

Syntax: NLP Procedure

Below are statements used in PROC NLP, listed in alphabetical order as they appear in the text that follows.

PROC NLP *options* ;
ARRAY *function names* ;
BOUNDS *boundary constraints* ;
BY *variables* ;
CRPJAC *variables* ;
DECVAR *function names* ;
GRADIENT *variables* ;
HESSIAN *variables* ;
INCLUDE *model files* ;
JACNLC *variables* ;
JACOBIAN *function names* ;
LABEL *decision variable labels* ;
LINCON *linear constraints* ;
MATRIX *matrix specification* ;
MIN, MAX, or LSQ *function names* ;
MINQUAD or MAXQUAD *matrix, vector, or number* ;
NLINCON *nonlinear constraints* ;
PROFILE *profile specification* ;
Program Statements ;

Functional Summary

The following table outlines the options in PROC NLP classified by function.

Table 4.1. Functional Summary

Description	Statement	Option
Input Data Set Options:		
input data set	PROC NLP	DATA=
initial values and constraints	PROC NLP	INEST=
quadratic objective function	PROC NLP	INQUAD=
program statements	PROC NLP	MODEL=
skip missing value observations	PROC NLP	NOMISS
Output Data Set Options:		
variables and derivatives	PROC NLP	OUT=
result parameter values	PROC NLP	OUTEST=
program statements	PROC NLP	OUTMODEL=
combine various OUT... statements	PROC NLP	OUTALL
CRP Jacobian in the OUTEST= data set	PROC NLP	OUTCRPJAC
derivatives in the OUT= data set	PROC NLP	OUTDER=

Description	Statement	Option
grid in the OUTEST= data set	PROC NLP	OUTGRID
Hessian in the OUTEST= data set	PROC NLP	OUTHESSIAN
iterative output in the OUTEST= data set	PROC NLP	OUTITER
Jacobian in the OUTEST= data set	PROC NLP	OUTJAC
NLC Jacobian in the OUTEST= data set	PROC NLP	OUTNLCJAC
time in the OUTEST= data set	PROC NLP	OUTTIME
Optimization Options:		
minimization method	PROC NLP	TECH=
update technique	PROC NLP	UPDATE=
version of optimization technique	PROC NLP	VERSION=
line-search method	PROC NLP	LINESEARCH=
line-search precision	PROC NLP	LSPRECISION=
type of Hessian scaling	PROC NLP	HESCAL=
start for approximated Hessian	PROC NLP	INHESSIAN=
iteration number for update restart	PROC NLP	RESTART=
Initial Value Options:		
produce best grid points	PROC NLP	BEST=
infeasible points in grid search	PROC NLP	INFEASIBLE
pseudorandom initial values	PROC NLP	RANDOM=
constant initial values	PROC NLP	INITIAL=
Derivative Options:		
finite-difference derivatives	PROC NLP	FD=
finite-difference derivatives	PROC NLP	FDHESSIAN=
compute finite-difference interval	PROC NLP	FDINT=
use only diagonal of Hessian	PROC NLP	DIAHES
test gradient specification	PROC NLP	GRADCHECK=
Constraint Options:		
range for active constraints	PROC NLP	LCEPSILON=
LM tolerance for deactivating	PROC NLP	LCDEACT=
tolerance for dependent constraints	PROC NLP	LCSINGULAR=
sum all observations for continuous functions	NLINCON	/ SUMOBS
evaluate each observation for continuous functions	NLINCON	/ EVERYOBS
Termination Criteria Options:		
maximum number of function calls	PROC NLP	MAXFUNC=
maximum number of iterations	PROC NLP	MAXITER=
minimum number of iterations	PROC NLP	MINITER=
upper limit on CPU time	PROC NLP	MAXTIME=
absolute function convergence criterion	PROC NLP	ABSCONV=
absolute function convergence criterion	PROC NLP	ABSFCONV=

Description	Statement	Option
absolute gradient convergence criterion	PROC NLP	ABSGCONV=
absolute parameter convergence criterion	PROC NLP	ABSXCONV=
relative function convergence criterion	PROC NLP	FCONV=
relative function convergence criterion	PROC NLP	FCONV2=
relative gradient convergence criterion	PROC NLP	GCONV=
relative gradient convergence criterion	PROC NLP	GCONV2=
relative parameter convergence criterion	PROC NLP	XCONV=
used in FCONV, GCONV criterion	PROC NLP	FSIZE=
used in XCONV criterion	PROC NLP	XSIZE=
Covariance Matrix Options:		
type of covariance matrix	PROC NLP	COV=
σ^2 factor of COV matrix	PROC NLP	SIGSQ=
determine factor of COV matrix	PROC NLP	VARDEF=
absolute singularity for inertia	PROC NLP	ASINGULAR=
relative M singularity for inertia	PROC NLP	MSINGULAR=
relative V singularity for inertia	PROC NLP	VSINGULAR=
threshold for Moore-Penrose inverse	PROC NLP	G4=
tolerance for singular COV matrix	PROC NLP	COVSING=
profile confidence limits	PROC NLP	CLPARM=
Printed Output Options:		
display (almost) all printed output	PROC NLP	PALL
suppress all printed output	PROC NLP	NOPRINT
reduce some default output	PROC NLP	PSHORT
reduce most default output	PROC NLP	PSUMMARY
display initial values and gradients	PROC NLP	PINIT
display optimization history	PROC NLP	PHISTORY
display Jacobian matrix	PROC NLP	PJACOBI
display crossproduct Jacobian matrix	PROC NLP	PCRPJAC
display Hessian matrix	PROC NLP	PHESSIAN
display Jacobian of nonlinear constraints	PROC NLP	PNLCJAC
display values of grid points	PROC NLP	PGRID
display values of functions in LSQ, MIN, MAX	PROC NLP	PFUNCTION
display approximate standard errors	PROC NLP	PSTDERR
display covariance matrix	PROC NLP	PCOV
display eigenvalues for covariance matrix	PROC NLP	PEIGVAL
print code evaluation problems	PROC NLP	PERROR
print measures of CPU time	PROC NLP	PTIME
display model program, variables	PROC NLP	LIST
display compiled model program	PROC NLP	LISTCODE
Step Length Options:		
damped steps in line search	PROC NLP	DAMPSTEP=
maximum trust region radius	PROC NLP	MAXSTEP=

Description	Statement	Option
initial trust region radius	PROC NLP	INSTEP=
Profile Point and Confidence Interval Options:		
factor relating discrepancy function to χ^2 quantile	PROFILE	FFACTOR=
scale for y values written to OUTEST= data set	PROFILE	FORCHI=
upper bound for confidence limit search	PROFILE	FEASRATIO=
write all confidence limit parameter estimates to OUTEST= data set	PROFILE	OUTTABLE
Miscellaneous Options:		
number of accurate digits in objective function	PROC NLP	FDIGITS=
number of accurate digits in nonlinear constraints	PROC NLP	CDIGITS=
general singularity criterion	PROC NLP	SINGULAR=
do not compute inertia of matrices	PROC NLP	NOEIGNUM
check optimality in neighborhood	PROC NLP	OPTCHECK=

PROC NLP Statement

PROC NLP *options* **;**

This statement invokes the NLP procedure. The following options are used with the PROC NLP statement.

ABSCONV=r
ABSTOL=r

specifies an absolute function convergence criterion. For minimization (maximization), termination requires $f(x^{(k)}) \leq (\geq)\ r$. The default value of ABSCONV is the negative (positive) square root of the largest double precision value.

ABSFCONV=$r[n]$
ABSFTOL=$r[n]$

specifies an absolute function convergence criterion. For all techniques except NMSIMP, termination requires a small change of the function value in successive iterations:

$$|f(x^{(k-1)}) - f(x^{(k)})| \leq r$$

For the NMSIMP technique the same formula is used, but $x^{(k)}$ is defined as the vertex with the lowest function value, and $x^{(k-1)}$ is defined as the vertex with the highest function value in the simplex. The default value is $r = 0$. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

ABSGCONV=$r[n]$
ABSGTOL=$r[n]$
specifies the absolute gradient convergence criterion. Termination requires the maximum absolute gradient element to be small:

$$\max_j |g_j(x^{(k)})| \leq r$$

This criterion is not used by the NMSIMP technique. The default value is $r = 1\text{E}{-5}$. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

ABSXCONV=$r[n]$
ABSXTOL=$r[n]$
specifies the absolute parameter convergence criterion. For all techniques except NMSIMP, termination requires a small Euclidean distance between successive parameter vectors:

$$\| x^{(k)} - x^{(k-1)} \|_2 \leq r$$

For the NMSIMP technique, termination requires either a small length $\alpha^{(k)}$ of the vertices of a restart simplex

$$\alpha^{(k)} \leq r$$

or a small simplex size

$$\delta^{(k)} \leq r$$

where the simplex size $\delta^{(k)}$ is defined as the L_1 distance of the simplex vertex $y^{(k)}$ with the smallest function value to the other n simplex points $x_l^{(k)} \neq y^{(k)}$:

$$\delta^{(k)} = \sum_{x_l \neq y} \| x_l^{(k)} - y^{(k)} \|_1$$

The default value is $r = 1\text{E}{-4}$ for the COBYLA NMSIMP technique, $r = 1\text{E}{-8}$ for the standard NMSIMP technique, and $r = 0$ otherwise. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

ASINGULAR=r
ASING=r
specifies an absolute singularity criterion for measuring singularity of Hessian and crossproduct Jacobian and their projected forms, which may have to be converted to compute the covariance matrix. The default is the square root of the smallest positive double precision value. For more information, see the section "Covariance Matrix" on page 370.

BEST=i
produces the i best grid points only. This option not only restricts the output, it also can significantly reduce the computation time needed for sorting the grid point information.

CDIGITS=r

specifies the number of accurate digits in nonlinear constraint evaluations. Fractional values such as CDIGITS=4.7 are allowed. The default value is $r = -\log_{10}(\epsilon)$, where ϵ is the machine precision. The value of r is used to compute the interval length h for the computation of finite-difference approximations of the Jacobian matrix of nonlinear constraints.

CLPARM= PL | WALD | BOTH

is similar to but not the same as that used by other SAS procedures. Using CLPARM=BOTH is equivalent to specifying

```
PROFILE / ALPHA=0.5 0.1 0.05 0.01 OUTTABLE;
```

The CLPARM=BOTH option specifies that profile confidence limits (PL CLs) for all parameters and for $\alpha = .5, .1, .05, .01$ are computed and displayed or written to the OUTEST= data set. Computing the profile confidence limits for all parameters can be very expensive and should be avoided when a difficult optimization problem or one with many parameters is solved. The OUTTABLE option is valid only when an OUTEST= data set is specified in the PROC NLP statement. For CLPARM=BOTH, the table of displayed output contains the Wald confidence limits computed from the standard errors as well as the PL CLs. The Wald confidence limits are not computed (displayed or written to the OUTEST= data set) unless the approximate covariance matrix of parameters is computed.

COV= 1 | 2 | 3 | 4 | 5 | 6 | M | H | J | B | E | U
COVARIANCE= 1 | 2 | 3 | 4 | 5 | 6 | M | H | J | B | E | U

specifies one of six formulas for computing the covariance matrix. For more information, see the section "Covariance Matrix" on page 370.

COVSING=r

specifies a threshold $r > 0$ that determines whether the eigenvalues of a singular Hessian matrix or crossproduct Jacobian matrix are considered to be zero. For more information, see the section "Covariance Matrix" on page 370.

DAMPSTEP[=r**]**
DS[=r**]**

specifies that the initial step length value $\alpha^{(0)}$ for each line search (used by the QUANEW, HYQUAN, CONGRA, or NEWRAP technique) cannot be larger than r times the step length value used in the former iteration. If the DAMPSTEP option is specified but r is not specified, the default is $r = 2$. The DAMPSTEP=r option can prevent the line-search algorithm from repeatedly stepping into regions where some objective functions are difficult to compute or where they could lead to floating point overflows during the computation of objective functions and their derivatives. The DAMPSTEP=r option can save time-costly function calls during the line searches of objective functions that result in very small steps. For more information, see the section "Restricting the Step Length" on page 366.

DATA=*SAS-data-set*

allows variables from the specified data set to be used in the specification of the objective function f. For more information, see the section "DATA= Input Data Set" on page 373.

DIAHES

specifies that only the diagonal of the Hessian or crossproduct Jacobian is used. This saves function evaluations but may slow the convergence process considerably. Note that the DIAHES option refers to both the Hessian and the crossproduct Jacobian when using the LSQ statement. When derivatives are specified using the HESSIAN or CRPJAC statement, these statements must refer only to the n diagonal derivative elements (otherwise, the $n(n+1)/2$ derivatives of the lower triangle must be specified). The DIAHES option is ignored if a quadratic programming with a constant Hessian is specified by TECH=QUADAS or TECH=LICOMP.

FCONV=*r*[*n*]
FTOL=*r*[*n*]

specifies the relative function convergence criterion. For all techniques except NMSIMP, termination requires a small relative change of the function value in successive iterations:

$$\frac{|f(x^{(k)}) - f(x^{(k-1)})|}{\max(|f(x^{(k-1)})|, \mathit{FSIZE})} \leq r$$

where *FSIZE* is defined by the FSIZE= option. For the NMSIMP technique, the same formula is used, but $x^{(k)}$ is defined as the vertex with the lowest function value, and $x^{(k-1)}$ is defined as the vertex with the highest function value in the simplex. The default value is $r = 10^{-\mathit{FDIGITS}}$ where *FDIGITS* is the value of the FDIGITS= option. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

FCONV2=*r*[*n*]
FTOL2=*r*[*n*]

specifies another function convergence criterion. For least-squares problems and all techniques except NMSIMP, termination requires a small predicted reduction

$$df^{(k)} \approx f(x^{(k)}) - f(x^{(k)} + s^{(k)})$$

of the objective function. The predicted reduction

$$\begin{aligned} df^{(k)} &= -g^{(k)T}s^{(k)} - \frac{1}{2}s^{(k)T}G^{(k)}s^{(k)} \\ &= -\frac{1}{2}s^{(k)T}g^{(k)} \\ &\leq r \end{aligned}$$

is based on approximating the objective function f by the first two terms of the Taylor series and substituting the Newton step

$$s^{(k)} = -G^{(k)-1}g^{(k)}$$

For the NMSIMP technique, termination requires a small standard deviation of the function values of the $n+1$ simplex vertices $x_l^{(k)},\ l = 0, \ldots, n$,

$$\sqrt{\frac{1}{n+1}\sum_l (f(x_l^{(k)}) - \overline{f}(x^{(k)}))^2} \leq r$$

where $\overline{f}(x^{(k)}) = \frac{1}{n+1}\sum_l f(x_l^{(k)})$. If there are n_{act} boundary constraints active at $x^{(k)}$, the mean and standard deviation are computed only for the $n+1-n_{act}$ unconstrained vertices. The default value is $r = 1\text{E}{-6}$ for the NMSIMP technique and the QUANEW technique with nonlinear constraints, and $r = 0$ otherwise. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

FD[=FORWARD | CENTRAL | *number*]
specifies that all derivatives be computed using finite-difference approximations. The following specifications are permitted:

FD=FORWARD uses forward differences.

FD=CENTRAL uses central differences.

FD=*number* uses central differences for the initial and final evaluations of the gradient, Jacobian, and Hessian. During iteration, start with forward differences and switch to a corresponding central-difference formula during the iteration process when one of the following two criteria is satisfied:

- The absolute maximum gradient element is less than or equal to *number* times the ABSGCONV threshold.
- The term left of the GCONV criterion is less than or equal to $\max(1.0\text{E}-6, number \times$ GCONV threshold). The $1.0\text{E}{-6}$ ensures that the switch is done, even if you set the GCONV threshold to zero.

FD is equivalent to FD=100.

Note that the FD and FDHESSIAN options cannot apply at the same time. The FDHESSIAN option is ignored when only first-order derivatives are used, for example, when the LSQ statement is used and the HESSIAN is not explicitly needed (displayed or written to a data set). For more information, see the section "Finite-Difference Approximations of Derivatives" on page 357.

FDHESSIAN[=FORWARD | CENTRAL]
FDHES[=FORWARD | CENTRAL]
FDH[=FORWARD | CENTRAL]
specifies that second-order derivatives be computed using finite-difference approximations based on evaluations of the gradients.

FDHESSIAN=FORWARD	uses forward differences.
FDHESSIAN=CENTRAL	uses central differences.
FDHESSIAN	uses forward differences for the Hessian except for the initial and final output.

Note that the FD and FDHESSIAN options cannot apply at the same time. For more information, see the section "Finite-Difference Approximations of Derivatives" on page 357

FDIGITS=r

specifies the number of accurate digits in evaluations of the objective function. Fractional values such as FDIGITS=4.7 are allowed. The default value is $r = -\log_{10}(\epsilon)$, where ϵ is the machine precision. The value of r is used to compute the interval length h for the computation of finite-difference approximations of the derivatives of the objective function and for the default value of the FCONV= option.

FDINT= OBJ | CON | ALL

specifies how the finite-difference intervals h should be computed. For FDINT=OBJ, the interval h is based on the behavior of the objective function; for FDINT=CON, the interval h is based on the behavior of the nonlinear constraints functions; and for FDINT=ALL, the interval h is based on the behavior of the objective function and the nonlinear constraints functions. For more information, see the section "Finite-Difference Approximations of Derivatives" on page 357.

FSIZE=r

specifies the FSIZE parameter of the relative function and relative gradient termination criteria. The default value is $r = 0$. For more details, refer to the FCONV= and GCONV= options.

G4=n

is used when the covariance matrix is singular. The value $n > 0$ determines which generalized inverse is computed. The default value of n is 60. For more information, see the section "Covariance Matrix" on page 370.

GCONV=$r[n]$
GTOL=$r[n]$

specifies the relative gradient convergence criterion. For all techniques except the CONGRA and NMSIMP techniques, termination requires that the normalized predicted function reduction is small:

$$\frac{g(x^{(k)})^T[G^{(k)}]^{-1}g(x^{(k)})}{\max(|f(x^{(k)})|, FSIZE)} \leq r$$

where *FSIZE* is defined by the FSIZE= option. For the CONGRA technique (where a reliable Hessian estimate G is not available),

$$\frac{\| g(x^{(k)}) \|_2^2 \quad \| s(x^{(k)}) \|_2}{\| g(x^{(k)}) - g(x^{(k-1)}) \|_2 \max(|f(x^{(k)})|, \text{FSIZE})} \leq r$$

is used. This criterion is not used by the NMSIMP technique. The default value is r = 1E−8. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

GCONV2=$r[n]$
GTOL2=$r[n]$

specifies another relative gradient convergence criterion,

$$\max_j \frac{|g_j(x^{(k)})|}{\sqrt{f(x^{(k)})G^{(k)}_{j,j}}} \leq r$$

This option is valid only when using the TRUREG, LEVMAR, NRRIDG, and NEWRAP techniques on least-squares problems. The default value is $r = 0$. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

GRADCHECK[= NONE | FAST | DETAIL]
GC[= NONE | FAST | DETAIL]

Specifying GRADCHECK=DETAIL computes a test vector and test matrix to check whether the gradient g specified by a GRADIENT statement (or indirectly by a JACOBIAN statement) is appropriate for the function f computed by the program statements. If the specification of the first derivatives is correct, the elements of the test vector and test matrix should be relatively small. For very large optimization problems, the algorithm can be too expensive in terms of computer time and memory. If the GRADCHECK option is not specified, a fast derivative test identical to the GRADCHECK=FAST specification is performed by default. It is possible to suppress the default derivative test by specifying GRADCHECK=NONE. For more information, see the section "Testing the Gradient Specification" on page 360.

HESCAL= 0 | 1 | 2 | 3
HS= 0 | 1 | 2 | 3

specifies the scaling version of the Hessian or crossproduct Jacobian matrix used in NRRIDG, TRUREG, LEVMAR, NEWRAP, or DBLDOG optimization. If the value of the HESCAL= option is not equal to zero, the first iteration and each restart iteration sets the diagonal scaling matrix $D^{(0)} = \text{diag}(d_i^{(0)})$:

$$d_i^{(0)} = \sqrt{\max(|G^{(0)}_{i,i}|, \epsilon)}$$

where $G^{(0)}_{i,i}$ are the diagonal elements of the Hessian or crossproduct Jacobian matrix. In all other iterations, the diagonal scaling matrix $D^{(0)} = \text{diag}(d_i^{(0)})$ is updated depending on the HESCAL= option:

HESCAL=0 specifies that no scaling is done

HESCAL=1 specifies the Moré (1978) scaling update:

$$d_i^{(k+1)} = \max\left(d_i^{(k)}, \sqrt{\max(|G^{(k)}_{i,i}|, \epsilon)}\right)$$

HESCAL=2 specifies the Dennis, Gay, and Welsch (1981) scaling update:

$$d_i^{(k+1)} = \max\left(0.6d_i^{(k)}, \sqrt{\max(|G^{(k)}_{i,i}|, \epsilon)}\right)$$

HESCAL=3 specifies that d_i is reset in each iteration:

$$d_i^{(k+1)} = \sqrt{\max(|G_{i,i}^{(k)}|, \epsilon)}$$

where ϵ is the relative machine precision. The default value is HESCAL=1 for LEVMAR minimization and HESCAL=0 otherwise. Scaling of the Hessian or crossproduct Jacobian matrix can be time-consuming in the case where general linear constraints are active.

INEST=*SAS-data-set*
INVAR=*SAS-data-set*
ESTDATA=*SAS-data-set*

can be used to specify the initial values of the parameters defined in a DECVAR statement as well as simple boundary constraints and general linear constraints. The INEST= data set can contain additional variables with names corresponding to constants used in the program statements. At the beginning of each run of PROC NLP, the values of the constants are read from the PARMS observation, initializing the constants in the program statements. For more information, see the section "INEST= Input Data Set" on page 374.

INFEASIBLE
IFP

specifies that the function values of both feasible and infeasible grid points are to be computed, displayed, and written to the OUTEST= data set, although only the feasible grid points are candidates for the starting point $x^{(0)}$. This option enables you to explore the shape of the objective function of points surrounding the feasible region. For the output, the grid points are sorted first with decreasing values of the maximum constraint violation. Points with the same value of the maximum constraint violation are then sorted with increasing (minimization) or decreasing (maximization) value of the objective function. Using the BEST= option restricts only the number of best grid points in the displayed output, not those in the data set. The INFEASIBLE option affects both the displayed output and the output saved to the OUTEST= data set. The OUTGRID option can be used to write the grid points and their function values to an OUTEST= data set. After small modifications (deleting unneeded information), this data set can be used with the G3D procedure of SAS/GRAPH to generate a three-dimensional surface plot of the objective function depending on two selected parameters. For more information on grids, see the section "DECVAR Statement" on page 327.

INHESSIAN[=r]
INHESS[=r]

specifies how the initial estimate of the approximate Hessian is defined for the quasi-Newton techniques QUANEW, DBLDOG, and HYQUAN. There are two alternatives:

- The $= r$ specification is not used: the initial estimate of the approximate Hessian is set to the true Hessian or crossproduct Jacobian at $x^{(0)}$.
- The $= r$ specification is used: the initial estimate of the approximate Hessian is set to the multiple of the identity matrix rI.

By default, if INHESSIAN=r is not specified, the initial estimate of the approximate Hessian is set to the multiple of the identity matrix rI, where the scalar r is computed from the magnitude of the initial gradient. For most applications, this is a sufficiently good first approximation.

INITIAL=*r*

specifies a value r as the common initial value for all parameters for which no other initial value assignments by the DECVAR statement or an INEST= data set are made.

INQUAD=*SAS-data-set*

can be used to specify (the nonzero elements of) the matrix H, the vector g, and the scalar c of a quadratic programming problem, $f(x) = \frac{1}{2}x^T Hx + g^T x + c$. This option cannot be used together with the NLINCON statement. Two forms (*dense* and *sparse*) of the INQUAD= data set can be used. For more information, see the section "INQUAD= Input Data Set" on page 375.

INSTEP=*r*

For highly nonlinear objective functions, such as the EXP function, the default initial radius of the trust region algorithms TRUREG, DBLDOG, or LEVMAR or the default step length of the line-search algorithms can result in arithmetic overflows. If this occurs, decreasing values of $0 < r < 1$ should be specified, such as INSTEP=1E$-$1, INSTEP=1E$-$2, INSTEP=1E$-$4, and so on, until the iteration starts successfully.

- For trust region algorithms (TRUREG, DBLDOG, LEVMAR), the INSTEP= option specifies a factor $r > 0$ for the initial radius $\Delta^{(0)}$ of the trust region. The default initial trust region radius is the length of the scaled gradient. This step corresponds to the default radius factor of $r = 1$.
- For line-search algorithms (NEWRAP, CONGRA, QUANEW, HYQUAN), the INSTEP= option specifies an upper bound for the initial step length for the line search during the first five iterations. The default initial step length is $r = 1$.
- For the Nelder-Mead simplex algorithm (NMSIMP), the INSTEP=r option defines the size of the initial simplex.

For more details, see the section "Computational Problems" on page 367.

LCDEACT=*r*
LCD=*r*

specifies a threshold r for the Lagrange multiplier that decides whether an active inequality constraint remains active or can be deactivated. For a maximization (minimization), an active inequality constraint can be deactivated only if its Lagrange multiplier is greater (less) than the threshold value r. For maximization, r must be greater than zero; for minimization, r must be smaller than zero. The default value is

$$r = \pm \min(0.01, \max(0.1 \times ABSGCONV, 0.001 \times gmax^{(k)}))$$

where the $+$ stands for maximization, the $-$ for minimization, ABSGCONV is the value of the absolute gradient criterion, and $gmax^{(k)}$ is the maximum absolute element of the (projected) gradient $g^{(k)}$ or $Z^T g^{(k)}$.

LCEPSILON=r
LCEPS=r
LCE=r

specifies the range $r > 0$ for active and violated boundary and linear constraints. During the optimization process, the introduction of rounding errors can force PROC NLP to increase the value of r by a factor of $10, 100, \ldots$. If this happens it is indicated by a message written to the log. For more information, see the section "Linear Complementarity (LICOMP)" on page 350.

LCSINGULAR=r
LCSING=r
LCS=r

specifies a criterion $r > 0$ used in the update of the QR decomposition that decides whether an active constraint is linearly dependent on a set of other active constraints. The default value is $r = 1\text{E}{-8}$. The larger r becomes, the more the active constraints are recognized as being linearly dependent. If the value of r is larger than 0.1, it is reset to 0.1.

LINESEARCH=i
LIS=i

specifies the line-search method for the CONGRA, QUANEW, HYQUAN, and NEWRAP optimization techniques. Refer to Fletcher (1987) for an introduction to line-search techniques. The value of i can be $1, \ldots, 8$. For CONGRA, QUANEW, and NEWRAP, the default value is $i = 2$. A special line-search method is the default for the least-squares technique HYQUAN that is based on an algorithm developed by Lindström and Wedin (1984). Although it needs more memory, this default line-search method sometimes works better with large least-squares problems. However, by specifying LIS=i, $i = 1, \ldots, 8$, it is possible to use one of the standard techniques with HYQUAN.

LIS=1 specifies a line-search method that needs the same number of function and gradient calls for cubic interpolation and cubic extrapolation.

LIS=2 specifies a line-search method that needs more function than gradient calls for quadratic and cubic interpolation and cubic extrapolation; this method is implemented as shown in Fletcher (1987) and can be modified to an exact line search by using the LSPRECISION= option.

LIS=3 specifies a line-search method that needs the same number of function and gradient calls for cubic interpolation and cubic extrapolation; this method is implemented as shown in Fletcher (1987) and can be modified to an exact line search by using the LSPRECISION= option.

LIS=4 specifies a line-search method that needs the same number of function and gradient calls for stepwise extrapolation and cubic interpolation.

LIS=5 specifies a line-search method that is a modified version of LIS=4.

LIS=6 specifies golden section line search (Polak 1971), which uses only function values for linear approximation.

LIS=7 specifies bisection line search (Polak 1971), which uses only function values for linear approximation.

LIS=8 specifies the Armijo line-search technique (Polak 1971), which uses only function values for linear approximation.

LIST
displays the model program and variable lists. The LIST option is a debugging feature and is not normally needed. This output is not included in either the default output or the output specified by the PALL option.

LISTCODE
displays the derivative tables and the compiled program code. The LISTCODE option is a debugging feature and is not normally needed. This output is not included in either the default output or the output specified by the PALL option. The option is similar to that used in MODEL procedure in SAS/ETS software.

LSPRECISION=r
LSP=r
specifies the degree of accuracy that should be obtained by the line-search algorithms LIS=2 and LIS=3. Usually an imprecise line search is inexpensive and sufficient for convergence to the optimum. For difficult optimization problems, a more precise and expensive line search may be necessary (Fletcher 1987). The second (default for NEWRAP, QUANEW, and CONGRA) and third line-search methods approach exact line search for small LSPRECISION= values. In the presence of numerical problems, it is advised to decrease the LSPRECISION= value to obtain a more precise line search. The default values are as follows:

TECH=	UPDATE=	LSP default
QUANEW	DBFGS, BFGS	$r = 0.4$
QUANEW	DDFP, DFP	$r = 0.06$
HYQUAN	DBFGS	$r = 0.1$
HYQUAN	DDFP	$r = 0.06$
CONGRA	all	$r = 0.1$
NEWRAP	no update	$r = 0.9$

For more details, refer to Fletcher (1987).

MAXFUNC=i
MAXFU=i
specifies the maximum number i of function calls in the optimization process. The default values are

- TRUREG, LEVMAR, NRRIDG, NEWRAP: 125
- QUANEW, HYQUAN, DBLDOG: 500

- CONGRA, QUADAS: 1000
- NMSIMP: 3000

Note that the optimization can be terminated only after completing a full iteration. Therefore, the number of function calls that are actually performed can exceed the number that is specified by the MAXFUNC= option.

MAXITER=$i[n]$
MAXIT=$i[n]$
specifies the maximum number i of iterations in the optimization process. The default values are:

- TRUREG, LEVMAR, NRRIDG, NEWRAP: 50
- QUANEW, HYQUAN, DBLDOG: 200
- CONGRA, QUADAS: 400
- NMSIMP: 1000

This default value is valid also when i is specified as a missing value. The optional second value n is valid only for TECH=QUANEW with nonlinear constraints. It specifies an upper bound n for the number of iterations of an algorithm used to reduce the violation of nonlinear constraints at a starting point. The default value is $n = 20$.

MAXSTEP=$r[n]$
specifies an upper bound for the step length of the line-search algorithms during the first n iterations. By default, r is the largest double precision value and n is the largest integer available. Setting this option can increase the speed of convergence for TECH=CONGRA, TECH=QUANEW, TECH=HYQUAN, and TECH=NEWRAP.

MAXTIME=r
specifies an upper limit of r seconds of CPU time for the optimization process. The default value is the largest floating point double representation of the computer. Note that the time specified by the MAXTIME= option is checked only once at the end of each iteration. Therefore, the actual running time of the PROC NLP job may be longer than that specified by the MAXTIME= option. The actual running time includes the rest of the time needed to finish the iteration, time for the output of the (temporary) results, and (if required) the time for saving the results in an OUTEST= data set. Using the MAXTIME= option with a permanent OUTEST= data set enables you to separate large optimization problems into a series of smaller problems that need smaller amounts of CPU time.

MINITER=i
MINIT=i
specifies the minimum number of iterations. The default value is zero. If more iterations than are actually needed are requested for convergence to a stationary point, the optimization algorithms can behave strangely. For example, the effect of rounding errors can prevent the algorithm from continuing for the required number of iterations.

MODEL=*model-name, model-list*
MOD=*model-name, model-list*
MODFILE=*model-name, model-list*

reads the program statements from one or more input model files created by previous PROC NLP steps using the OUTMODEL= option. If it is necessary to include the program code at a special location in newly written code, the INCLUDE statement can be used instead of using the MODEL= option. Using both the MODEL= option and the INCLUDE statement with the same model file will include the same model twice, which can produce different results than including it once. The MODEL= option is similar to the option used in PROC MODEL in SAS/ETS software.

MSINGULAR=r
MSING=r

specifies a relative singularity criterion $r > 0$ for measuring singularity of Hessian and crossproduct Jacobian and their projected forms. The default value is $1\mathrm{E}{-12}$ if the SINGULAR= option is not specified and $\max(10 \times \epsilon, 1\mathrm{E} - 4 \times SINGULAR)$ otherwise. For more information, see the section "Covariance Matrix" on page 370.

NOEIGNUM

suppresses the computation and output of the determinant and the inertia of the Hessian, crossproduct Jacobian, and covariance matrices. The inertia of a symmetric matrix are the numbers of negative, positive, and zero eigenvalues. For large applications, the NOEIGNUM option can save computer time.

NOMISS

is valid only for those variables of the DATA= data set that are referred to in program statements. If the NOMISS option is specified, observations with any missing value for those variables are skipped. If the NOMISS option is not specified, the missing value may result in a missing value of the objective function, implying that the corresponding BY group of data is not processed.

NOPRINT
NOP

suppresses the output.

OPTCHECK[=r**]**

computes the function values $f(x_l)$ of a grid of points x_l in a small neighborhood of x^*. The x_l are located in a ball of radius r about x^*. If the OPTCHECK option is specified without r, the default value is $r = 0.1$ at the starting point and $r = 0.01$ at the terminating point. If a point x_l^* is found with a better function value than $f(x^*)$, then optimization is restarted at x_l^*. For more information on grids, see the section "DECVAR Statement" on page 327.

OUT=*SAS-data-set*

creates an output data set that contains those variables of a DATA= input data set referred to in the program statements plus additional variables computed by performing the program statements of the objective function, derivatives, and nonlinear constraints. The OUT= data set can also contain first- and second-order derivatives of these variables if the OUTDER= option is specified. The variables and derivatives are evaluated at x^*; for TECH=NONE, they are evaluated at x^0.

OUTALL
If an OUTEST= data set is specified, this option sets the OUTHESSIAN option if the MIN or MAX statement is used. If the LSQ statement is used, the OUTALL option sets the OUTCRPJAC option. If nonlinear constraints are specified using the NLINCON statement, the OUTALL option sets the OUTNLCJAC option.

OUTCRPJAC
If an OUTEST= data set is specified, the crossproduct Jacobian matrix of the m functions composing the least-squares function is written to the OUTEST= data set.

OUTDER= 0 | 1 | 2
specifies whether or not derivatives are written to the OUT= data set. For OUTDER=2, first- and second-order derivatives are written to the data set; for OUTDER=1, only first-order derivatives are written; for OUTDER=0, no derivatives are written to the data set. The default value is OUTDER=0. Derivatives are evaluated at x^*.

OUTEST=*SAS-data-set*
OUTVAR=*SAS-data-set*
creates an output data set that contains the results of the optimization. This is useful for reporting and for restarting the optimization in a subsequent execution of the procedure. Information in the data set can include parameter estimates, gradient values, constraint information, Lagrangian values, Hessian values, Jacobian values, covariance, standard errors, and confidence intervals.

OUTGRID
writes the grid points and their function values to the OUTEST= data set. By default, only the feasible grid points are saved; however, if the INFEASIBLE option is specified, all feasible and infeasible grid points are saved. Note that the BEST= option does not affect the output of grid points to the OUTEST= data set. For more information on grids, see the section "DECVAR Statement" on page 327.

OUTHESSIAN
OUTHES
writes the Hessian matrix of the objective function to the OUTEST= data set. If the Hessian matrix is computed for some other reason (if, for example, the PHESSIAN option is specified), the OUTHESSIAN option is set by default.

OUTITER
writes during each iteration the parameter estimates, the value of the objective function, the gradient (if available), and (if OUTTIME is specified) the time in seconds from the start of the optimization to the OUTEST= data set.

OUTJAC
writes the Jacobian matrix of the m functions composing the least-squares function to the OUTEST= data set. If the PJACOBI option is specified, the OUTJAC option is set by default.

OUTMODEL=*model-name*
OUTMOD=*model-name*
OUTM=*model-name*

specifies the name of an output model file to which the program statements are to be written. The program statements of this file can be included into the program statements of a succeeding PROC NLP run using the MODEL= option or the INCLUDE program statement. The OUTMODEL= option is similar to the option used in PROC MODEL in SAS/ETS software. Note that the following statements are not part of the program code that is written to an OUTMODEL= data set: MIN, MAX, LSQ, MINQUAD, MAXQUAD, DECVAR, BOUNDS, BY, CRPJAC, GRADIENT, HESSIAN, JACNLC, JACOBIAN, LABEL, LINCON, MATRIX, and NLINCON.

OUTNLCJAC

If an OUTEST= data set is specified, the Jacobian matrix of the nonlinear constraint functions specified by the NLINCON statement is written to the OUTEST= data set. If the Jacobian matrix of the nonlinear constraint functions is computed for some other reason (if, for example, the PNLCJAC option is specified), the OUTNLCJAC option is set by default.

OUTTIME

is used if an OUTEST= data set is specified and if the OUTITER option is specified. If OUTTIME is specified, the time in seconds from the start of the optimization to the start of each iteration is written to the OUTEST= data set.

PALL
ALL

displays all optional output except the output generated by the PSTDERR, PCOV, LIST, or LISTCODE option.

PCOV

displays the covariance matrix specified by the COV= option. The PCOV option is set automatically if the PALL and COV= options are set.

PCRPJAC
PJTJ

displays the $n \times n$ crossproduct Jacobian matrix $J^T J$. If the PALL option is specified and the LSQ statement is used, this option is set automatically. If general linear constraints are active at the solution, the projected crossproduct Jacobian matrix is also displayed.

PEIGVAL

displays the distribution of eigenvalues if a G4 inverse is computed for the covariance matrix. The PEIGVAL option is useful for observing which eigenvalues of the matrix are recognized as zero eigenvalues when the generalized inverse is computed, and it is the basis for setting the COVSING= option in a subsequent execution of PROC NLP. For more information, see the section "Covariance Matrix" on page 370.

PERROR

specifies additional output for such applications where the program code for objective function or nonlinear constraints cannot be evaluated during the iteration process.

The PERROR option is set by default during the evaluations at the starting point but not during the optimization process.

PFUNCTION
displays the values of all functions specified in a LSQ, MIN, or MAX statement for each observation read fom the DATA= input data set. The PALL option sets the PFUNCTION option automatically.

PGRID
displays the function values from the grid search. For more information on grids, see the section "DECVAR Statement" on page 327.

PHESSIAN
PHES
displays the $n \times n$ Hessian matrix G. If the PALL option is specified and the MIN or MAX statement is used, this option is set automatically. If general linear constraints are active at the solution, the projected Hessian matrix is also displayed.

PHISTORY
PHIS
displays the optimization history. No optimization history is displayed for TECH=LICOMP. This output is included in both the default output and the output specified by the PALL option.

PINIT
PIN
displays the initial values and derivatives (if available). This output is included in both the default output and the output specified by the PALL option.

PJACOBI
PJAC
displays the $m \times n$ Jacobian matrix J. Because of the memory requirement for large least-squares problems, this option is not invoked when using the PALL option.

PNLCJAC
displays the Jacobian matrix of nonlinear constraints specified by the NLINCON statement. The PNLCJAC option is set automatically if the PALL option is specified.

PSHORT
SHORT
PSH
restricts the amount of default output. If PSHORT is specified, then

- The initial values are not displayed.
- The listing of constraints is not displayed.
- If there is more than one function in the MIN, MAX, or LSQ statement, their values are not displayed.
- If the GRADCHECK[=DETAIL] option is used, only the test vector is displayed.

PSTDERR
STDERR
SE
computes standard errors that are defined as square roots of the diagonal elements of the covariance matrix. The t values and probabilities $> |t|$ are displayed together with the approximate standard errors. The type of covariance matrix must be specified using the COV= option. The SIGSQ= option, the VARDEF= option, and the special variables _NOBS_ and _DF_ defined in the program statements can be used to define a scalar factor σ^2 of the covariance matrix and the approximate standard errors. For more information, see the section "Covariance Matrix" on page 370.

PSUMMARY
SUMMARY
SUM
restricts the amount of default displayed output to a short form of iteration history and notes, warnings, and errors.

PTIME
specifies the output of four different but partially overlapping differences of CPU time:

- total running time
- total time for the evaluation of objective function, nonlinear constraints, and derivatives: shows the total time spent executing the programming statements specifying the objective function, derivatives, and nonlinear constraints, and (if necessary) their first- and second-order derivatives. This is the total time needed for code evaluation before, during, and after iterating.
- total time for optimization: shows the total time spent iterating.
- time for some CMP parsing: shows the time needed for parsing the program statements and its derivatives. In most applications this is a negligible number, but for applications that contain ARRAY statements or DO loops or use an optimization technique with analytic second-order derivatives, it can be considerable.

RANDOM=i
specifies a positive integer as a seed value for the pseudorandom number generator. Pseudorandom numbers are used as the initial value $x^{(0)}$.

RESTART=i
REST=i
specifies that the QUANEW, HYQUAN, or CONGRA algorithm is restarted with a steepest descent/ascent search direction after at most $i > 0$ iterations. Default values are as follows:

- CONGRA with UPDATE=PB: restart is done automatically so specification of i is not used
- CONGRA with UPDATE≠PB: $i = \min(10n, 80)$, where n is the number of parameters

- QUANEW, HYQUAN: i is the largest integer available

SIGSQ=sq
specifies a scalar factor $sq > 0$ for computing the covariance matrix. If the SIGSQ= option is specified, VARDEF=N is the default. For more information, see the section "Covariance Matrix" on page 370.

SINGULAR=r
SING=r
specifies the singularity criterion $r > 0$ for the inversion of the Hessian matrix and crossproduct Jacobian. The default value is $1\text{E}{-8}$. For more information, refer to the MSINGULAR= and VSINGULAR= options.

TECH=*name*
TECHNIQUE=*name*
specifies the optimization technique. Valid values for it are as follows:

- CONGRA
  chooses one of four different conjugate gradient optimization algorithms, which can be more precisely specified with the UPDATE= option and modified with the LINESEARCH= option. When this option is selected, UPDATE=PB by default. For $n \geq 400$, CONGRA is the default optimization technique.
- DBLDOG
  performs a version of double dogleg optimization, which can be more precisely specified with the UPDATE= option. When this option is selected, UPDATE=DBFGS by default.
- HYQUAN
  chooses one of three different hybrid quasi-Newton optimization algorithms which can be more precisely defined with the VERSION= option and modified with the LINESEARCH= option. By default, VERSION=2 and UPDATE=DBFGS.
- LEVMAR
  performs the Levenberg-Marquardt minimization. For $n < 40$, this is the default minimization technique for least-squares problems.
- LICOMP
  solves a quadratic program as a linear complementarity problem.
- NMSIMP
  performs the Nelder-Mead simplex optimization method.
- NONE
  does not perform any optimization. This option can be used
  - to do grid search without optimization
  - to compute and display derivatives and covariance matrices which cannot be obtained efficiently with any of the optimization techniques
- NEWRAP
  performs the Newton-Raphson optimization technique. The algorithm

combines a line-search algorithm with ridging. The line-search algorithm LINESEARCH=2 is the default.

- NRRIDG
  performs the Newton-Raphson optimization technique. For $n \leq 40$ and nonlinear least-squares, this is the default.
- QUADAS
  performs a special quadratic version of the active set strategy.
- QUANEW
  chooses one of four quasi-Newton optimization algorithms which can be defined more precisely with the UPDATE= option and modified with the LINESEARCH= option. This is the default for $40 < n < 400$ or if there are nonlinear constraints.
- TRUREG
  performs the trust region optimization technique.

UPDATE=*method*
UPD=*method*

specifies the update method for the (dual) quasi-Newton, double dogleg, hybrid quasi-Newton, or conjugate gradient optimization technique. Not every update method can be used with each optimizer. For more information, see the section "Optimization Algorithms" on page 346. Valid values for *method* are as follows:

BFGS	performs the original BFGS (Broyden, Fletcher, Goldfarb, & Shanno) update of the inverse Hessian matrix.
DBFGS	performs the dual BFGS (Broyden, Fletcher, Goldfarb, & Shanno) update of the Cholesky factor of the Hessian matrix.
DDFP	performs the dual DFP (Davidon, Fletcher, & Powell) update of the Cholesky factor of the Hessian matrix.
DFP	performs the original DFP (Davidon, Fletcher, & Powell) update of the inverse Hessian matrix.
PB	performs the automatic restart update method of Powell (1977) and Beale (1972).
FR	performs the Fletcher-Reeves update (Fletcher 1987).
PR	performs the Polak-Ribiere update (Fletcher 1987).
CD	performs a conjugate-descent update of Fletcher (1987).

VARDEF= DF | N

specifies the divisor d used in the calculation of the covariance matrix and approximate standard errors. If the SIGSQ= option is not specified, the default value is VARDEF=DF; otherwise, VARDEF=N is the default. For more information, see the section "Covariance Matrix" on page 370.

VERSION= 1 | 2 | 3
VS= 1 | 2 | 3

specifies the version of the hybrid quasi-Newton optimization technique or the version of the quasi-Newton optimization technique with nonlinear constraints.

For the hybrid quasi-Newton optimization technique,

VS=1 specifies version HY1 of Fletcher and Xu (1987).

VS=2 specifies version HY2 of Fletcher and Xu (1987).

VS=3 specifies version HY3 of Fletcher and Xu (1987).

For the quasi-Newton optimization technique with nonlinear constraints,

VS=1 specifies update of the μ vector like Powell (1978a, b) (update like VF02AD).

VS=2 specifies update of the μ vector like Powell (1982b) (update like VMCWD).

In both cases, the default value is VS=2.

VSINGULAR=r
VSING=r

specifies a relative singularity criterion $r > 0$ for measuring singularity of Hessian and crossproduct Jacobian and their projected forms, which may have to be converted to compute the covariance matrix. The default value is $1\text{E}{-8}$ if the SINGULAR= option is not specified and the value of SINGULAR otherwise. For more information, see the section "Covariance Matrix" on page 370.

XCONV=$r[n]$
XTOL=$r[n]$

specifies the relative parameter convergence criterion. For all techniques except NMSIMP, termination requires a small relative parameter change in subsequent iterations:

$$\frac{\max_j |x_j^{(k)} - x_j^{(k-1)}|}{\max(|x_j^{(k)}|, |x_j^{(k-1)}|, \text{XSIZE})} \leq r$$

For the NMSIMP technique, the same formula is used, but $x_j^{(k)}$ is defined as the vertex with the lowest function value and $x_j^{(k-1)}$ is defined as the vertex with the highest function value in the simplex. The default value is $r = 1\text{E}{-8}$ for the NMSIMP technique and $r = 0$ otherwise. The optional integer value n specifies the number of successive iterations for which the criterion must be satisfied before the process can be terminated.

XSIZE=r

specifies the parameter $r > 0$ of the relative parameter termination criterion. The default value is $r = 0$. For more details, see the XCONV= option.

ARRAY Statement

ARRAY *arrayname [{ dimensions }] [$] [variables and constants]* **;**

The ARRAY statement is similar to, but not the same as, the ARRAY statement in the SAS DATA step. The ARRAY statement is used to associate a name (of no more than eight characters) with a list of variables and constants. The array name is used with subscripts in the program to refer to the array elements. The following code illustrates this:

```
array r[8] r1-r8;

do i = 1 to 8;
   r[i] = 0;
   end;
```

The ARRAY statement does not support all the features of the DATA step ARRAY statement. It cannot be used to give initial values to array elements. Implicit indexing of variables cannot be used; all array references must have explicit subscript expressions. Only exact array dimensions are allowed; lower-bound specifications are not supported and a maximum of six dimensions is allowed.

On the other hand, the ARRAY statement does allow both variables and constants to be used as array elements. (Constant array elements cannot have values assigned to them.) Both dimension specification and the list of elements are optional, but at least one must be given. When the list of elements is not given or fewer elements than the size of the array are listed, array variables are created by suffixing element numbers to the array name to complete the element list.

BOUNDS Statement

BOUNDS *b_con [, b_con...]* **;**

where *b_con* is given in one of the following formats:

- number *operator* parameter_list *operator* number
- number *operator* parameter_list
- parameter_list *operator* number

and *operator* is $\leq$, $<$, $\geq$, $>$, or $=$.

Boundary constraints are specified with a BOUNDS statement. One- or two-sided boundary constraints are allowed. The list of boundary constraints are separated by commas. For example,

```
bounds 0 <= a1-a9 X <= 1, -1 <= c2-c5;
bounds b1-b10 y >= 0;
```

More than one BOUNDS statement can be used. If more than one lower (upper) bound for the same parameter is specified, the maximum (minimum) of these is taken. If the maximum l_j of all lower bounds is larger than the minimum of all upper bounds u_j for the same variable x_j, the boundary constraint is replaced by $x_j = l_j = \min(u_j)$ defined by the minimum of all upper bounds specified for x_j.

BY Statement

BY *variables* **;**

A BY statement can be used with PROC NLP to obtain separate analyses on DATA= data set observations in groups defined by the BY variables. That means, for values of the TECH= option other than NONE, an optimization problem is solved for each BY group separately. When a BY statement appears, the procedure expects the input DATA= data set to be sorted in order of the BY variables. If the input data set is not sorted in ascending order, it is necessary to use one of the following alternatives:

- Use the SORT procedure with a similar BY statement to sort the data.
- Use the BY statement option NOTSORTED or DESCENDING in the BY statement for the NLP procedure. As a cautionary note, the NOTSORTED option does not mean that the data are unsorted but rather that the data are arranged in groups (according to values of the BY variables) and that these groups are not necessarily in alphabetical or increasing numeric order.
- Use the DATASETS procedure (in Base SAS software) to create an index on the BY variables.

For more information on the BY statement, refer to the discussion in *SAS Language Reference: Concepts*. For more information on the DATASETS procedure, refer to the *SAS Procedures Guide*.

CRPJAC Statement

CRPJAC *variables* **;**

The CRPJAC statement defines the crossproduct Jacobian matrix $J^T J$ used in solving least-squares problems. For more information, see the section "Derivatives" on page 344. If the DIAHES option is not specified, the CRPJAC statement lists $n(n+1)/2$ variable names, which correspond to the elements $(J^T J)_{j,k},\ j \geq k$ of the lower triangle of the symmetric crossproduct Jacobian matrix listed by rows. For example, the statements

```
lsq f1-f3;
decvar x1-x3;
crpjac jj1-jj6;
```

correspond to the crossproduct Jacobian matrix

$$J^T J = \begin{bmatrix} JJ1 & JJ2 & JJ4 \\ JJ2 & JJ3 & JJ5 \\ JJ4 & JJ5 & JJ6 \end{bmatrix}$$

If the DIAHES option is specified, only the n diagonal elements must be listed in the CRPJAC statement. The n rows and n columns of the crossproduct Jacobian matrix must be in the same order as the n corresponding parameter names listed in the DECVAR statement. To specify the values of nonzero derivatives, the variables specified in the CRPJAC statement have to be defined at the left-hand side of algebraic expressions in programming statements. For example, consider the Rosenbrock function:

```
proc nlp tech=levmar;
   lsq f1 f2;
   decvar x1 x2;
   gradient g1 g2;
   crpjac cpj1-cpj3;

   f1   = 10 * (x2 - x1 * x1);
   f2   = 1 - x1;
   g1   = -200 * x1 * (x2 - x1 * x1) - (1 - x1);
   g2   =  100 * (x2 - x1 * x1);

   cpj1 = 400 * x1 * x1 + 1 ;
   cpj2 = -200 * x1;
   cpj3 = 100;
run;
```

DECVAR Statement

DECVAR *name_list [=numbers] [, name_list [=numbers] ...]* **;**

VAR *name_list [=numbers] [, name_list [=numbers] ...]* **;**

PARMS *name_list [=numbers] [, name_list [=numbers] ...]* **;**

PARAMETERS *name_list [=numbers] [, name_list [=numbers] ...]* **;**

The DECVAR statement lists the names of the $n > 0$ decision variables and specifies grid search and initial values for an iterative optimization process. The decision variables listed in the DECVAR statement cannot also be used in the MIN, MAX, MINQUAD, MAXQUAD, LSQ, GRADIENT, HESSIAN, JACOBIAN, CRPJAC, or NLINCON statement.

The DECVAR statement contains a list of decision variable names (not separated by commas) optionally followed by an equals sign and a list of numbers. If the number list consists of only one number, this number defines the initial value for all the decision variables listed to the left of the equals sign.

If the number list consists of more than one number, these numbers specify the grid locations for each of the decision variables listed left of the equals sign. The TO and BY keywords can be used to specify a number list for a grid search. When a grid of points is specified with a DECVAR statement, PROC NLP computes the objective function value at each grid point and chooses the best (feasible) grid point as a starting point for the optimization process. The use of the BEST= option is recommended to save computing time and memory for the storing and sorting of all

grid point information. Usually only feasible grid points are included in the grid search. If the specified grid contains points located outside the feasible region and you are interested in the function values at those points, it is possible to use the INFEASIBLE option to compute (and display) their function values as well.

GRADIENT Statement

GRADIENT *variables* **;**

The GRADIENT statement defines the gradient vector which contains the first-order derivatives of the objective function f with respect to $x_1, \ldots, x_n$. For more information, see the section "Derivatives" on page 344. To specify the values of nonzero derivatives, the variables specified in the GRADIENT statement must be defined on the left-hand side of algebraic expressions in programming statements. For example, consider the Rosenbrock function:

```
proc nlp tech=congra;
   min y;
   decvar x1 x2;
   gradient g1 g2;

   y1 = 10 * (x2 - x1 * x1);
   y2 = 1 - x1;

   y  = .5 * (y1 * y1 + y2 * y2);

   g1 = -200 * x1 * (x2 - x1 * x1) - (1 - x1);
   g2 = 100 * (x2 - x1 * x1);
run;
```

HESSIAN Statement

HESSIAN *variables* **;**

The HESSIAN statement defines the Hessian matrix G containing the second-order derivatives of the objective function f with respect to $x_1, \ldots, x_n$. For more information, see the section "Derivatives" on page 344.

If the DIAHES option is not specified, the HESSIAN statement lists $n(n+1)/2$ variable names which correspond to the elements $G_{j,k},\ j \geq k$, of the lower triangle of the symmetric Hessian matrix listed by rows. For example, the statements

```
min f;
decvar x1 - x3;
hessian g1-g6;
```

correspond to the Hessian matrix

$$G = \begin{bmatrix} G1 & G2 & G4 \\ G2 & G3 & G5 \\ G4 & G5 & G6 \end{bmatrix} = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 & \partial^2 f/\partial x_1 \partial x_3 \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 & \partial^2 f/\partial x_2 \partial x_3 \\ \partial^2 f/\partial x_3 \partial x_1 & \partial^2 f/\partial x_3 \partial x_2 & \partial^2 f/\partial x_3^2 \end{bmatrix}$$

If the DIAHES option is specified, only the n diagonal elements must be listed in the HESSIAN statement. The n rows and n columns of the Hessian matrix G must correspond to the order of the n parameter names listed in the DECVAR statement. To specify the values of nonzero derivatives, the variables specified in the HESSIAN statement must be defined on the left-hand side of algebraic expressions in the programming statements. For example, consider the Rosenbrock function:

```
proc nlp tech=nrridg;
   min f;
   decvar x1 x2;
   gradient g1 g2;
   hessian h1-h3;

   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;

   f = .5 * (f1 * f1 + f2 * f2);

   g1 = -200 * x1 * (x2 - x1 * x1) - (1 - x1);
   g2 = 100 * (x2 - x1 * x1);

   h1 = -200 * (x2 - 3 * x1 * x1) + 1;
   h2 = -200 * x1;
   h3 = 100;
run;
```

INCLUDE Statement

INCLUDE *model files* **;**

The INCLUDE statement can be used to append model code to the current model code. The contents of included model files, created using the OUTMODEL= option, are inserted into the model program at the position in which the INCLUDE statement appears.

JACNLC Statement

JACNLC *variables* **;**

The JACNLC statement defines the Jacobian matrix for the system of constraint functions $c_1(x), \ldots, c_{mc}(x)$. The statements list the $mc \times n$ variable names which correspond to the elements $CJ_{i,j}$, $i = 1, \ldots, mc$; $j = 1, \ldots, n$, of the Jacobian matrix by rows.

For example, the statements

```
nlincon c1-c3;
decvar  x1-x2;
jacnlc  cj1-cj6;
```

correspond to the Jacobian matrix

$$CJ = \begin{bmatrix} CJ1 & CJ2 \\ CJ3 & CJ4 \\ CJ5 & CJ6 \end{bmatrix} = \begin{bmatrix} \partial c_1/\partial x_1 & \partial c_1/\partial x_2 \\ \partial c_2/\partial x_1 & \partial c_2/\partial x_2 \\ \partial c_3/\partial x_1 & \partial c_3/\partial x_2 \end{bmatrix}$$

The mc rows of the Jacobian matrix must be in the same order as the mc corresponding names of nonlinear constraints listed in the NLINCON statement. The n columns of the Jacobian matrix must be in the same order as the n corresponding parameter names listed in the DECVAR statement. To specify the values of nonzero derivatives, the variables specified in the JACNLC statement must be defined on the left-hand side of algebraic expressions in programming statements.

For example,

```
array cd[3,4] cd1-cd12;
nlincon c1-c3 >= 0;
jacnlc cd1-cd12;

c1 = 8 - x1 * x1 - x2 * x2 - x3 * x3 - x4 * x4 -
       x1 + x2 - x3 + x4;
c2 = 10 - x1 * x1 - 2 * x2 * x2 - x3 * x3 - 2 * x4 * x4 +
       x1 + x4;
c3 = 5 - 2 * x1 * x2 - x2 * x2 - x3 * x3 - 2 * x1 + x2 + x4;

cd[1,1]= -1 - 2 * x1;    cd[1,2]= 1 - 2 * x2;
cd[1,3]= -1 - 2 * x3;    cd[1,4]= 1 - 2 * x4;
cd[2,1]=  1 - 2 * x1;    cd[2,2]= -4 * x2;
cd[2,3]= -2 * x3;        cd[2,4]= 1 - 4 * x4;
cd[3,1]= -2 - 4 * x1;    cd[3,2]= 1 - 2 * x2;
cd[3,3]= -2 * x3;        cd[3,4]= 1;
```

JACOBIAN Statement

JACOBIAN *variables* **;**

The JACOBIAN statement defines the JACOBIAN matrix J for a system of objective functions. For more information, see the section "Derivatives" on page 344.

The JACOBIAN statement lists $m \times n$ variable names that correspond to the elements $J_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, of the Jacobian matrix listed by rows.

For example, the statements

```
lsq f1-f3;
decvar x1 x2;
jacobian j1-j6;
```

correspond to the Jacobian matrix

$$J = \begin{bmatrix} J1 & J2 \\ J3 & J4 \\ J5 & J6 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 \end{bmatrix}$$

The m rows of the Jacobian matrix must correspond to the order of the m function names listed in the MIN, MAX, or LSQ statement. The n columns of the Jacobian matrix must correspond to the order of the n decision variables listed in the DECVAR statement. To specify the values of nonzero derivatives, the variables specified in the JACOBIAN statement must be defined on the left-hand side of algebraic expressions in programming statements.

For example, consider the Rosenbrock function:

```
proc nlp tech=levmar;
   array j[2,2] j1-j4;
   lsq f1 f2;
   decvar x1 x2;
   jacobian j1-j4;

   f1 = 10 * (x2 - x1 * x1);
   f2 = 1 - x1;

   j[1,1] = -20 * x1;
   j[1,2] = 10;
   j[2,1] = -1;
   j[2,2] = 0;   /* is not needed */
run;
```

The JACOBIAN statement is useful only if more than one objective function is given in the MIN, MAX, or LSQ statement, or if a DATA= input data set specifies more than one function. If the MIN, MAX, or LSQ statement contains only one objective function and no DATA= input data set is used, the JACOBIAN and GRADIENT statements are equivalent. In the case of least-squares minimization, the crossproduct Jacobian is used as an approximate Hessian matrix.

LABEL Statement

LABEL *variable='label' [,variable='label'...]* **;**

The LABEL statement can be used to assign labels (up to 40 chararcters) to the decision variables listed in the DECVAR statement. The INEST= data set can also be used to assign labels. The labels are attached to the output and are used in an OUTEST= data set.

LINCON Statement

LINCON *l_con [, l_con ...]* **;**

where *l_con* is given in one of the following formats:

- linear_term *operator* number
- number *operator* linear_term

and linear_term is of the following form:

$$< +|- >< \text{number}* > \text{variable} < +|- < \text{number}* > \text{variable} \ldots >$$

The value of *operator* can be one of the following: $\leq$, $<$, $\geq$, $>$, or $=$.

The LINCON statement specifies equality or inequality constraints

$$\sum_{j=1}^{n} a_{ij}x_j \; \{\leq \mid = \mid \geq\} \; b_i \quad \text{for } i = 1, \ldots, m$$

separated by commas. For example, the constraint $4x_1 - 3x_2 = 0$ is expressed as

```
decvar x1 x2;
lincon 4 * x1 - 3 * x2 = 0;
```

and the constraints

$$10x_1 - x_2 \geq 10$$

$$x_1 + 5x_2 \geq 15$$

are expressed as

```
decvar x1 x2;
lincon 10 <= 10 * x1 - x2,
       x1 + 5 * x2 >= 15;
```

MATRIX Statement

MATRIX *M_name pattern_definitions* **;**

The MATRIX statement defines a matrix H and the vector g, which can be given in the MINQUAD or MAXQUAD statement. The matrix H and vector g are initialized to zero, so that only the nonzero elements are given. The five different forms of the MATRIX statement are illustrated with the following example:

$$H = \begin{bmatrix} 100 & 10 & 1 & 0 \\ 10 & 100 & 10 & 1 \\ 1 & 10 & 100 & 10 \\ 0 & 1 & 10 & 100 \end{bmatrix} \quad g = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad c = 0$$

Each MATRIX statement first names the matrix or vector and then lists its elements. If more than one MATRIX statement is given for the same matrix, the later definitions override the earlier ones.

The rows and columns in matrix H and vector g correspond to the order of decision variables in the DECVAR statement.

- **Full Matrix Definition:** The MATRIX statement consists of *H_name* or *g_name* followed by an equals sign and all (nonredundant) numerical values of the matrix H or vector g. Assuming symmetry, only the elements of the lower triangular part of the matrix H must be listed. This specification should be used mainly for small problems with almost dense H matrices.

```
MATRIX H=  100
           10 100
            1  10 100
            0   1  10 100;
MATRIX G= 1  2  3  4;
```

- **Band-diagonal Matrix Definition:** This form of *pattern definition* is useful if the H matrix has (almost) constant band-diagonal structure. The MATRIX statement consists of *H_name* followed by empty brackets $[,]$, an equals sign, and a list of numbers to be assigned to the diagonal and successive subdiagonals.

```
MATRIX H[,]=  100 10 1;
MATRIX G= 1  2  3  4;
```

- **Sparse Matrix Definitions:** In each of the following three specification types, the *H_name* or *g_name* is followed by a list of *pattern definitions* separated by commas. Each *pattern definition* consists of a location specification in brackets on the left side of an equals sign that is followed by a list of numbers.

  - **(Sub)Diagonalwise:** This form of *pattern definition* is useful if the H matrix contains nonzero elements along diagonals or subdiagonals. The starting location is specified by an index pair in brackets $[i, j]$. The expression $k * num$ on the right-hand side specifies that num is assigned to the elements $[i, j], \ldots, [i + k - 1, j + k - 1]$ in a diagonal direction of the H matrix. The special case $k = 1$ can be used to assign values to single nonzero element locations in H.

```
MATRIX H [1,1]= 4 * 100,
         [2,1]= 3 *  10,
         [3,1]= 2 *   1;
MATRIX G [1,1]= 1  2  3  4;
```

  - **Columnwise Starting in Diagonal:** This form of *pattern definition* is useful if the H matrix contains nonzero elements columnwise starting in the diagonal. The starting location is specified by only one index j in brackets $[, j]$. The k numbers at the right-hand side are assigned to the elements $[j, j], \ldots, [\min(j + k - 1, n), j]$.

```
MATRIX H [,1]= 100  10   1,
         [,2]= 100  10   1,
         [,3]= 100  10,
         [,4]= 100;
MATRIX G [,1]= 1  2  3  4;
```

  - **Rowwise Starting in First Column:** This form of *pattern definition* is useful if the H matrix contains nonzero elements rowwise ending in the diagonal. The starting location is specified by only one index i in brackets $[i,]$. The k numbers at the right-hand side are assigned to the elements $[i, 1], \ldots, [i, \min(k, i)]$.

```
MATRIX H [1,]= 100,
         [2,]=  10 100,
         [3,]=   1  10 100,
         [4,]=   0   1  10 100;
MATRIX G [1,]= 1  2  3  4;
```

MIN, MAX, and LSQ Statements

MIN *variables* **;**

MAX *variables* **;**

LSQ *variables* **;**

The MIN, MAX, or LSQ statement specifies the objective functions. Only one of the three statements can be used at a time and at least one must be given. The MIN and LSQ statements are for minimizing the objective function, and the MAX statement is for maximizing the objective function. The MIN, MAX, or LSQ statement lists one or more variables naming the objective functions f_i, $i = 1, \ldots, m$ (later defined by SAS program code).

- If the MIN or MAX statement lists m function names $f_1, \ldots, f_m$, the objective function f is
$$f(x) = \sum_{i=1}^{m} f_i$$
- If the LSQ statement lists m function names $f_1, \ldots, f_m$, the objective function f is
$$f(x) = \frac{1}{2} \sum_{i=1}^{m} f_i^2(x)$$

Note that the LSQ statement can be used only if TECH=LEVMAR or TECH=HYQUAN.

MINQUAD and MAXQUAD Statements

MINQUAD *H_name [, g_name [, c_number]]* **;**

MAXQUAD *H_name [, g_name [, c_number]]* **;**

The MINQUAD and MAXQUAD statements specify the matrix H, vector g, and scalar c that define a quadratic objective function. The MINQUAD statement is for minimizing the objective function and the MAXQUAD statement is for maximizing the objective function.

The rows and columns in H and g correspond to the order of decision variables given in the DECVAR statement. Specifying the objective function with a MINQUAD or MAXQUAD statement indirectly defines the analytic derivatives for the objective function. Therefore, statements specifying derivatives are not valid in these cases. Also, only use these statements when TECH=LICOMP or TECH=QUADAS and no nonlinear constraints are imposed.

There are three ways of using the MINQUAD or MAXQUAD statement:

- **Using ARRAY Statements**:
The names *H_name* and *g_name* specified in the MINQUAD or MAXQUAD statement can be used in ARRAY statements. This specification is mainly for small problems with almost dense H matrices.

```
proc nlp pall;
   array h[2,2] .4 0
                0 4;
   minquad h, -100;
   decvar x1 x2 = -1;
   bounds  2 <= x1 <= 50,
         -50 <= x2 <= 50;
   lincon 10 <= 10 * x1 - x2;
run;
```

- **Using Elementwise Setting**:
  The names *H_name* and *g_name* specified in the MINQUAD or MAXQUAD statement can be followed directly by one-dimensional indices specifying the corresponding elements of the matrix H and vector g. These element names can be used on the left side of numerical assignments. The one-dimensional index value l following *H_name*, which corresponds to the element H_{ij}, is computed by $l = (i-1)n + j, i \geq j$. The matrix H and vector g are initialized to zero, so that only the nonzero elements must be given. This specification is efficient for small problems with sparse H matrices.

```
proc nlp pall;
   minquad h, -100;
   decvar x1 x2;
   bounds  2 <= x1 <= 50,
         -50 <= x2 <= 50;
   lincon 10 <= 10 * x1 - x2;
   h1 = .4; h4 = 4;
run;
```

- **Using MATRIX Statements**:
  The names *H_name* and *g_name* specified in the MINQUAD or MAXQUAD statement can be used in MATRIX statements. There are different ways to specify the nonzero elements of the matrix H and vector g by MATRIX statements. The following example illustrates one way to use the MATRIX statement.

```
proc nlp all;
   matrix h[1,1] = .4 4;
   minquad h, -100;
   decvar x1 x2 = -1;
   bounds 2 <= x1 <= 50;
        -50 <= x2 <= 50;
   lincon 10 <= 10 * x1 - x2;
run;
```

NLINCON Statement

NLINCON *nlcon [, nlcon ...] [/ option]* **;**

NLC *nlcon [, nlcon ...] [/ option]* **;**

where *nlcon* is given in one of the following formats:

- number *operator* variable_list *operator* number
- -number *operator* variable_list
- variable_list *operator* number

and *operator* is $\leq, <, \geq, >,$ or $=$. The value of *option* can be SUMOBS or EVERYOBS.

General nonlinear equality and inequality constraints are specified with the NLINCON statement. The syntax of the NLINCON statement is similar to that of the BOUNDS statement with two small additions:

- The BOUNDS statement can contain only the names of decision variables. The NLINCON statement can also contain the names of continuous functions of the decision variables. These functions must be computed in the program statements, and since they can depend on the values of some of the variables in the DATA= data set, there are two possibilities:
  - If the continuous functions should be summed across all observations read from the DATA= data set, the NLINCON statement must be terminated by the / SUMOBS option.
  - If the continuous functions should be evaluated separately for each observation in the data set, the NLINCON statement must be terminated by the / EVERYOBS option. One constraint is generated for each observation in the data set.
- If the continuous function should be evaluated only once for the entire data set, the NLINCON statement has the same form as the BOUNDS statement. If this constraint does depend on the values of variables in the DATA= data set, it is evaluated using the data of the first observation.

One- or two-sided constraints can be specified in the NLINCON statement. However, equality constraints must be one-sided. The pairs of operators (<,<=) and (>,>=) are treated in the same way.

These three statements require the values of the three functions v_1, v_2, v_3 to be between zero and ten, and they are equivalent:

```
nlincon 0 <= v1-v3,
        v1-v3 <= 10;

nlincon 0 <= v1-v3 <= 10;

nlincon 10 >= v1-v3 >= 0;
```

Also, consider the Rosen-Suzuki problem. It has three nonlinear inequality constraints:

$$\begin{aligned} 8 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 &\geq 0 \\ 10 - x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 &\geq 0 \\ 5 - 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 &\geq 0 \end{aligned}$$

These are specified as

```
nlincon c1-c3 >= 0;

c1 = 8 - x1 * x1 - x2 * x2 - x3 * x3 - x4 * x4 -
         x1 + x2 - x3 + x4;
c2 = 10 - x1 * x1 - 2 * x2 * x2 - x3 * x3 - 2 * x4 * x4 +
         x1 + x4;
c3 = 5 - 2 * x1 * x1 - x2 * x2 - x3 * x3 - 2 * x1 + x2 + x4;
```

Note: QUANEW and NMSIMP are the only optimization subroutines that support the NLINCON statement.

PROFILE Statement

PROFILE *parms [/ [ALPHA= values] [options]]* **;**

where *parms* is given in the format *pnam_1 pnam_2 ... pnam_n*, and *values* is the list of α values in (0,1).

The PROFILE statement

- writes the (x, y) coordinates of profile points for each of the listed parameters to the OUTEST= data set
- displays, or writes to the OUTEST= data set, the profile likelihood confidence limits (PL CLs) for the listed parameters for the specified α values. If the approximate standard errors are available, the corresponding Wald confidence limits can be computed.

When computing the profile points or likelihood profile confidence intervals, PROC NLP assumes that a maximization of the log likelihood function is desired. Each point of the profile and each endpoint of the confidence interval is computed by solving corresponding nonlinear optimization problems.

The keyword PROFILE must be followed by the names of parameters for which the profile or the PL CLs should be computed. If the parameter name list is empty, the profiles and PL CLs for all parameters are computed. Then, optionally, the α values follow. The list of α values may contain TO and BY keywords. Each element must satisfy $0 < \alpha < 1$. The following is an example:

```
profile l11-l15 u1-u5 c /
        alpha= .9 to .1 by -.1 .09 to .01 by -.01;
```

Duplicate α values or values outside $(0, 1)$ are automatically eliminated from the list.

A number of additional options can be specified.

FFACTOR=r specifies the factor relating the discrepancy function $f(\theta)$ to the χ^2 quantile. The default value is $r = 2$.

FORCHI= F | CHI defines the scale for the y values written to the OUTEST= data set. For FORCHI=F, the y values are scaled to the values of the log likelihood function $f = f(\theta)$; for FORCHI=CHI, the y values are scaled so that $\hat{y} = \chi^2$. The default value is FORCHI=F.

FEASRATIO=r specifies a factor of the Wald confidence limit (or an approximation of it if standard errors are not computed) defining an upper bound for the search for confidence limits. In general, the range of x values in the profile graph is between $r = 1$ and $r = 2$ times the length of the corresponding Wald interval. For many examples, the χ^2 quantiles corresponding to small α values define a y level $\hat{y} - \frac{1}{2}q_1(1-\alpha)$, which is too far away from $\hat{y}$ to be reached by $y(x)$ for x within the range of twice the Wald confidence limit. The search for an intersection with such a y level at a practically infinite value of x can be computationally expensive. A smaller value for r can speed up computation time by restricting the search for confidence limits to a region closer to $\hat{x}$. The default value of $r = 1000$ practically disables the FEASRATIO= option.

OUTTABLE specifies that the complete set θ of parameter estimates rather than only $x = \theta_j$ for each confidence limit is written to the OUTEST= data set. This output can be helpful for further analyses on how small changes in $x = \theta_j$ affect the changes in the $\theta_i, i \neq j$.

For some applications, it may be computationally less expensive to compute the PL confidence limits for a few parameters than to compute the approximate covariance matrix of many parameters, which is the basis for the Wald confidence limits. However, the computation of the profile of the discrepancy function and the corresponding CLs in general will be much more time-consuming than that of the Wald CLs.

Program Statements

This section lists the program statements used to code the objective function and nonlinear constraints and their derivatives, and it documents the differences between program statements in the NLP procedure and program statements in the DATA step. The syntax of program statements used in PROC NLP is identical to that used in the CALIS, GENMOD, and MODEL procedures (refer to the *SAS/ETS User's Guide*).

Most of the program statements which can be used in the SAS DATA step can also be used in the NLP procedure. See the *SAS Language Guide* or base SAS documentation for a description of the SAS program statements.

```
ABORT;
CALL name [ ( expression [, expression ... ] ) ];
DELETE;
DO [ variable = expression
                [ TO expression ] [ BY expression ]
                [, expression [ TO expression ] [ BY expression ] ... ]
    ]
    [ WHILE expression ] [ UNTIL expression ];
END;
GOTO statement_label;
IF expression;
IF expression THEN program_statement;
    ELSE program_statement;
variable = expression;
variable + expression;
LINK statement_label;
PUT [ variable] [=] [...] ;
RETURN;
SELECT [( expression )];
STOP;
SUBSTR( variable, index, length ) = expression;
WHEN (expression) program_statement;
    OTHERWISE program_statement;
```

For the most part, the SAS program statements work as they do in the SAS DATA step as documented in the *SAS Language Guide*. However, there are several differences that should be noted.

- The ABORT statement does not allow any arguments.
- The DO statement does not allow a character index variable. Thus

  ```
  do i = 1,2,3;
  ```

  is supported; however,

  ```
  do i = 'A','B','C';
  ```

  is not.
- The PUT statement, used mostly for program debugging in PROC NLP, supports only some of the features of the DATA step PUT statement, and has some new features that the DATA step PUT statement does not:
  - The PROC NLP PUT statement does not support line pointers, factored lists, iteration factors, overprinting, _INFILE_, the colon (:) format modifier, or "$".
  - The PROC NLP PUT statement does support expressions, but the expression must be enclosed inside of parentheses. For example, the following statement displays the square root of x: `put (sqrt(x));`

  - The PROC NLP PUT statement supports the print item _PDV_ to print a formatted listing of all variables in the program. For example, the following statement displays a more readable listing of the variables than the _all_ print item: `put _pdv_;`

- The WHEN and OTHERWISE statements allow more than one target statement. That is, DO/END groups are not necessary for multiple statement WHENs. For example, the following syntax is valid:

```
SELECT;
WHEN ( exp1 )   stmt1;
                stmt2;
WHEN ( exp2 )   stmt3;
                stmt4;
END;
```

It is recommended to keep some kind of order in the input of NLP, that is, between the statements that define decision variables and constraints and the program code used to specify objective functions and derivatives.

Use of Special Variables in Program Code

Except for the quadratic programming techniques (QUADAS and LICOMP) that do not execute program statements during the iteration process, several special variables in the program code can be used to communicate with PROC NLP in special situations:

- **_OBS_** If a DATA= input data set is used, it is possible to access a variable _OBS_ which contains the number of the observation processed from the data set. You should not change the content of the _OBS_ variable. This variable enables you to modify the programming statements depending on the observation number processed in the DATA= input data set. For example, to set variable A to 1 when observation 10 is processed, and otherwise to 2, it is possible to specify

```
IF _OBS_ = 10 THEN A=1; ELSE A=2;
```

- **_ITER_** This variable is set by PROC NLP, and it contains the number of the current iteration of the optimization technique as it is displayed in the optimization history. You should not change the content of the _ITER_ variable. It is possible to read the value of this variable in order to modify the programming statements depending on the iteration number processed. For example, to display the content of the variables A, B, and C when there are more than 100 iterations processed, it is possible to use

```
IF _ITER_ > 100 THEN PUT A B C;
```

- **_DPROC_** This variable is set by PROC NLP to indicate whether the code is called only to obtain the values of the m objective functions f_i (_DPROC_=0)

or whether specified derivatives (defined by the GRADIENT, JACOBIAN, CRPJAC, or HESSIAN statement) also have to be computed (_DPROC_=1). You should not change the content of the _DPROC_ variable. Checking the _DPROC_ variable makes it possible to save computer time by not performing derivative code that is not needed by the current call. In particular, when a DATA= input data set is used, the code is processed many times to compute only the function values. If the programming statements in the program contain the specification of computationally expensive first- and second-order derivatives, you can put the derivative code in an IF statement that is processed only if _DPROC_ is not zero.

- **_INDF_** The _INDF_ variable is set by PROC NLP to inform you of the source of calls to the function or derivative programming.

  INDF=0 indicates the first function call in a grid search. This is also the first call evaluating the programming statements if there is a grid search defined by grid values in the DECVAR statement.

  INDF=1 indicates further function calls in a grid search.

  INDF=2 indicates the call for the feasible starting point. This is also the first call evaluating the programming statements if there is no grid search defined.

  INDF=3 indicates calls from a gradient-checking algorithm.

  INDF=4 indicates calls from the minimization algorithm. The _ITER_ variable contains the iteration number.

  INDF=5 If the active set algorithm leaves the feasible region (due to rounding errors), an algorithm tries to return it into the feasible region; _INDF_=5 indicates a call that is done when such a step is successful.

  INDF=6 indicates calls from a factorial test subroutine that tests the neighborhood of a point x for optimality.

  INDF=7, 8 indicates calls from subroutines needed to compute finite-difference derivatives using only values of the objective function. No nonlinear constraints are evaluated.

  INDF=9 indicates calls from subroutines needed to compute second-order finite-difference derivatives using analytic (specified) first-order derivatives. No nonlinear constraints are evaluated.

  INDF=10 indicates calls where only the nonlinear constraints but no objective function are needed. The analytic derivatives of the nonlinear constraints are computed.

  INDF=11 indicates calls where only the nonlinear constraints but no objective function are needed. The analytic derivatives of the nonlinear constraints are not computed.

  INDF=-1 indicates the last call at the final solution.

  You should not change the content of the _INDF_ variable.

- **_LIST_** You can set the _LIST_ variable to control the output during the iteration process:

LIST=0 is equivalent to the NOPRINT option. It suppresses all output.

LIST=1 is equivalent to the PSUMMARY but not the PHISTORY option. The optimization start and termination messages are displayed. However, the PSUMMARY option suppresses the output of the iteration history.

LIST=2 is equivalent to the PSHORT option or to a combination of the PSUMMARY and PHISTORY options. The optimization start information, the iteration history, and termination message are displayed.

LIST=3 is equivalent to not PSUMMARY, not PSHORT, and not PALL. The optimization start information, the iteration history, and the termination message are displayed.

LIST=4 is equivalent to the PALL option. The extended optimization start information (also containing settings of termination criteria and other control parameters) is displayed.

LIST=5 In addition to the iteration history, the vector $x^{(k)}$ of parameter estimates is displayed for each iteration k.

LIST=6 In addition to the iteration history, the vector $x^{(k)}$ of parameter estimates and the gradient $g^{(k)}$ (if available) of the objective function are displayed for each iteration k.

It is possible to set the _LIST_ variable in the program code to obtain more or less output in each iteration of the optimization process. For example,

```
IF _ITER_ = 11      THEN _LIST_=5;
ELSE IF _ITER_ > 11 THEN _LIST_=1;
                    ELSE _LIST_=3;
```

- **_TOOBIG_** The value of _TOOBIG_ is initialized to 0 by PROC NLP, but you can set it to 1 during the iteration, indicating problems evaluating the program statements. The objective function and derivatives must be computable at the starting point. However, during the iteration it is possible to set the _TOOBIG_ variable to 1, indicating that the programming statements (computing the value of the objective function or the specified derivatives) cannot be performed for the current value of x_k. Some of the optimization techniques check the value of _TOOBIG_ and try to modify the parameter estimates so that the objective function (or derivatives) can be computed in a following trial.
- **_NOBS_** The value of the _NOBS_ variable is initialized by PROC NLP to the product of the number of functions *mfun* specified in the MIN, MAX or LSQ statement and the number of valid observations *nobs* in the current BY group of the DATA= input data set. The value of the _NOBS_ variable is used for computing the scalar factor of the covariance matrix (see the COV=, VARDEF=, and SIGSQ= options). If you reset the value of the _NOBS_ variable, the value that is available at the end of the iteration is used by PROC NLP to compute the scalar factor of the covariance matrix.
- **_DF_** The value of the _DF_ variable is initialized by PROC NLP to the number n of parameters specified in the DECVAR statement. The value of the _DF_ variable is used for computing the scalar factor d of the covariance matrix (see the COV=, VARDEF=, and SIGSQ= options). If you reset the value

of the _DF_ variable, the value that is available at the end of the iteration is used by PROC NLP to compute the scalar factor of the covariance matrix.

- **_LASTF_** In each iteration (except the first one), the value of the _LASTF_ variable is set by PROC NLP to the final value of the objective function that was achieved during the last iteration. This value should agree with the value that is displayed in the iteration history and that is written in the OUTEST= data set when the OUTITER option is specified.

Details: NLP Procedure

Criteria for Optimality

PROC NLP solves

$$\begin{array}{ll} \min_{x \in \mathcal{R}^n} & f(x) \\ \text{subject to} & c_i(x) = 0, \quad i = 1, \ldots, m_e \\ & c_i(x) \geq 0, \quad i = m_e + 1, \ldots, m \end{array}$$

where f is the objective function and the c_i's are the constraint functions.

A point x is feasible if it satisfies all the constraints. The feasible region $\mathcal{G}$ is the set of all the feasible points. A feasible point x^* is a global solution of the preceding problem if no point in $\mathcal{G}$ has a smaller function value than $f(x^*)$. A feasible point x^* is a local solution of the problem if there exists some open neighborhood surrounding x^* in that no point has a smaller function value than $f(x^*)$. Nonlinear programming algorithms cannot consistently find global minima. All the algorithms in PROC NLP find a local minimum for this problem. If you need to check whether the obtained solution is a global minimum, you may have to run PROC NLP with different starting points obtained either at random or by selecting a point on a grid that contains $\mathcal{G}$.

Every local minimizer x^* of this problem satisfies the following local optimality conditions:

- The gradient (vector of first derivatives) $g(x^*) = \nabla f(x^*)$ of the objective function f (projected toward the feasible region if the problem is constrained) at the point x^* is zero.
- The Hessian (matrix of second derivatives) $G(x^*) = \nabla^2 f(x^*)$ of the objective function f (projected toward the feasible region $\mathcal{G}$ in the constrained case) at the point x^* is positive definite.

Most of the optimization algorithms in PROC NLP use iterative techniques that result in a sequence of points $x^0, ..., x^n, ...$, that converges to a local solution x^*. At the solution, PROC NLP performs tests to confirm that the (projected) gradient is close to zero and that the (projected) Hessian matrix is positive definite.

Karush-Kuhn-Tucker Conditions

An important tool in the analysis and design of algorithms in constrained optimization is the *Lagrangian function*, a linear combination of the objective function and the constraints:

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x)$$

The coefficients λ_i are called *Lagrange multipliers*. This tool makes it possible to state necessary and sufficient conditions for a local minimum. The various algorithms in PROC NLP create sequences of points, each of which is closer than the previous one to satisfying these conditions.

Assuming that the functions f and c_i are twice continuously differentiable, the point x^* is a *local minimum* of the nonlinear programming problem, if there exists a vector $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ that meets the following conditions.

1. First-order Karush-Kuhn-Tucker conditions:

$$\begin{array}{lll} c_i(x^*) = 0, & & i = 1, \ldots, m_e \\ c_i(x^*) \geq 0, & \lambda_i^* \geq 0, \quad \lambda_i^* c_i(x^*) = 0, & i = m_e + 1, \ldots, m \\ \nabla_x L(x^*, \lambda^*) = 0 & & \end{array}$$

2. Second-order conditions: Each nonzero vector $y \in \mathcal{R}^n$ that satisfies

$$y^T \nabla_x c_i(x^*) = 0 \left\{ \begin{array}{l} i = 1, \ldots, m_e \\ \forall i \in \{m_e + 1, \ldots, m : \lambda_i^* > 0\} \end{array} \right.$$

also satisfies

$$y^T \nabla_x^2 L(x^*, \lambda^*) y > 0$$

Most of the algorithms to solve this problem attempt to find a combination of vectors x and λ for which the gradient of the Lagrangian function with respect to x is zero.

Derivatives

The first- and second-order conditions of optimality are based on first and second derivatives of the objective function f and the constraints c_i.

The gradient vector contains the first derivatives of the objective function f with respect to the parameters $x_1, \ldots, x_n$, as follows:

$$g(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_j} \right)$$

The $n \times n$ symmetric Hessian matrix contains the second derivatives of the objective function f with respect to the parameters $x_1, \ldots, x_n$, as follows:

$$G(x) = \nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_j \partial x_k} \right)$$

For least-squares problems, the $m \times n$ Jacobian matrix contains the first-order derivatives of the m objective functions $f_i(x)$ with respect to the parameters $x_1, \ldots, x_n$, as follows:

$$J(x) = (\nabla f_1, \ldots, \nabla f_m) = \left(\frac{\partial f_i}{\partial x_j} \right)$$

In the case of least-squares problems, the crossproduct Jacobian

$$J^T J = \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} \right)$$

is used as an approximate Hessian matrix. It is a very good approximation of the Hessian if the residuals at the solution are "small." (If the residuals are not sufficiently small at the solution, this approach may result in slow convergence.) The fact that it is possible to obtain Hessian approximations for this problem that do not require any computation of second derivatives means that least-squares algorithms are more efficient than unconstrained optimization algorithms. Using the vector $f(x) = (f_1(x), \ldots, f_m(x))^T$ of function values, PROC NLP computes the gradient $g(x)$ by

$$g(x) = J^T(x) f(x)$$

The $mc \times n$ Jacobian matrix contains the first-order derivatives of the mc nonlinear constraint functions $c_i(x)$, $i = 1, \ldots, mc$, with respect to the parameters $x_1, \ldots, x_n$, as follows:

$$CJ(x) = (\nabla c_1, \ldots, \nabla c_{mc}) = \left(\frac{\partial c_i}{\partial x_j} \right)$$

PROC NLP provides three ways to compute derivatives:

- It computes analytical first- and second-order derivatives of the objective function f with respect to the n variables x_j.
- It computes first- and second-order finite-difference approximations to the derivatives. For more information, see the section "Finite-Difference Approximations of Derivatives" on page 357.
- The user supplies formulas for analytical or numerical first- and second-order derivatives of the objective function in the GRADIENT, JACOBIAN, CRPJAC, and HESSIAN statements. The JACNLC statement can be used to specify the derivatives for the nonlinear constraints.

Optimization Algorithms

There are three groups of optimization techniques available in PROC NLP. A particular optimizer can be selected with the TECH= option in the PROC NLP statement.

Algorithm	TECH=
Linear Complementarity Problem	LICOMP
Quadratic Active Set Technique	QUADAS
Trust-Region Method	TRUREG
Newton-Raphson Method with Line Search	NEWRAP
Newton-Raphson Method with Ridging	NRRIDG
Quasi-Newton Methods (DBFGS, DDFP, BFGS, DFP)	QUANEW
Double Dogleg Method (DBFGS, DDFP)	DBLDOG
Conjugate Gradient Methods (PB, FR, PR, CD)	CONGRA
Nelder-Mead Simplex Method	NMSIMP
Levenberg-Marquardt Method	LEVMAR
Hybrid Quasi-Newton Methods (DBFGS, DDFP)	HYQUAN

Since no single optimization technique is invariably superior to others, PROC NLP provides a variety of optimization techniques that work well in various circumstances. However, it is possible to devise problems for which none of the techniques in PROC NLP can find the correct solution. Moreover, nonlinear optimization can be computationally expensive in terms of time and memory, so care must be taken when matching an algorithm to a problem.

All optimization techniques in PROC NLP use $O(n^2)$ memory except the conjugate gradient methods, which use only $O(n)$ memory and are designed to optimize problems with many variables. Since the techniques are iterative, they require the repeated computation of

- the function value (optimization criterion)
- the gradient vector (first-order partial derivatives)
- for some techniques, the (approximate) Hessian matrix (second-order partial derivatives)
- values of linear and nonlinear constraints
- the first-order partial derivatives (Jacobian) of nonlinear constraints

However, since each of the optimizers requires different derivatives and supports different types of constraints, some computational efficiencies can be gained. The following table shows, for each optimization technique, which derivatives are needed (FOD: first-order derivatives; SOD: second-order derivatives) and what kinds of constraints (BC: boundary constraints; LIC: linear constraints; NLC: nonlinear constraints) are supported.

Algorithm	FOD	SOD	BC	LIC	NLC
LICOMP	-	-	x	x	-
QUADAS	-	-	x	x	-
TRUREG	x	x	x	x	-
NEWRAP	x	x	x	x	-
NRRIDG	x	x	x	x	-
QUANEW	x	-	x	x	x
DBLDOG	x	-	x	x	-
CONGRA	x	-	x	x	-
NMSIMP	-	-	x	x	x
LEVMAR	x	-	x	x	-
HYQUAN	x	-	x	x	-

Preparation for Using Optimization Algorithms

It is rare that a problem is submitted to an optimization algorithm "as is." By making a few changes in your problem, you can reduce its complexity, which would increase the chance of convergence and save execution time.

- Whenever possible, use linear functions instead of nonlinear functions. PROC NLP will reward you with faster and more accurate solutions.
- Most optimization algorithms are based on quadratic approximations to nonlinear functions. You should try to avoid the use of functions that cannot be properly approximated by quadratic functions. Try to avoid the use of rational functions.

  For example, the constraint

  $$\frac{\sin(x)}{x+1} > 0$$

  should be replaced by the equivalent constraint

  $$\sin(x)(x+1) > 0$$

  and the constraint

  $$\frac{\sin(x)}{x+1} = 1$$

  should be replaced by the equivalent constraint

  $$\sin(x) - (x+1) = 0$$

- Try to avoid the use of exponential functions, if possible.
- If you can reduce the complexity of your function by the addition of a small number of variables, it may help the algorithm avoid stationary points.

- Provide the best starting point you can. A good starting point leads to better quadratic approximations and faster convergence.

Choosing an Optimization Algorithm

The factors that go into choosing a particular optimizer for a particular problem are complex and may involve trial and error. Several things must be taken into account. First, the structure of the problem has to be considered: Is it quadratic? least-squares? Does it have linear or nonlinear constraints? Next, it is important to consider the type of derivatives of the objective function and the constraints that are needed and whether these are analytically tractable or not. This section provides some guidelines for making the right choices.

For many optimization problems, computing the gradient takes more computer time than computing the function value, and computing the Hessian sometimes takes *much* more computer time and memory than computing the gradient, especially when there are many decision variables. Optimization techniques that do not use the Hessian usually require more iterations than techniques that do use Hessian approximations (such as finite differences or BFGS update) and so are often slower. Techniques that do not use Hessians at all tend to be slow and less reliable.

The derivative compiler is not efficient in the computation of second-order derivatives. For large problems, memory and computer time can be saved by programming your own derivatives using the GRADIENT, JACOBIAN, CRPJAC, HESSIAN, and JACNLC statements. If you are not able to specify first- and second-order derivatives of the objective function, you can rely on finite-difference gradients and Hessian update formulas. This combination is frequently used and works very well for small and medium problems. For large problems, you are advised not to use an optimization technique that requires the computation of second derivatives.

The following provides some guidance for matching an algorithm to a particular problem.

- Quadratic Programming
  - **QUADAS**
  - **LICOMP**
- General Nonlinear Optimization
  - Nonlinear Constraints
    - **Small Problems: NMSIMP**
      Not suitable for highly nonlinear problems or for problems with $n > 20$.
    - **Medium Problems: QUANEW**
  - Only Linear Constraints
    - **Small Problems: TRUREG (NEWRAP, NRRIDG)**
      ($n \leq 40$) where the Hessian matrix is not expensive to compute.

Sometimes NRRIDG can be faster than TRUREG, but TRUREG can be more stable. NRRIDG needs only one matrix with $n(n+1)/2$ double words; TRUREG and NEWRAP need two such matrices.

* **Medium Problems: QUANEW (DBLDOG)**
  ($n \leq 200$) where the objective function and the gradient are much faster to evaluate than the Hessian. QUANEW and DBLDOG in general need more iterations than TRUREG, NRRIDG, and NEWRAP, but each iteration can be much faster. QUANEW and DBLDOG need only the gradient to update an approximate Hessian. QUANEW and DBLDOG need slightly less memory than TRUREG or NEWRAP (essentially one matrix with $n(n+1)/2$ double words).
* **Large Problems: CONGRA**
  ($n > 200$) where the objective function and the gradient can be computed much faster than the Hessian and where too much memory is needed to store the (approximate) Hessian. CONGRA in general needs more iterations than QUANEW or DBLDOG, but each iteration can be much faster. Since CONGRA needs only a factor of n double-word memory, many large applications of PROC NLP can be solved only by CONGRA.
* **No Derivatives: NMSIMP**
  ($n \leq 20$) where derivatives are not continuous or are very difficult to compute.

- Least-Squares Minimization

  - **Small Problems: LEVMAR (HYQUAN)**
    ($n \leq 60$) where the crossproduct Jacobian matrix is inexpensive to compute. In general, LEVMAR is more reliable, but there are problems with high residuals where HYQUAN can be faster than LEVMAR.
  - **Medium Problems: QUANEW (DBLDOG)**
    ($n \leq 200$) where the objective function and the gradient are much faster to evaluate than the crossproduct Jacobian. QUANEW and DBLDOG in general need more iterations than LEVMAR or HYQUAN, but each iteration can be much faster.
  - **Large Problems: CONGRA**
  - **No Derivatives: NMSIMP**

Quadratic Programming Method

The QUADAS and LICOMP algorithms can be used to minimize or maximize a quadratic objective function,

$$f(x) = \frac{1}{2}x^T G x + g^T x + c, \quad \text{with} \quad G^T = G$$

subject to linear or boundary constraints

$$Ax \geq b \quad \text{or} \quad l_j \leq x_j \leq u_j$$

where $x = (x_1, \ldots, x_n)^T$, $g = (g_1, \ldots, g_n)^T$, G is an $n \times n$ symmetric matrix, A is an $m \times n$ matrix of general linear constraints, and $b = (b_1, \ldots, b_m)^T$. The value of c modifies only the value of the objective function, not its derivatives, and the location of the optimizer x^* does not depend on the value of the constant term c. For QUADAS or LICOMP, the objective function must be specified using the MINQUAD or MAXQUAD statement or using an INQUAD= data set. In this case, derivatives do not need to be specified because the gradient vector

$$\nabla f(x) = Gx + g$$

and the $n \times n$ Hessian matrix

$$\nabla^2 f(x) = G$$

are easily obtained from the data input.

Simple boundary and general linear constraints can be specified using the BOUNDS or LINCON statement or an INQUAD= or INEST= data set.

General Quadratic Programming (QUADAS)

The QUADAS algorithm is an active set method that iteratively updates the QT decomposition of the matrix A_k of active linear constraints and the Cholesky factor of the projected Hessian $Z_k^T G Z_k$ simultaneously. The update of active boundary and linear constraints is done separately; refer to Gill et al. (1984). Here Q is an $n_{free} \times n_{free}$ orthogonal matrix composed of vectors spanning the null space Z of A_k in its first $n_{free} - n_{alc}$ columns and range space Y in its last n_{alc} columns; T is an $n_{alc} \times n_{alc}$ triangular matrix of special form, $t_{ij} = 0$ for $i < n - j$, where n_{free} is the number of free parameters (n minus the number of active boundary constraints), and n_{alc} is the number of active linear constraints. The Cholesky factor of the projected Hessian matrix $Z_k^T G Z_k$ and the QT decomposition are updated simultaneously when the active set changes.

Linear Complementarity (LICOMP)

The LICOMP technique solves a quadratic problem as a linear complementarity problem. It can be used only if G is positive (negative) semidefinite for minimization (maximization) and if the parameters are restricted to be positive.

This technique finds a point that meets the Karush-Kuhn-Tucker conditions by solving the linear complementary problem

$$w = Mz + q$$

with constraints

$$w^T z \geq 0, \quad w \geq 0, \quad z \geq 0,$$

where

$$z = \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad M = \begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \quad q = \begin{bmatrix} g \\ -b \end{bmatrix}$$

Only the LCEPSILON= option can be used to specify a tolerance used in computations.

General Nonlinear Optimization

Trust-Region Optimization (TRUREG)

The trust region method uses the gradient $g(x^{(k)})$ and Hessian matrix $G(x^{(k)})$ and thus requires that the objective function $f(x)$ have continuous first- and second-order derivatives inside the feasible region.

The trust region method iteratively optimizes a quadratic approximation to the nonlinear objective function within a hyperelliptic trust region with radius Δ that constrains the step length corresponding to the quality of the quadratic approximation. The trust region method is implemented using Dennis, Gay, and Welsch (1981), Gay (1983).

The trust region method performs well for small to medium problems and does not require many function, gradient, and Hessian calls. If the computation of the Hessian matrix is computationally expensive, use the UPDATE= option for update formulas (that gradually build the second-order information in the Hessian). For larger problems, the conjugate gradient algorithm may be more appropriate.

Newton-Raphson Optimization With Line-Search (NEWRAP)

The NEWRAP technique uses the gradient $g(x^{(k)})$ and Hessian matrix $G(x^{(k)})$ and thus requires that the objective function have continuous first- and second-order derivatives inside the feasible region. If second-order derivatives are computed efficiently and precisely, the NEWRAP method may perform well for medium to large problems, and it does not need many function, gradient, and Hessian calls.

This algorithm uses a pure Newton step when the Hessian is positive definite and when the Newton step reduces the value of the objective function successfully. Otherwise, a combination of ridging and line search is done to compute successful steps. If the Hessian is not positive definite, a multiple of the identity matrix is added to the Hessian matrix to make it positive definite (Eskow and Schnabel 1991).

In each iteration, a line search is done along the search direction to find an approximate optimum of the objective function. The default line-search method uses quadratic interpolation and cubic extrapolation (LIS=2).

Newton-Raphson Ridge Optimization (NRRIDG)

The NRRIDG technique uses the gradient $g(x^{(k)})$ and Hessian matrix $G(x^{(k)})$ and thus requires that the objective function have continuous first- and second-order derivatives inside the feasible region.

This algorithm uses a pure Newton step when the Hessian is positive definite and when the Newton step reduces the value of the objective function successfully. If at least one of these two conditions is not satisfied, a multiple of the identity matrix is added to the Hessian matrix. If this algorithm is used for least-squares problems, it performs a ridged Gauss-Newton minimization.

The NRRIDG method performs well for small to medium problems and does not need many function, gradient, and Hessian calls. However, if the computation of

the Hessian matrix is computationally expensive, one of the (dual) quasi-Newton or conjugate gradient algorithms may be more efficient.

Since NRRIDG uses an orthogonal decomposition of the approximate Hessian, each iteration of NRRIDG can be slower than that of NEWRAP, which works with Cholesky decomposition. However, usually NRRIDG needs fewer iterations than NEWRAP.

Quasi-Newton Optimization (QUANEW)

The (dual) quasi-Newton method uses the gradient $g(x^{(k)})$ and does not need to compute second-order derivatives since they are approximated. It works well for medium to moderately large optimization problems where the objective function and the gradient are much faster to compute than the Hessian, but in general it requires more iterations than the techniques TRUREG, NEWRAP, and NRRIDG, which compute second-order derivatives.

The QUANEW algorithm depends on whether or not there are nonlinear constraints.

Unconstrained or Linearly Constrained Problems

If there are no nonlinear constraints, QUANEW is either

- the original quasi-Newton algorithm that updates an approximation of the inverse Hessian, or
- the dual quasi-Newton algorithm that updates the Cholesky factor of an approximate Hessian (default),

depending on the value of the UPDATE= option. For problems with general linear inequality constraints, the dual quasi-Newton methods can be more efficient than the original ones.

Four update formulas can be specified with the UPDATE= option:

DBFGS	performs the dual BFGS (Broyden, Fletcher, Goldfarb, & Shanno) update of the Cholesky factor of the Hessian matrix. This is the default.
DDFP	performs the dual DFP (Davidon, Fletcher, & Powell) update of the Cholesky factor of the Hessian matrix.
BFGS	performs the original BFGS (Broyden, Fletcher, Goldfarb, & Shanno) update of the inverse Hessian matrix.
DFP	performs the original DFP (Davidon, Fletcher, & Powell) update of the inverse Hessian matrix.

In each iteration, a line search is done along the search direction to find an approximate optimum. The default line-search method uses quadratic interpolation and cubic extrapolation to obtain a step length α satisfying the Goldstein conditions. One of the Goldstein conditions can be violated if the feasible region defines an upper limit of the step length. Violating the left-side Goldstein condition can affect the positive

definiteness of the quasi-Newton update. In those cases, either the update is skipped or the iterations are restarted with an identity matrix resulting in the steepest descent or ascent search direction. Line-search algorithms other than the default one can be specified with the LINESEARCH= option.

Nonlinearly Constrained Problems

The algorithm used for nonlinearly constrained quasi-Newton optimization is an efficient modification of Powell's (1978a, 1982b) *Variable Metric Constrained WatchDog* (VMCWD) algorithm. A similar but older algorithm (VF02AD) is part of the Harwell library. Both VMCWD and VF02AD use Fletcher's VE02AD algorithm (part of the Harwell library) for positive-definite quadratic programming. The PROC NLP QUANEW implementation uses a quadratic programming subroutine that updates and downdates the approximation of the Cholesky factor when the active set changes. The nonlinear QUANEW algorithm is not a feasible-point algorithm, and the value of the objective function need not decrease (minimization) or increase (maximization) monotonically. Instead, the algorithm tries to reduce a linear combination of the objective function and constraint violations, called the *merit function*.

The following are similarities and differences between this algorithm and the VMCWD algorithm:

- A modification of this algorithm can be performed by specifying VERSION=1, which replaces the update of the Lagrange vector μ with the original update of Powell (1978a, b) that is used in VF02AD. This can be helpful for some applications with linearly dependent active constraints.
- If the VERSION option is not specified or if VERSION=2 is specified, the evaluation of the Lagrange vector μ is performed in the same way as Powell (1982b) describes.
- Instead of updating an approximate Hessian matrix, this algorithm uses the dual BFGS (or DFP) update that updates the Cholesky factor of an approximate Hessian. If the condition of the updated matrix gets too bad, a restart is done with a positive diagonal matrix. At the end of the first iteration after each restart, the Cholesky factor is scaled.
- The Cholesky factor is loaded into the quadratic programming subroutine, automatically ensuring positive definiteness of the problem. During the quadratic programming step, the Cholesky factor of the projected Hessian matrix $Z_k^T G Z_k$ and the QT decomposition are updated simultaneously when the active set changes. Refer to Gill et al. (1984) for more information.
- The line-search strategy is very similar to that of Powell (1982b). However, this algorithm does not call for derivatives during the line search, so the algorithm generally needs fewer derivative calls than function calls. VMCWD always requires the same number of derivative and function calls. Sometimes Powell's line-search method uses steps that are too long. In these cases, use the INSTEP= option to restrict the step length α.
- The watchdog strategy is similar to that of Powell (1982b); however, it doesn't return automatically after a fixed number of iterations to a former better point.

A return here is further delayed if the observed function reduction is close to the expected function reduction of the quadratic model.

- The Powell termination criterion still is used (as FCONV2) but the QUANEW implementation uses two additional termination criteria (GCONV and ABSGCONV).

The nonlinear QUANEW algorithm needs the Jacobian matrix of the first-order derivatives (constraints normals) of the constraints $CJ(x)$.

You can specify two update formulas with the UPDATE= option:

DBFGS	performs the dual BFGS update of the Cholesky factor of the Hessian matrix. This is the default.
DDFP	performs the dual DFP update of the Cholesky factor of the Hessian matrix.

This algorithm uses its own line-search technique. No options or parameters (except the INSTEP= option) controlling the line search in the other algorithms apply here. In several applications, large steps in the first iterations were troublesome. You can use the INSTEP= option to impose an upper bound for the step length α during the first five iterations. You may also use the INHESSIAN= option to specify a different starting approximation for the Hessian. Choosing simply the INHESSIAN option will use the Cholesky factor of a (possibly ridged) finite-difference approximation of the Hessian to initialize the quasi-Newton update process. The values of the LCSINGULAR=, LCEPSILON=, and LCDEACT= options, which control the processing of linear and boundary constraints, are valid only for the quadratic programming subroutine used in each iteration of the nonlinear constraints QUANEW algorithm.

Double Dogleg Optimization (DBLDOG)

The double dogleg optimization method combines the ideas of the quasi-Newton and trust region methods. The double dogleg algorithm computes in each iteration the step $s^{(k)}$ as a linear combination of the steepest descent or ascent search direction $s_1^{(k)}$ and a quasi-Newton search direction $s_2^{(k)}$:

$$s^{(k)} = \alpha_1 s_1^{(k)} + \alpha_2 s_2^{(k)}$$

The step is requested to remain within a prespecified trust region radius; refer to Fletcher (1987, p. 107). Thus, the DBLDOG subroutine uses the dual quasi-Newton update but does not perform a line search. Two update formulas can be specified with the UPDATE= option:

DBFGS	performs the dual BFGS (Broyden, Fletcher, Goldfarb, & Shanno) update of the Cholesky factor of the Hessian matrix. This is the default.
DDFP	performs the dual DFP (Davidon, Fletcher, & Powell) update of the Cholesky factor of the Hessian matrix.

The double dogleg optimization technique works well for medium to moderately large optimization problems where the objective function and the gradient are much faster to compute than the Hessian. The implementation is based on Dennis and Mei (1979) and Gay (1983) but is extended for dealing with boundary and linear constraints. DBLDOG generally needs more iterations than the techniques TRUREG, NEWRAP, or NRRIDG that need second-order derivatives, but each of the DBLDOG iterations is computationally cheap. Furthermore, DBLDOG needs only gradient calls for the update of the Cholesky factor of an approximate Hessian.

Conjugate Gradient Optimization (CONGRA)

Second-order derivatives are not used by CONGRA. The CONGRA algorithm can be expensive in function and gradient calls but needs only $O(n)$ memory for unconstrained optimization. In general, many iterations are needed to obtain a precise solution, but each of the CONGRA iterations is computationally cheap. Four different update formulas for generating the conjugate directions can be specified using the UPDATE= option:

PB	performs the automatic restart update method of Powell (1977) and Beale (1972). This is the default.
FR	performs the Fletcher-Reeves update (Fletcher 1987).
PR	performs the Polak-Ribiere update (Fletcher 1987).
CD	performs a conjugate-descent update of Fletcher (1987).

The default value is UPDATE=PB, since it behaved best in most test examples. You are advised to avoid the option UPDATE=CD, a it behaved worst in most test examples.

The CONGRA subroutine should be used for optimization problems with large n. For the unconstrained or boundary constrained case, CONGRA needs only $O(n)$ bytes of working memory, whereas all other optimization methods require order $O(n^2)$ bytes of working memory. During n successive iterations, uninterrupted by restarts or changes in the working set, the conjugate gradient algorithm computes a cycle of n conjugate search directions. In each iteration, a line search is done along the search direction to find an approximate optimum of the objective function. The default line-search method uses quadratic interpolation and cubic extrapolation to obtain a step length α satisfying the Goldstein conditions. One of the Goldstein conditions can be violated if the feasible region defines an upper limit for the step length. Other line-search algorithms can be specified with the LINESEARCH= option.

Nelder-Mead Simplex Optimization (NMSIMP)

The Nelder-Mead simplex method does not use any derivatives and does not assume that the objective function has continuous derivatives. The objective function itself needs to be continuous. This technique requires a large number of function evaluations. It is unlikely to give accurate results for $n \gg 40$.

Depending on the kind of constraints, one of the following Nelder-Mead simplex algorithms is used:

- unconstrained or only boundary constrained problems

  The original Nelder-Mead simplex algorithm is implemented and extended to boundary constraints. This algorithm does not compute the objective for infeasible points. This algorithm is automatically invoked if the LINCON or NLINCON statement is not specified.

- general linearly constrained or nonlinearly constrained problems

  A slightly modified version of Powell's (1992) COBYLA (Constrained Optimization BY Linear Approximations) implementation is used. This algorithm is automatically invoked if either the LINCON or the NLINCON statement is specified.

The original Nelder-Mead algorithm cannot be used for general linear or nonlinear constraints but can be faster for the unconstrained or boundary constrained case. The original Nelder-Mead algorithm changes the shape of the simplex adapting the nonlinearities of the objective function which contributes to an increased speed of convergence. The two NMSIMP subroutines use special sets of termination criteria. For more details, refer to the section "Termination Criteria" on page 361.

Powell's COBYLA Algorithm (COBYLA)

Powell's COBYLA algorithm is a sequential trust region algorithm (originally with a monotonically decreasing radius ρ of a spheric trust region) that tries to maintain a regular-shaped simplex over the iterations. A small modification was made to the original algorithm that permits an increase of the trust region radius ρ in special situations. A sequence of iterations is performed with a constant trust region radius ρ until the computed objective function reduction is much less than the predicted reduction. Then, the trust region radius ρ is reduced. The trust region radius is increased only if the computed function reduction is relatively close to the predicted reduction and the simplex is well-shaped. The start radius ρ_{beg} and the final radius ρ_{end} can be specified using ρ_{beg}=*INSTEP* and ρ_{end}=*ABSXTOL*. The convergence to small values of ρ_{end} (high precision) may take many calls of the function and constraint modules and may result in numerical problems. There are two main reasons for the slow convergence of the COBYLA algorithm:

- Only linear approximations of the objective and constraint functions are used locally.
- Maintaining the regular-shaped simplex and not adapting its shape to nonlinearities yields very small simplices for highly nonlinear functions (for example, fourth-order polynomials).

Nonlinear Least-Squares Optimization

Levenberg-Marquardt Least-Squares Method (LEVMAR)

The Levenberg-Marquardt method is a modification of the trust region method for nonlinear least-squares problems and is implemented as in Moré (1978).

This is the recommended algorithm for small to medium least-squares problems. Large least-squares problems can be transformed into minimization problems, which

can be processed with conjugate gradient or (dual) quasi-Newton techniques. In each iteration, LEVMAR solves a quadratically constrained quadratic minimization problem that restricts the step to stay at the surface of or inside an n- dimensional elliptical (or spherical) trust region. In each iteration, LEVMAR uses the crossproduct Jacobian matrix $J^T J$ as an approximate Hessian matrix.

Hybrid Quasi-Newton Least-Squares Methods (HYQUAN)

In each iteration of one of the Fletcher and Xu (1987) (refer also to Al-Baali and Fletcher (1985,1986)) hybrid quasi-Newton methods, a criterion is used to decide whether a Gauss-Newton or a dual quasi-Newton search direction is appropriate. The VERSION= option can be used to choose one of three criteria (HY1, HY2, HY3) proposed by Fletcher and Xu (1987). The default is VERSION=2; that is, HY2. In each iteration, HYQUAN computes the crossproduct Jacobian (used for the Gauss-Newton step), updates the Cholesky factor of an approximate Hessian (used for the quasi-Newton step), and does a line search to compute an approximate minimum along the search direction. The default line-search technique used by HYQUAN is especially designed for least-squares problems (refer to Lindström and Wedin (1984) and Al-Baali and Fletcher (1986)). Using the LINESEARCH= option you can choose a different line-search algorithm than the default one.

Two update formulas can be specified with the UPDATE= option:

DBFGS performs the dual BFGS (Broyden, Fletcher, Goldfarb, and Shanno) update of the Cholesky factor of the Hessian matrix. This is the default.

DDFP performs the dual DFP (Davidon, Fletcher, and Powell) update of the Cholesky factor of the Hessian matrix.

The HYQUAN subroutine needs about the same amount of working memory as the LEVMAR algorithm. In most applications, LEVMAR seems to be superior to HYQUAN, and using HYQUAN is recommended only when problems are experienced with the performance of LEVMAR.

Finite-Difference Approximations of Derivatives

The FD= and FDHESSIAN= options specify the use of finite-difference approximations of the derivatives. The FD= option specifies that all derivatives are approximated using function evaluations, and the FDHESSIAN= option specifies that second-order derivatives are approximated using gradient evaluations.

Computing derivatives by finite-difference approximations can be very time-consuming, especially for second-order derivatives based only on values of the objective function (FD= option). If analytical derivatives are difficult to obtain (for example, if a function is computed by an iterative process), you might consider one of the optimization techniques that uses first-order derivatives only (TECH=QUANEW, TECH=DBLDOG, or TECH=CONGRA).

Forward-Difference Approximations

The forward-difference derivative approximations consume less computer time but are usually not as precise as those using central-difference formulas.

- First-order derivatives: n additional function calls are needed:

$$g_i = \frac{\partial f}{\partial x_i} = \frac{f(x + h_i e_i) - f(x)}{h_i}$$

- Second-order derivatives based on function calls only (Dennis and Schnabel 1983, p. 80, 104): for dense Hessian, $n(n+3)/2$ additional function calls are needed:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{f(x + h_i e_i + h_j e_j) - f(x + h_i e_i) - f(x + h_j e_j) + f(x)}{h_j}$$

- Second-order derivatives based on gradient calls (Dennis and Schnabel 1983, p. 103): n additional gradient calls are needed:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{g_i(x + h_j e_j) - g_i(x)}{2h_j} + \frac{g_j(x + h_i e_i) - g_j(x)}{2h_i}$$

Central-Difference Approximations

- First-order derivatives: $2n$ additional function calls are needed:

$$g_i = \frac{\partial f}{\partial x_i} = \frac{f(x + h_i e_i) - f(x - h_i e_i)}{2h_i}$$

- Second-order derivatives based on function calls only (Abramowitz and Stegun 1972, p. 884): for dense Hessian, $2n(n+1)$ additional function calls are needed:

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{-f(x + 2h_i e_i) + 16 f(x + h_i e_i) - 30 f(x) + 16 f(x - h_i e_i) - f(x - 2h_i e_i)}{12 h_i^2}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{f(x + h_i e_i + h_j e_j) - f(x + h_i e_i - h_j e_j) - f(x - h_i e_i + h_j e_j) + f(x - h_i e_i - h_j e_j)}{4 h_i h_j}$$

- Second-order derivatives based on gradient: $2n$ additional gradient calls are needed:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{g_i(x + h_j e_j) - g_i(x - h_j e_j)}{4h_j} + \frac{g_j(x + h_i e_i) - g_j(x - h_i e_i)}{4h_i}$$

The FDIGITS= and CDIGITS= options can be used for specifying the number of accurate digits in the evaluation of objective function and nonlinear constraints. These specifications are helpful in determining an appropriate interval length h to be used in the finite-difference formulas.

The FDINT= option specifies whether the finite-difference intervals h should be computed using an algorithm of Gill, Murray, Saunders, and Wright (1983) or based only on the information of the FDIGITS= and CDIGITS= options. For FDINT=OBJ, the interval h is based on the behavior of the objective function; for FDINT=CON, the interval h is based on the behavior of the nonlinear constraints functions; and for FDINT=ALL, the interval h is based on the behaviors of both the objective function and the nonlinear constraints functions. Note that the algorithm of Gill, Murray, Saunders, and Wright (1983) to compute the finite-difference intervals h_j can be very expensive in the number of function calls. If the FDINT= option is specified, it is currently performed twice, the first time before the optimization process starts and the second time after the optimization terminates.

If FDINT= is not specified, the step lengths h_j, $j = 1, \ldots, n$, are defined as follows:

- for the forward-difference approximation of first-order derivatives using function calls and second-order derivatives using gradient calls: $h_j = \sqrt[2]{\eta_j}(1 + |x_j|)$,
- for the forward-difference approximation of second-order derivatives that use only function calls and all central-difference formulas: $h_j = \sqrt[3]{\eta_j}(1 + |x_j|)$,

where η is defined using the FDIGITS= option:

- If the number of accurate digits is specified with FDIGITS=r, η is set to 10^{-r}.
- If FDIGITS= is not specified, η is set to the machine precision ϵ.

For FDINT=OBJ and FDINT=ALL, the FDIGITS= specification is used in computing the forward and central finite-difference intervals.

If the problem has nonlinear constraints and the FD= option is specified, the first-order formulas are used to compute finite-difference approximations of the Jacobian matrix $JC(x)$. You can use the CDIGITS= option to specify the number of accurate digits in the constraint evaluations to define the step lengths h_j, $j = 1, \ldots, n$. For FDINT=CON and FDINT=ALL, the CDIGITS= specification is used in computing the forward and central finite-difference intervals.

Note: If you are unable to specify analytic derivatives and the finite-difference approximations provided by PROC NLP are not good enough to solve your problem, you may program better finite-difference approximations using the GRADIENT, JACOBIAN, CRPJAC, or HESSIAN statement and the program statements.

Hessian and CRP Jacobian Scaling

The rows and columns of the Hessian and crossproduct Jacobian matrix can be scaled when using the trust region, Newton-Raphson, double dogleg, and Levenberg-Marquardt optimization techniques. Each element $G_{i,j}$, $i,j = 1,\ldots,n$, is divided by the scaling factor $d_i \times d_j$, where the scaling vector $d = (d_1,\ldots,d_n)$ is iteratively updated in a way specified by the HESCAL=i option, as follows:

$i = 0$ No scaling is done (equivalent to $d_i = 1$).

$i \neq 0$ First iteration and each restart iteration:

$$d_i^{(0)} = \sqrt{\max(|G_{i,i}^{(0)}|, \epsilon)}$$

$i = 1$ refer to Moré (1978):

$$d_i^{(k+1)} = \max\left(d_i^{(k)}, \sqrt{\max(|G_{i,i}^{(k)}|, \epsilon)}\right)$$

$i = 2$ refer to Dennis, Gay, and Welsch (1981):

$$d_i^{(k+1)} = \max\left(0.6d_i^{(k)}, \sqrt{\max(|G_{i,i}^{(k)}|, \epsilon)}\right)$$

$i = 3$ d_i is reset in each iteration:

$$d_i^{(k+1)} = \sqrt{\max(|G_{i,i}^{(k)}|, \epsilon)}$$

where ϵ is the relative machine precision or, equivalently, the largest double precision value that when added to 1 results in 1.

Testing the Gradient Specification

There are three main ways to check the correctness of derivative specifications:

- Specify the FD= or FDHESSIAN= option in the PROC NLP statement to compute finite-difference approximations of first- and second-order derivatives. In many applications, the finite-difference approximations are computed with high precision and do not differ too much from the derivatives that are computed by specified formulas.
- Specify the GRADCHECK[=DETAIL] option in the PROC NLP statement to compute and display a test vector and a test matrix of the gradient values at the starting point $x^{(0)}$ by the method of Wolfe (1982). If you do not specify the GRADCHECK option, a fast derivative test identical to the GRADCHECK=FAST specification is done by default.

- If the default analytical derivative compiler is used or if derivatives are specified using the GRADIENT or JACOBIAN statement, the gradient or Jacobian computed at the initial point $x^{(0)}$ is tested by default using finite-difference approximations. In some examples, the relative test can show significant differences between the two forms of derivatives, resulting in a warning message indicating that the specified derivatives could be wrong, even if they are correct. This happens especially in cases where the magnitude of the gradient at the starting point $x^{(0)}$ is small.

The algorithm of Wolfe (1982) is used to check whether the gradient $g(x)$ specified by a GRADIENT statement (or indirectly by a JACOBIAN statement) is appropriate for the objective function $f(x)$ specified by the program statements.

Using function and gradient evaluations in the neighborhood of the starting point $x^{(0)}$, second derivatives are approximated by finite-difference formulas. Forward differences of gradient values are used to approximate the Hessian element G_{jk},

$$G_{jk} \approx H_{jk} = \frac{g_j(x + \delta e_k) - g_j(x)}{\delta}$$

where δ is a small step length and $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ is the unit vector along the kth coordinate axis. The test vector s, with

$$s_j = H_{jj} - \frac{2}{\delta}\left\{\frac{f(x + \delta e_j) - f(x)}{\delta} - g_j(x)\right\}$$

contains the differences between two sets of finite-difference approximations for the diagonal elements of the Hessian matrix

$$G_{jj} = \partial^2 f(x^{(0)})/\partial x_j^2\,, \quad j = 1, \ldots, n$$

The test matrix ΔH contains the absolute differences of symmetric elements in the approximate Hessian $|H_{jk} - H_{kj}|$, $j, k = 1, \ldots, n$, generated by forward differences of the gradient elements.

If the specification of the first derivatives is correct, the elements of the test vector and test matrix should be relatively small. The location of large elements in the test matrix points to erroneous coordinates in the gradient specification. For very large optimization problems, this algorithm can be too expensive in terms of computer time and memory.

Termination Criteria

All optimization techniques stop iterating at $x^{(k)}$ if at least one of a set of termination criteria is satisfied. PROC NLP also terminates if the point $x^{(k)}$ is fully constrained by n linearly independent active linear or boundary constraints, and all Lagrange multiplier estimates of active inequality constraints are greater than a small negative tolerance.

Since the Nelder-Mead simplex algorithm does not use derivatives, no termination criterion is available based on the gradient of the objective function. Powell's COBYLA algorithm uses only one more termination criterion. COBYLA is a trust region algorithm that sequentially reduces the radius ρ of a spherical trust region from a start radius ρ_{beg} = *INSTEP* to the final radius ρ_{end} = *ABSXTOL*. The default value is $\rho_{end} = 1e{-}4$. The convergence to small values of ρ_{end} (high precision) may take many calls of the function and constraint modules and may result in numerical problems.

In some applications, the small default value of the ABSGCONV= criterion is too difficult to satisfy for some of the optimization techniques. This occurs most often when finite-difference approximations of derivatives are used.

The default setting for the GCONV= option sometimes leads to early termination far from the location of the optimum. This is especially true for the special form of this criterion used in the CONGRA optimization.

The QUANEW algorithm for nonlinearly constrained optimization does not monotonically reduce the value of either the objective function or some kind of merit function which combines objective and constraint functions. Furthermore, the algorithm uses the watchdog technique with backtracking (Chamberlain et al. 1982). Therefore, no termination criteria were implemented that are based on the values (x or f) of successive iterations. In addition to the criteria used by all optimization techniques, three more termination criteria are currently available. They are based on satisfying the Karush-Kuhn-Tucker conditions, which require that the gradient of the Lagrange function is zero at the optimal point (x^*, λ^*):

$$\nabla_x L(x^*, \lambda^*) = 0$$

For more information, refer to the section "Criteria for Optimality" on page 343.

Active Set Methods

The parameter vector $x \in \mathcal{R}^n$ may be subject to a set of m linear equality and inequality constraints:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m_e$$
$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = m_e + 1, \ldots, m$$

The coefficients a_{ij} and right-hand sides b_i of the equality and inequality constraints are collected in the $m \times n$ matrix A and the $m-$vector b.

The m linear constraints define a feasible region $\mathcal{G}$ in $\mathcal{R}^n$ that must contain the point x^* that minimizes the problem. If the feasible region $\mathcal{G}$ is empty, no solution to the optimization problem exists.

All optimization techniques in PROC NLP (except those processing nonlinear constraints) are *active set methods*. The iteration starts with a feasible point $x^{(0)}$, which

either is provided by the user or can be computed by the Schittkowski and Stoer (1979) algorithm implemented in PROC NLP. The algorithm then moves from one feasible point $x^{(k-1)}$ to a better feasible point $x^{(k)}$ along a feasible search direction $s^{(k)}$:

$$x^{(k)} = x^{(k-1)} + \alpha^{(k)} s^{(k)}, \quad \alpha^{(k)} > 0$$

Theoretically, the path of points $x^{(k)}$ never leaves the feasible region $\mathcal{G}$ of the optimization problem, but it can hit its boundaries. The active set $\mathcal{A}^{(k)}$ of point $x^{(k)}$ is defined as the index set of all linear equality constraints and those inequality constraints that are satisfied at $x^{(k)}$. If no constraint is active for $x^{(k)}$, the point is located in the interior of $\mathcal{G}$, and the active set $\mathcal{A}^{(k)}$ is empty. If the point $x^{(k)}$ in iteration k hits the boundary of inequality constraint i, this constraint i becomes active and is added to $\mathcal{A}^{(k)}$. Each equality or active inequality constraint reduces the dimension (degrees of freedom) of the optimization problem.

In practice, the active constraints can be satisfied only with finite precision. The LCEPSILON=r option specifies the range for active and violated linear constraints. If the point $x^{(k)}$ satisfies the condition

$$\left| \sum_{j=1}^{n} a_{ij} x_j^{(k)} - b_i \right| \leq t$$

where $t = r \times (|b_i| + 1)$, the constraint i is recognized as an active constraint. Otherwise, the constraint i is either an inactive inequality or a violated inequality or equality constraint. Due to rounding errors in computing the projected search direction, error can be accumulated so that an iterate $x^{(k)}$ steps out of the feasible region. In those cases, PROC NLP may try to pull the iterate $x^{(k)}$ into the feasible region. However, in some cases the algorithm needs to increase the feasible region by increasing the LCEPSILON=r value. If this happens it is indicated by a message displayed in the log output.

If you cannot expect an improvement in the value of the objective function by moving from an active constraint back into the interior of the feasible region, you use this inequality constraint as an equality constraint in the next iteration. That means the active set $\mathcal{A}^{(k+1)}$ still contains the constraint i. Otherwise you release the active inequality constraint and increase the dimension of the optimization problem in the next iteration.

A serious numerical problem can arise when some of the active constraints become (nearly) linearly dependent. Linearly dependent equality constraints are removed before entering the optimization. You can use the LCSINGULAR= option to specify a criterion r used in the update of the QR decomposition that decides whether an active constraint is linearly dependent relative to a set of other active constraints.

If the final parameter set x^* is subjected to n_{act} linear equality or active inequality constraints, the QR decomposition of the $n \times n_{act}$ matrix $\hat{A}^T$ of the linear constraints is computed by $\hat{A}^T = QR$, where Q is an $n \times n$ orthogonal matrix and R is an $n \times n_{act}$ upper triangular matrix. The n columns of matrix Q can be separated into two matrices, $Q = [Y, Z]$, where Y contains the first n_{act} orthogonal columns of

Q and Z contains the last $n - n_{act}$ orthogonal columns of Q. The $n \times (n - n_{act})$ column-orthogonal matrix Z is also called the nullspace matrix of the active linear constraints $\hat{A}^T$. The $n - n_{act}$ columns of the $n \times (n - n_{act})$ matrix Z form a basis orthogonal to the rows of the $n_{act} \times n$ matrix $\hat{A}$.

At the end of the iteration process, the PROC NLP can display the *projected gradient*

$$g_Z = Z^T g$$

In the case of boundary constrained optimization, the elements of the projected gradient correspond to the gradient elements of the free parameters. A necessary condition for x^* to be a local minimum of the optimization problem is

$$g_Z(x^*) = Z^T g(x^*) = 0$$

The symmetric $n_{act} \times n_{act}$ matrix

$$G_Z = Z^T G Z$$

is called a *projected Hessian matrix*. A second-order necessary condition for x^* to be a local minimizer requires that the projected Hessian matrix is positive semidefinite. If available, the projected gradient and projected Hessian matrix can be displayed and written in an OUTEST= data set.

Those elements of the n_{act} vector of first-order estimates of *Lagrange multipliers*

$$\lambda = (\hat{A}\hat{A}^T)^{-1}\hat{A}ZZ^T g$$

which correspond to active inequality constraints indicate whether an improvement of the objective function can be obtained by releasing this active constraint. For minimization (maximization), a significant negative (positive) Lagrange multiplier indicates that a possible reduction (increase) of the objective function can be obtained by releasing this active linear constraint. The LCDEACT=r option can be used to specify a threshold r for the Lagrange multiplier that decides whether an active inequality constraint remains active or can be deactivated. The Lagrange multipliers are displayed (and written in an OUTEST= data set) only if linear constraints are active at the solution x^*. (In the case of boundary-constrained optimization, the Lagrange multipliers for active lower (upper) constraints are the negative (positive) gradient elements corresponding to the active parameters.)

Feasible Starting Point

Two algorithms are used to obtain a feasible starting point.

- When only boundary constraints are specified:

– If the parameter x_j, $1 \le j \le n$, violates a two-sided boundary constraint (or an equality constraint) $l_j \le x_j \le u_j$, the parameter is given a new value inside the feasible interval, as follows:

$$x_j = \begin{cases} l_j, & \text{if } u_j \le l_j \\ l_j + \frac{1}{2}(u_j - l_j), & \text{if } u_j - l_j < 4 \\ l_j + \frac{1}{10}(u_j - l_j), & \text{if } u_j - l_j \ge 4 \end{cases}$$

– If the parameter x_j, $1 \le j \le n$, violates a one-sided boundary constraint $l_j \le x_j$ or $x_j \le u_j$, the parameter is given a new value near the violated boundary, as follows:

$$x_j = \begin{cases} l_j + \max(1, \frac{1}{10}l_j), & \text{if } x_j < l_j \\ u_j - \max(1, \frac{1}{10}u_j), & \text{if } x_j > u_j \end{cases}$$

- When general linear constraints are specified, the algorithm of Schittkowski and Stoer (1979) computes a feasible point, which may be quite far from a user-specified infeasible point.

Line-Search Methods

In each iteration k, the (dual) quasi-Newton, hybrid quasi-Newton, conjugate gradient, and Newton-Raphson minimization techniques use iterative line-search algorithms that try to optimize a linear, quadratic, or cubic approximation of f along a feasible descent search direction $s^{(k)}$

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)} s^{(k)}, \quad \alpha^{(k)} > 0$$

by computing an approximately optimal scalar $\alpha^{(k)}$.

Therefore, a line-search algorithm is an iterative process that optimizes a nonlinear function $f = f(\alpha)$ of one parameter (α) within each iteration k of the optimization technique, which itself tries to optimize a linear or quadratic approximation of the nonlinear objective function $f = f(x)$ of n parameters x. Since the outside iteration process is based only on the approximation of the objective function, the inside iteration of the line-search algorithm does not have to be perfect. Usually, the choice of α significantly reduces (in a minimization) the objective function. Criteria often used for termination of line-search algorithms are the Goldstein conditions (refer to Fletcher (1987)).

Various line-search algorithms can be selected using the LINESEARCH= option. The line-search method LINESEARCH=2 seems to be superior when function evaluation consumes significantly less computation time than gradient evaluation. Therefore, LINESEARCH=2 is the default value for Newton-Raphson, (dual) quasi-Newton, and conjugate gradient optimizations.

A special default line-search algorithm for TECH=HYQUAN is useful only for least-squares problems and cannot be chosen by the LINESEARCH= option. This method uses three columns of the $m \times n$ Jacobian matrix, which for large m can require

more memory than using the algorithms designated by LINESEARCH=1 through LINESEARCH=8.

The line-search methods LINESEARCH=2 and LINESEARCH=3 can be modified to exact line search by using the LSPRECISION= option (specifying the σ parameter in Fletcher (1987)). The line-search methods LINESEARCH=1, LINESEARCH=2, and LINESEARCH=3 satisfy the left-hand-side and right-hand-side Goldstein conditions (refer to Fletcher (1987)). When derivatives are available, the line-search methods LINESEARCH=6, LINESEARCH=7, and LINESEARCH=8 try to satisfy the right-hand-side Goldstein condition; if derivatives are not available, these line-search algorithms use only function calls.

Restricting the Step Length

Almost all line-search algorithms use iterative extrapolation techniques which can easily lead them to (feasible) points where the objective function f is no longer defined. (e.g., resulting in indefinite matrices for ML estimation) or difficult to compute (e.g., resulting in floating point overflows). Therefore, PROC NLP provides options restricting the step length α or trust region radius Δ, especially during the first main iterations.

The inner product $g^T s$ of the gradient g and the search direction s is the slope of $f(\alpha) = f(x + \alpha s)$ along the search direction s. The default starting value $\alpha^{(0)} = \alpha^{(k,0)}$ in each line-search algorithm $(\min_{\alpha>0} f(x + \alpha s))$ during the main iteration k is computed in three steps:

1. The first step uses either the difference $df = |f^{(k)} - f^{(k-1)}|$ of the function values during the last two consecutive iterations or the final step length value α^- of the last iteration $k - 1$ to compute a first value of $\alpha_1^{(0)}$.

   - Not using the DAMPSTEP=r option:

   $$\alpha_1^{(0)} = \begin{cases} step, & \text{if } 0.1 \leq step \leq 10 \\ 10, & \text{if } step > 10 \\ 0.1, & \text{if } step < 0.1 \end{cases}$$

   with

   $$step = \begin{cases} df/|g^T s|, & \text{if } |g^T s| \geq \epsilon \max(100\, df, 1) \\ 1, & \text{otherwise} \end{cases}$$

   This value of $\alpha_1^{(0)}$ can be too large and lead to a difficult or impossible function evaluation, especially for highly nonlinear functions such as the EXP function.

   - Using the DAMPSTEP=r option:

   $$\alpha_1^{(0)} = \min(1, r\alpha^-)$$

   The initial value for the new step length can be no larger than r times the final step length α^- of the previous iteration. The default value is $r = 2$.

2. During the first five iterations, the second step enables you to reduce $\alpha_1^{(0)}$ to a smaller starting value $\alpha_2^{(0)}$ using the INSTEP=r option:

$$\alpha_2^{(0)} = \min(\alpha_1^{(0)}, r)$$

   After more than five iterations, $\alpha_2^{(0)}$ is set to $\alpha_1^{(0)}$.

3. The third step can further reduce the step length by

$$\alpha_3^{(0)} = \min(\alpha_2^{(0)}, \min(10, u))$$

   where u is the maximum length of a step inside the feasible region.

The INSTEP=r option lets you specify a smaller or larger radius Δ of the trust region used in the first iteration of the trust region, double dogleg, and Levenberg-Marquardt algorithms. The default initial trust region radius $\Delta^{(0)}$ is the length of the scaled gradient (Moré 1978). This step corresponds to the default radius factor of $r = 1$. In most practical applications of the TRUREG, DBLDOG, and LEVMAR algorithms, this choice is successful. However, for bad initial values and highly nonlinear objective functions (such as the EXP function), the default start radius can result in arithmetic overflows. If this happens, you may try decreasing values of INSTEP=r, $0 < r < 1$, until the iteration starts successfully. A small factor r also affects the trust region radius $\Delta^{(k+1)}$ of the next steps because the radius is changed in each iteration by a factor $0 < c \leq 4$, depending on the ratio ρ expressing the goodness of quadratic function approximation. Reducing the radius Δ corresponds to increasing the ridge parameter λ, producing smaller steps directed more closely toward the (negative) gradient direction.

Computational Problems

First Iteration Overflows

If you use bad initial values for the parameters, the computation of the value of the objective function (and its derivatives) can lead to arithmetic overflows in the first iteration. The line-search algorithms that work with cubic extrapolation are especially sensitive to arithmetic overflows. If an overflow occurs with an optimization technique that uses line search, you can use the INSTEP= option to reduce the length of the first trial step during the line search of the first five iterations or use the DAMPSTEP or MAXSTEP= option to restrict the step length of the initial α in subsequent iterations. If an arithmetic overflow occurs in the first iteration of the trust region, double dogleg, or Levenberg-Marquardt algorithm, you can use the INSTEP= option to reduce the default trust region radius of the first iteration. You can also change the minimization technique or the line-search method. If none of these methods helps, consider the following actions:

- scale the parameters
- provide better initial values
- use boundary constraints to avoid the region where overflows may happen
- change the algorithm (specified in program statements) which computes the objective function

Problems in Evaluating the Objective Function

The starting point $x^{(0)}$ must be a point that can be evaluated by all the functions involved in your problem. However, during optimization the optimizer may iterate to a point $x^{(k)}$ where the objective function or nonlinear constraint functions and their derivatives cannot be evaluated. If you can identify the problematic region, you can prevent the algorithm from reaching it by adding another constraint to the problem. Another possibility is a modification of the objective function that will produce a large, undesired function value. As a result, the optimization algorithm reduces the step length and stays closer to the point that has been evaluated successfully in the previous iteration. For more information, refer to the section "Missing Values in Program Statements" on page 384.

Problems with Quasi-Newton Methods for Nonlinear Constraints

The sequential quadratic programming algorithm in QUANEW, which is used for solving nonlinearly constrained problems, can have problems updating the Lagrange multiplier vector μ. This usually results in very high values of the Lagrangian function and in *watchdog* restarts indicated in the iteration history. If this happens, there are three actions you can try:

- By default, the Lagrange vector μ is evaluated in the same way as Powell (1982b) describes. This corresponds to VERSION=2. By specifying VERSION=1, a modification of this algorithm replaces the update of the Lagrange vector μ with the original update of Powell (1978a, b), which is used in VF02AD.
- You can use the INSTEP= option to impose an upper bound for the step length α during the first five iterations.
- You can use the INHESSIAN= option to specify a different starting approximation for the Hessian. Choosing only the INHESSIAN option will use the Cholesky factor of a (possibly ridged) finite-difference approximation of the Hessian to initialize the quasi-Newton update process.

Other Convergence Difficulties

There are a number of things to try if the optimizer fails to converge.

- Check the derivative specification:
  If derivatives are specified by using the GRADIENT, HESSIAN, JACOBIAN, CRPJAC, or JACNLC statement, you can compare the specified derivatives with those computed by finite-difference approximations (specifying the FD and FDHESSIAN option). Use the GRADCHECK option to check if the gradient g is correct. For more information, refer to the section "Testing the Gradient Specification" on page 360.
- Forward-difference derivatives specified with the FD= or FDHESSIAN= option may not be precise enough to satisfy strong gradient termination criteria. You may need to specify the more expensive central-difference formulas or use analytical derivatives. The finite-difference intervals may be too small or too

big and the finite-difference derivatives may be erroneous. You can specify the FDINT= option to compute better finite-difference intervals.

- Change the optimization technique:
  For example, if you use the default TECH=LEVMAR, you can
  - change to TECH=QUANEW or to TECH=NRRIDG
  - run some iterations with TECH=CONGRA, write the results in an OUTEST= data set, and use them as initial values specified by an INEST= data set in a second run with a different TECH= technique
- Change or modify the update technique and the line-search algorithm:
  This method applies only to TECH=QUANEW, TECH=HYQUAN, or TECH=CONGRA. For example, if you use the default update formula and the default line-search algorithm, you can
  - change the update formula with the UPDATE= option
  - change the line-search algorithm with the LINESEARCH= option
  - specify a more precise line search with the LSPRECISION= option, if you use LINESEARCH=2 or LINESEARCH=3
- Change the initial values by using a grid search specification to obtain a set of good feasible starting values.

Convergence to Stationary Point

The (projected) gradient at a stationary point is zero and that results in a zero step length. The stopping criteria are satisfied.

There are two ways to avoid this situation:

- Use the DECVAR statement to specify a grid of feasible starting points.
- Use the OPTCHECK= option to avoid terminating at the stationary point.

The signs of the eigenvalues of the (reduced) Hessian matrix contain information regarding a stationary point:

- If all eigenvalues are positive, the Hessian matrix is positive definite and the point is a minimum point.
- If some of the eigenvalues are positive and all remaining eigenvalues are zero, the Hessian matrix is positive semidefinite and the point is a minimum or saddle point.
- If all eigenvalues are negative, the Hessian matrix is negative definite and the point is a maximum point.
- If some of the eigenvalues are negative and all remaining eigenvalues are zero, the Hessian matrix is negative semidefinite and the point is a maximum or saddle point.
- If all eigenvalues are zero, the point can be a minimum, maximum, or saddle point.

Precision of Solution

In some applications, PROC NLP may result in parameter estimates that are not precise enough. Usually this means that the procedure terminated too early at a point too far from the optimal point. The termination criteria define the size of the termination region around the optimal point. Any point inside this region can be accepted for terminating the optimization process. The default values of the termination criteria are set to satisfy a reasonable compromise between the computational effort (computer time) and the precision of the computed estimates for the most common applications. However, there are a number of circumstances where the default values of the termination criteria specify a region that is either too large or too small. If the termination region is too large, it can contain points with low precision. In such cases, you should inspect the log or list output to find the message stating which termination criterion terminated the optimization process. In many applications, you can obtain a solution with higher precision by simply using the old parameter estimates as starting values in a subsequent run where you specify a smaller value for the termination criterion that was satisfied at the previous run.

If the termination region is too small, the optimization process may take longer to find a point inside such a region or may not even find such a point due to rounding errors in function values and derivatives. This can easily happen in applications where finite-difference approximations of derivatives are used and the GCONV and ABSGCONV termination criteria are too small to respect rounding errors in the gradient values.

Covariance Matrix

The COV= option must be specified to compute an approximate covariance matrix for the parameter estimates under asymptotic theory for least-squares, maximum-likelihood, or Bayesian estimation, with or without corrections for degrees of freedom as specified by the VARDEF= option.

Two groups of six different forms of covariance matrices (and therefore approximate standard errors) can be computed corresponding to the following two situations:

- The LSQ statement is specified, which means that least-squares estimates are being computed:

$$\min f(x) = \sum_{i=1}^{m} f_i^2(x)$$

- The MIN or MAX statement is specified, which means that maximum-likelihood or Bayesian estimates are being computed:

$$\mathrm{opt}\, f(x) = \sum_{i=1}^{m} f_i(x)$$

  where opt is either $\min$ or $\max$.

In either case, the following matrices are used:

$$G = \nabla^2 f(x)$$

$$J(f) = (\nabla f_1, \ldots, \nabla f_m) = \left(\frac{\partial f_i}{\partial x_j} \right)$$

$$JJ(f) = J(f)^T J(f)$$

$$V = J(f)^T \mathrm{diag}(f_i^2) J(f)$$

$$W = J(f)^T \mathrm{diag}(f_i^\dagger) J(f)$$

where

$$f_i^\dagger = \begin{cases} 0, & \text{if } f_i = 0 \\ 1/f_i, & \text{otherwise} \end{cases}$$

For unconstrained minimization, or when none of the final parameter estimates are subjected to linear equality or active inequality constraints, the formulas of the six types of covariance matrices are as follows:

COV		MIN or MAX Statement	LSQ Statement
1	M	$(\_\mathrm{NOBS}\_/d)G^{-1}JJ(f)G^{-1}$	$(\_\mathrm{NOBS}\_/d)G^{-1}VG^{-1}$
2	H	$(\_\mathrm{NOBS}\_/d)G^{-1}$	$\sigma^2 G^{-1}$
3	J	$(1/d)W^{-1}$	$\sigma^2 JJ(f)^{-1}$
4	B	$(1/d)G^{-1}WG^{-1}$	$\sigma^2 G^{-1}JJ(f)G^{-1}$
5	E	$(\_\mathrm{NOBS}\_/d)JJ(f)^{-1}$	$(1/d)V^{-1}$
6	U	$(\_\mathrm{NOBS}\_/d)W^{-1}JJ(f)W^{-1}$	$(\_\mathrm{NOBS}\_/d)JJ(f)^{-1}VJJ(f)^{-1}$

The value of d depends on the VARDEF= option and on the value of the _NOBS_ variable:

$$d = \begin{cases} \max(1, \_\mathrm{NOBS}\_ - \_\mathrm{DF}\_), & \text{for VARDEF=DF} \\ \_\mathrm{NOBS}\_, & \text{for VARDEF=N} \end{cases}$$

where _DF_ is either set in the program statements or set by default to n (the number of parameters) and _NOBS_ is either set in the program statements or set by default to *nobs* $\times$ *mfun*, where *nobs* is the number of observations in the data set and *mfun* is the number of functions listed in the LSQ, MIN, or MAX statement.

The value σ^2 depends on the specification of the SIGSQ= option and on the value of d:

$$\sigma^2 = \begin{cases} sq \times \_NOBS\_/d, & \text{if SIGSQ=}sq \text{ is specified} \\ 2f(x^*)/d, & \text{if SIGSQ= is not specified} \end{cases}$$

where $f(x^*)$ is the value of the objective function at the optimal parameter estimates x^*.

The two groups of formulas distinguish between two situations:

- For least-squares estimates, the error variance can be estimated from the objective function value and is used in three of the six different forms of covariance matrices. If you have an independent estimate of the error variance, you can specify it with the SIGSQ= option.
- For maximum-likelihood or Bayesian estimates, the objective function should be the logarithm of the likelihood or of the posterior density when using the MAX statement.

For minimization, the inversion of the matrices in these formulas is done so that negative eigenvalues are considered zero, resulting always in a positive semidefinite covariance matrix.

In small samples, estimates of the covariance matrix based on asymptotic theory are often too small and should be used with caution.

If the final parameter estimates are subjected to $n_{act} > 0$ linear equality or active linear inequality constraints, the formulas of the covariance matrices are modified similar to Gallant (1987) and Cramer (1986, p. 38) and additionally generalized for applications with singular matrices. In the constrained case, the value of d used in the scalar factor σ^2 is defined by

$$d = \begin{cases} \max(1, \text{\_NOBS\_} - \text{\_DF\_} + n_{act}), & \text{for VARDEF=DF} \\ \text{\_NOBS\_}, & \text{for VARDEF=N} \end{cases}$$

where n_{act} is the number of active constraints and _NOBS_ is set as in the unconstrained case.

For minimization, the covariance matrix should be positive definite; for maximization it should be negative definite. There are several options available to check for a rank deficiency of the covariance matrix:

- The ASINGULAR=, MSINGULAR=, and VSINGULAR= options can be used to set three singularity criteria for the inversion of the matrix A needed to compute the covariance matrix, when A is either the Hessian or one of the crossproduct Jacobian matrices. The singularity criterion used for the inversion is

  $$|d_{j,j}| \leq \max(ASING, VSING \times |A_{j,j}|, MSING \times \max(|A_{1,1}|, \ldots, |A_{n,n}|))$$

  where $d_{j,j}$ is the diagonal pivot of the matrix A, and *ASING*, *VSING* and *MSING* are the specified values of the ASINGULAR=, VSINGULAR=, and MSINGULAR= options. The default values are

  - *ASING*: the square root of the smallest positive double precision value
  - *MSING*: 1E−12 if the SINGULAR= option is not specified and $\max(10 \times \epsilon, 1\text{E} - 4 \times SINGULAR)$ otherwise, where ϵ is the machine precision
  - *VSING*: 1E−8 if the SINGULAR= option is not specified and the value of SINGULAR otherwise

Note: In many cases, a normalized matrix $D^{-1}AD^{-1}$ is decomposed and the singularity criteria are modified correspondingly.

- If the matrix A is found singular in the first step, a generalized inverse is computed. Depending on the G4= option, a generalized inverse is computed that satisfies either all four or only two Moore-Penrose conditions. If the number of parameters n of the application is less than or equal to G4=i, a G4 inverse is computed; otherwise only a G2 inverse is computed. The G4 inverse is computed by (the computationally very expensive but numerically stable) eigenvalue decomposition; the G2 inverse is computed by Gauss transformation. The G4 inverse is computed using the eigenvalue decomposition $A = Z\Lambda Z^T$, where Z is the orthogonal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues, $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. If the PEIGVAL option is specified, the eigenvalues λ_i are displayed. The G4 inverse of A is set to

  $$A^- = Z\Lambda^- Z^T$$

  where the diagonal matrix $\Lambda^- = \mathrm{diag}(\lambda_1^-, ..., \lambda_n^-)$ is defined using the COVSING= option:

  $$\lambda_i^- = \begin{cases} 1/\lambda_i, & \text{if } |\lambda_i| > COVSING \\ 0, & \text{if } |\lambda_i| \leq COVSING \end{cases}$$

  If the COVSING= option is not specified, the nr smallest eigenvalues are set to zero, where nr is the number of rank deficiencies found in the first step.

For optimization techniques that do not use second-order derivatives, the covariance matrix is usually computed using finite-difference approximations of the derivatives. By specifying TECH=NONE, any of the covariance matrices can be computed using analytical derivatives. The covariance matrix specified by the COV= option can be displayed (using the PCOV option) and is written to the OUTEST= data set.

Input and Output Data Sets

DATA= Input Data Set

The DATA= data set is used only to specify an objective function f that is a combination of m other functions f_i. For each function f_i, $i = 1, \ldots, m$, listed in a MAX, MIN, or LSQ statement, each observation l, $l = 1, \ldots, nobs$, in the DATA= data set defines a specific function f_{il} that is evaluated by substituting the values of the variables of this observation into the program statements. If the MAX or MIN statement is used, the $m \times nobs$ specific functions f_{il} are added to a single objective function f. If the LSQ statement is used, the sum-of-squares f of the $m \times nobs$ specific functions f_{il} is minimized. The NOMISS option causes observations with missing values to be skipped.

INEST= Input Data Set

The INEST= (or INVAR=, or ESTDATA=) input data set can be used to specify the initial values of the parameters defined in a DECVAR statement as well as boundary constraints and the more general linear constraints which could be imposed on these parameters. This form of input is similar to the dense format input used in PROC LP.

The variables of the INEST= data set are

- a character variable _TYPE_ that indicates the type of the observation
- n numeric variables with the parameter names used in the DECVAR statement
- the BY variables that are used in a DATA= input data set
- a numeric variable _RHS_ specifying the right-hand-side constants (needed only if linear constraints are used)
- additional variables with names corresponding to constants used in the program statements

The content of the _TYPE_ variable defines the meaning of the observation of the INEST= data set. PROC NLP recognizes the following _TYPE_ values:

- PARMS, which specifies initial values for parameters. Additional variables can contain the values of constants that are referred to in program statements. The values of the constants in the PARMS observation initialize the constants in the program statements.
- UPPERBD | UB, which specifies upper bounds. A missing value indicates that no upper bound is specified for the parameter.
- LOWERBD | LB, which specifies lower bounds. A missing value indicates that no lower bound is specified for the parameter.
- LE | <= | <, which specifies linear constraint $\sum_j a_{ij}x_j \leq b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.
- GE | >= | >, which specifies linear constraint $\sum_j a_{ij}x_j \geq b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.
- EQ | =, which specifies linear constraint $\sum_j a_{ij}x_j = b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.

The constraints specified in an INEST= data set are added to the constraints specified in the BOUNDS and LINCON statements. You can use an OUTEST= data set as an INEST= data set in a subsequent run of PROC NLP. However, be aware that the OUTEST= data set also contains the boundary and general linear constraints specified in the previous run of PROC NLP. When you are using this OUTEST= data set without changes as an INEST= data set, PROC NLP adds the constraints from the data set to the constraints specified by a BOUNDS and LINCON statement. Although PROC NLP automatically eliminates multiple identical constraints you should avoid specifying the same constraint twice.

INQUAD= Input Data Set

Two types of INQUAD= data sets can be used to specify the objective function of a quadratic programming problem for TECH=QUADAS or TECH=LICOMP,

$$f(x) = \frac{1}{2}x^T Gx + g^T x + c, \quad \text{with} \quad G^T = G$$

The *dense* INQUAD= data set must contain all numerical values of the symmetric matrix G, the vector g, and the scalar c. Using the *sparse* INQUAD= data set allows you to specify only the nonzero positions in matrix G and vector g. Those locations that are not set by the *sparse* INQUAD= data set are assumed to be zero.

Dense INQUAD= Data Set

A dense INQUAD= data set must contain two character variables, _TYPE_ and _NAME_, and at least n numeric variables whose names are the parameter names. The _TYPE_ variable takes the following values:

- QUAD lists the n values of the row of the G matrix that is defined by the parameter name used in the _NAME_ variable.
- LINEAR lists the n values of the g vector.
- CONST sets the value of the scalar c and cannot contain different numerical values; however, it could contain up to $n - 1$ missing values.
- PARMS specifies initial values for parameters.
- UPPERBD | UB specifies upper bounds. A missing value indicates that no upper bound is specified.
- LOWERBD | LB specifies lower bounds. A missing value indicates that no lower bound is specified.
- LE | <= | < specifies linear constraint $\sum_j a_{ij}x_j \leq b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.
- GE | >= | > specifies linear constraint $\sum_j a_{ij}x_j \geq b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.
- EQ | = specifies linear constraint $\sum_j a_{ij}x_j = b_i$. The n parameter values contain the coefficients a_{ij}, and the _RHS_ variable contains the right-hand side b_i. Missing values indicate zeros.

Constraints specified in a dense INQUAD= data set are added to the constraints specified in BOUNDS and LINCON statements.

Sparse INQUAD= Data Set

A sparse INQUAD= data set must contain three character variables _TYPE_, _ROW_, and _COL_, and one numeric variable _VALUE_. The _TYPE_ variable can assume two values:

- QUAD specifies that the _ROW_ and _COL_ variables define the row and column locations of the values in the G matrix.
- LINEAR specifies that the _ROW_ variable defines the row locations of the values in the g vector. The _COL_ variable is not used.

Using both the MODEL= option and the INCLUDE statement with the same model file will include the file twice (erroneous in most cases).

OUT= Output Data Set

The OUT= data set contains those variables of a DATA= input data set that are referred to in the program statements and additional variables computed by the program statements for the objective function. Specifying the NOMISS option enables you to skip observations with missing values in variables used in the program statements. The OUT= data set can also contain first- and second-order derivatives of these variables if the OUTDER= option is specified. The variables and derivatives are the final parameter estimates x^* or (for TECH=NONE) the initial value x^0.

The variables of the OUT= data set are

- the BY variables and all other variables that are used in a DATA= input data set and referred to in the program code
- a variable _OBS_ containing the number of observations read from a DATA= input data set, where the counting is restarted with the start of each BY group. If there is no DATA= input data set, then _OBS_=1.
- a character variable _TYPE_ naming the type of the observation
- the parameter variables listed in the DECVAR statement
- the function variables listed in the MIN, MAX, or LSQ statement
- all other variables computed in the program statements
- the character variable _WRT_ (if OUTDER=1) containing the *with respect to* variable for which the first-order derivatives are written in the function variables
- the two character variables _WRT1_ and _WRT2_ (if OUTDER=2) containing the two *with respect to* variables for which the first- and second-order derivatives are written in the function variables

OUTEST= Output Data Set

The OUTEST= or OUTVAR= output data set saves the optimization solution of PROC NLP. You can use the OUTEST= or OUTVAR= data set as follows:

- to save the values of the objective function on grid points to examine, for example, surface plots using PROC G3D (use the OUTGRID option)
- to avoid any costly computation of analytical (first- or second-order) derivatives during optimization when they are needed only upon termination. In this case a two-step approach is recommended:

1. In a first execution, the optimization is done; that is, optimal parameter estimates are computed, and the results are saved in an OUTEST= data set.
2. In a subsequent execution, the optimal parameter estimates in the previous OUTEST= data set are read in an INEST= data set and used with TECH=NONE to compute further results, such as analytical second-order derivatives or some kind of covariance matrix.

- to restart the procedure using parameter estimates as initial values
- to split a time-consuming optimization problem into a series of smaller problems using intermediate results as initial values in subsequent runs. (Refer to the MAXTIME=, MAXIT=, and MAXFUNC= options to trigger stopping.)
- to write the value of the objective function, the parameter estimates, the time in seconds starting at the beginning of the optimization process and (if available) the gradient to the OUTEST= data set during the iterations. After the PROC NLP run is completed, the convergence progress can be inspected by graphically displaying the iterative information. (Refer to the OUTITER option.)

The variables of the OUTEST= data set are

- the BY variables that are used in a DATA= input data set
- a character variable _TECH_ naming the optimization technique used
- a character variable _TYPE_ specifying the type of the observation
- a character variable _NAME_ naming the observation. For a linear constraint, the _NAME_ variable indicates whether the constraint is active at the solution. For the initial observations, the _NAME_ variable indicates if the number in the _RHS_ variable corresponds to the number of positive, negative, or zero eigenvalues.
- n numeric variables with the parameter names used in the DECVAR statement. These variables contain a point x of the parameter space, lower or upper bound constraints, or the coefficients of linear constraints.
- a numeric variable _RHS_ (right-hand side) that is used for the right-hand-side value b_i of a linear constraint or for the value $f = f(x)$ of the objective function at a point x of the parameter space
- a numeric variable _ITER_ that is zero for initial values, equal to the iteration number for the OUTITER output, and missing for the result output

The _TYPE_ variable identifies how to interpret the observation. If _TYPE_ is

- PARMS then parameter-named variables contain the coordinates of the resulting point x^*. The _RHS_ variable contains $f(x^*)$.
- INITIAL then parameter-named variables contain the feasible starting point $x^{(0)}$. The _RHS_ variable contains $f(x^{(0)})$.

- GRIDPNT then (if the OUTGRID option is specified) parameter-named variables contain the coordinates of any point $x^{(k)}$ used in the grid search. The _RHS_ variable contains $f(x^{(k)})$.
- GRAD then parameter-named variables contain the gradient at the initial or final estimates.
- STDERR then parameter-named variables contain the approximate standard errors (square roots of the diagonal elements of the covariance matrix) if the COV= option is specified.
- _NOBS_ then (if the COV= option is specified) all parameter variables contain the value of _NOBS_ used in computing the σ^2 value in the formula of the covariance matrix.
- UPPERBD | UB then (if there are boundary constraints) the parameter variables contain the upper bounds.
- LOWERBD | LB then (if there are boundary constraints) the parameter variables contain the lower bounds.
- NACTBC then all parameter variables contain the number n_{abc} of active boundary constraints at the solution x^*.
- ACTBC then (if there are active boundary constraints) the observation indicate which parameters are actively constrained, as follows:

  NAME=GE the active lower bounds

  NAME=LE the active upper bounds

  NAME=EQ the active equality constraints

- NACTLC then all parameter variables contain the number n_{alc} of active linear constraints that are recognized as linearly independent.
- NLDACTLC then all parameter variables contain the number of active linear constraints that are recognized as linearly dependent.
- LE then (if there are linear constraints) the observation contains the ith linear constraint $\sum_j a_{ij}x_j \leq b_i$. The parameter variables contain the coefficients a_{ij}, $j = 1, \ldots, n$, and the _RHS_ variable contains b_i. If the constraint i is active at the solution x^*, then _NAME_=ACTLC or _NAME_=LDACTLC.
- GE then (if there are linear constraints) the observation contains the ith linear constraint $\sum_j a_{ij}x_j \geq b_i$. The parameter variables contain the coefficients a_{ij}, $j = 1, \ldots, n$, and the _RHS_ variable contains b_i. If the constraint i is active at the solution x^*, then _NAME_=ACTLC or _NAME_=LDACTLC.
- EQ then (if there are linear constraints) the observation contains the ith linear constraint $\sum_j a_{ij}x_j = b_i$. The parameter variables contain the coefficients a_{ij}, $j = 1, \ldots, n$, the _RHS_ variable contains b_i, and _NAME_=ACTLC or _NAME_=LDACTLC.
- LAGRANGE then (if at least one of the linear constraints is an equality constraint or an active inequality constraint) the observation contains the vector of Lagrange multipliers. The Lagrange multipliers of active boundary constraints are listed first followed by those of active linear constraints and those of active

nonlinear constraints. Lagrange multipliers are available only for the set of linearly independent active constraints.

- PROJGRAD then (if there are linear constraints) the observation contains the $n - n_{act}$ values of the projected gradient $g_Z = Z^T g$ in the variables corresponding to the first $n - n_{act}$ parameters.
- JACOBIAN then (if the PJACOBI or OUTJAC option is specified) the m observations contain the m rows of the $m \times n$ Jacobian matrix. The _RHS_ variable contains the row number l, $l = 1, \ldots, m$.
- HESSIAN then the first n observations contain the n rows of the (symmetric) Hessian matrix. The _RHS_ variable contains the row number j, $j = 1, \ldots, n$, and the _NAME_ variable contains the corresponding parameter name.
- PROJHESS then the first $n - n_{act}$ observations contain the $n - n_{act}$ rows of the projected Hessian matrix $Z^T G Z$. The _RHS_ variable contains the row number j, $j = 1, \ldots, n - n_{act}$, and the _NAME_ variable is blank.
- CRPJAC then the first n observations contain the n rows of the (symmetric) crossproduct Jacobian matrix at the solution. The _RHS_ variable contains the row number j, $j = 1, \ldots, n$, and the _NAME_ variable contains the corresponding parameter name.
- PROJCRPJ then the first $n - n_{act}$ observations contain the $n - n_{act}$ rows of the projected crossproduct Jacobian matrix $Z^T (J^T J) Z$. The _RHS_ variable contains the row number j, $j = 1, \ldots, n - n_{act}$, and the _NAME_ variable is blank.
- COV1, COV2, COV3, COV4, COV5, or COV6 then (depending on the COV= option) the first n observations contain the n rows of the (symmetric) covariance matrix of the parameter estimates. The _RHS_ variable contains the row number j, $j = 1, \ldots, n$, and the _NAME_ variable contains the corresponding parameter name.
- DETERMIN contains the determinant $det = a \times 10^b$ of the matrix specified by the value of the _NAME_ variable where a is the value of the first variable in the DECVAR statement and b is in _RHS_.
- NEIGPOS, NEIGNEG, or NEIGZER then the _RHS_ variable contains the number of positive, negative, or zero eigenvalues of the matrix specified by the value of the _NAME_ variable.
- COVRANK then the _RHS_ variable contains the rank of the covariance matrix.
- SIGSQ then the _RHS_ variable contains the scalar factor of the covariance matrix.
- _TIME_ then (if the OUTITER option is specified) the _RHS_ variable contains the number of seconds passed since the start of the optimization.
- TERMINAT then if optimization terminated at a point satisfying one of the termination criteria, an abbreviation of the corresponding criteria is given to the _NAME_ variable. Otherwise _NAME_=PROBLEMS.

If for some reason the procedure does not terminate successfully (for example, no feasible initial values can be computed or the function value or derivatives at the starting point cannot be computed), the OUTEST= data set may contain only part of the observations (usually only the PARMS and GRAD observation).

Note: Generally you can use an OUTEST= data set as an INEST= data set in a further run of PROC NLP. However, be aware that the OUTEST= data set also contains the boundary and general linear constraints specified in the previous run of PROC NLP. When you are using this OUTEST= data set without changes as an INEST= data set, PROC NLP adds the constraints from the data set to the constraints specified by a BOUNDS or LINCON statement. Although PROC NLP automatically eliminates multiple identical constraints you should avoid specifying the same constraint twice.

Output of Profiles

The following observations are written to the OUTEST= data set only when the PROFILE statement or CLPARM option is specified.

TYPE	_NAME_	_RHS_	Meaning of Observation
PLC_LOW	parname	y value	coordinates of lower CL for α
PLC_UPP	parname	y value	coordinates of upper CL for α
WALD_CL	LOWER	y value	lower Wald CL for α in _ALPHA_
WALD_CL	UPPER	y value	upper Wald CL for α in _ALPHA_
PL_CL	LOWER	y value	lower PL CL for α in _ALPHA_
PL_CL	UPPER	y value	upper PL CL for α in _ALPHA_
PROFILE	L(THETA)	missing	y value corresponding to x in following _NAME_=THETA
PROFILE	THETA	missing	x value corresponding to y in previous _NAME_=L(THETA)

Assume that the PROFILE statement specifies n_p parameters and n_α confidence levels. For CLPARM, $n_p = n$ and $n_\alpha = 4$.

- _TYPE_=PLC_LOW and _TYPE_=PLC_UPP:
  If the CLPARM= option or the PROFILE statement with the OUTTABLE option is specified, then the complete set θ of parameter estimates (rather than only the confidence limit $x = \theta_j$) is written to the OUTEST= data set for each side of the confidence interval. This output may be helpful for further analyses on how small changes in $x = \theta_j$ affect the changes in the other $\theta_i, i \neq j$. The _ALPHA_ variable contains the corresponding value of α. There should be no more than $2n_\alpha n_p$ observations. If the confidence limit cannot be computed, the corresponding observation is not available.
- _TYPE_=WALD_CL:
  If CLPARM=WALD, CLPARM=BOTH, or the PROFILE statement with α values is specified, then the Wald confidence limits are written to the OUTEST= data set for each of the default or specified values of α. The

ALPHA variable contains the corresponding value of α. There should be $2n_\alpha$ observations.

- _TYPE_=PL_CL:
  If CLPARM=PL, CLPARM=BOTH, or the PROFILE statement with α values is specified, then the PL confidence limits are written to the OUTEST= data set for each of the default or specified values of α. The _ALPHA_ variable contains the corresponding values of α. There should be $2n_\alpha$ observations; some observations may have missing values.
- _TYPE_=PROFILE:
  If CLPARM=PL, CLPARM=BOTH, or the CLPARM= statement with or without α values is specified, then a set of (x, y) point coordinates in two adjacent observations with _NAME_=L(THETA) (y value) and _NAME_=THETA (x value) is written to the OUTEST= data set. The _RHS_ and _ALPHA_ variables are not used (are set to missing). The number of observations depends on the difficulty of the optimization problems.

OUTMODEL= Output Data Set

The program statements for objective functions, nonlinear constraints, and derivatives can be saved into an OUTMODEL= output data set. This data set can be used in an INCLUDE program statement or as a MODEL= input data set in subsequent calls of PROC NLP. The OUTMODEL= option is similar to the option used in PROC MODEL in SAS/ETS software.

Storing Programs in Model Files

Models can be saved to and recalled from SAS catalog files. SAS catalogs are special files which can store many kinds of data structures as separate units in one SAS file. Each separate unit is called an entry, and each entry has an entry type that identifies its structure to the SAS system.

In general, to save a model, use the OUTMODEL=*name* option in the PROC NLP statement, where *name* is specified as *libref.catalog.entry*, *libref.entry*, or *entry*. The *libref*, *catalog*, and *entry* names must be valid SAS names no more than 8 characters long. The *catalog* name is restricted to 7 characters on the CMS operating system. If not given, the *catalog* name defaults to MODELS, and the *libref* defaults to WORK. The entry type is always MODEL. Thus, OUTMODEL=X writes the model to the file WORK.MODELS.X.MODEL.

The MODEL= option is used to read in a model. A list of model files can be specified in the MODEL= option, and a range of names with numeric suffixes can be given, as in MODEL=(MODEL1-MODEL10). When more than one model file is given, the list must be placed in parentheses, as in MODEL=(A B C). If more than one model file is specified, the files are combined in the order listed in the MODEL= option.

When the MODEL= option is specified in the PROC NLP statement and model definition statements are also given later in the PROC NLP step, the model files are read in first, in the order listed, and the model program specified in the PROC NLP step is appended after the model program read from the MODEL= files.

The INCLUDE statement can be used to append model code to the current model code. The contents of the model files are inserted into the current model at the position where the INCLUDE statement appears.

Note that the following statements are not part of the program code that is written to an OUTMODEL= data set: MIN, MAX, LSQ, MINQUAD, MAXQUAD, DECVAR, BOUNDS, BY, CRPJAC, GRADIENT, HESSIAN, JACNLC, JACOBIAN, LABEL, LINCON, MATRIX, and NLINCON.

Displayed Output

Procedure Initialization

After the procedure has processed the problem, it displays summary information about the problem and the options that you have selected. It may also display a list of linearly dependent constraints and other information about the constraints and parameters.

Optimization Start

At the start of optimization the procedure displays

- the number of constraints that are active at the starting point, or more precisely, the number of constraints that are currently members of the working set. If this number is followed by a plus sign, there are more active constraints, of which at least one is temporarily released from the working set due to negative Lagrange multipliers.
- the value of the objective function at the starting point
- if the (projected) gradient is available, the value of the largest absolute (projected) gradient element
- for the TRUREG and LEVMAR subroutines, the initial radius of the trust region around the starting point

Iteration History

In general, the iteration history consists of one line of output containing the most important information for each iteration. The iteration-extensive Nelder-Mead simplex method, however, displays only one line for several internal iterations. This technique skips the output for some iterations because

- some of the termination tests (size and standard deviation) are rather time-consuming compared to the simplex operations and are done once every five simplex operations
- the resulting history output is smaller

The _LIST_ variable (refer to the section "Program Statements" on page 338) also enables you to display the parameter estimates $x^{(k)}$ and the gradient $g^{(k)}$ in all or some selected iterations k.

The iteration history always includes the following (the words in parentheses indicate the column header output):

- the iteration number (iter)
- the number of iteration restarts (nrest)
- the number of function calls (nfun)
- the number of active constraints (act)
- the value of the optimization criterion (optcrit)
- the difference between adjacent function values (difcrit)
- the maximum of the absolute (projected) gradient components (maxgrad)

An apostrophe trailing the number of active constraints indicates that at least one of the active constraints was released from the active set due to a significant Lagrange multiplier.

The optimization history is displayed by default because it is important to check for possible convergence problems.

Optimization Termination

The output of the optimization history ends with a short output of information concerning the optimization result:

- the number of constraints that are active at the final point, or more precisely, the number of constraints that are currently members of the working set. When this number is followed by a plus sign, it indicates that there are more active constraints of which at least one is temporarily released from the working set due to negative Lagrange multipliers.
- the value of the objective function at the final point
- if the (projected) gradient is available, the value of the largest absolute (projected) gradient element
- other information that is specific for the optimization technique

The NOPRINT option suppresses all output to the list file and only errors, warnings, and notes are displayed to the log file. The PALL option sets a large group of some of the commonly used specific displaying options, the PSHORT option suppresses some, and the PSUMMARY option suppresses almost all of the default output. The following table summarizes the correspondence between the general and the specific print options.

Output Options	PALL	default	PSHORT	PSUMMARY	Output
	y	y	y	y	summary of optimization
	y	y	y	n	parameter estimates
	y	y	y	n	gradient of objective func
PHISTORY	y	y	y	n	iteration history
PINIT	y	y	n	n	setting of initial values
	y	y	n	n	listing of constraints

PGRID	y	n	n	n	results of grid search
PNLCJAC	y	n	n	n	Jacobian nonlin. constr.
PFUNCTION	y	n	n	n	values of functions
PEIGVAL	y	n	n	n	eigenvalue distribution
PCRPJAC	y	n	n	n	crossproduct Jacobian
PHESSIAN	y	n	n	n	Hessian matrix
PSTDERR	y	n	n	n	approx. standard errors
PCOV	y	n	n	n	covariance matrices
PJACOBI	n	n	n	n	Jacobian
LIST	n	n	n	n	model program, variables
LISTCODE	n	n	n	n	compiled model program

Convergence Status

Upon termination, the NLP procedure creates an ODS table called "ConvergenceStatus." You can use this name to reference the table when using the Output Delivery System (ODS) to select tables and create output data sets. Within the "ConvergenceStatus" table there are two variables, "Status" and "Reason," which contain the status of the optimization run. For the "Status" variable, a value of zero indicates that one of the convergence criteria is satisfied; a nonzero value indicates otherwise. In all cases, an explicit interpretation of the status code is displayed as a string stored in the "Reason" variable. For more information about ODS, see *SAS Output Delivery System: User's Guide*.

Missing Values

Missing Values in Program Statements

There is one very important reason for using missing values in program statements specifying the values of the objective functions and derivatives: it may not be possible to evaluate the program statements for a particular point x. For example, the extrapolation formula of one of the line-search algorithms may generate large x values for which the EXP function cannot be evaluated without floating point overflow. The compiler of the program statements may check for such situations automatically, but it would be safer if you check the feasibility of your program statements. In some cases, the specification of boundary or linear constraints for parameters can avoid such situations. In many other cases, you can indicate that x is a *bad* point simply by returning a missing value for the objective function. In such cases the optimization algorithms in PROC NLP shorten the step length α or reduce the trust region radius so that the next point will be closer to the point that was already successfully evaluated at the last iteration. Note that the starting point $x^{(0)}$ must be a point for which the program statements can be evaluated.

Missing Values in Input Data Sets

Observations with missing values in the DATA= data set for variables used in the objective function can lead to a missing value of the objective function implying that the corresponding BY group of data is not processed. The NOMISS option can be

used to skip those observations of the DATA= data set for which relevant variables have missing values. Relevant variables are those that are referred to in program statements.

There can be different reasons to include observations with missing values in the INEST= data set. The value of the _RHS_ variable is not used in some cases and can be missing. Missing values for the variables corresponding to parameters in the _TYPE_ variable are as follows:

- PARMS observations cause those parameters to have initial values assigned by the DECVAR statement or by the RANDOM= or INITIAL= option.
- UPPERBD or LOWERBD observations cause those parameters to be unconstrained by upper or lower bounds.
- LE, GE, or EQ observations cause those parameters to have zero values in the constraint.

In general, missing values are treated as zeros.

Computational Resources

Since nonlinear optimization is an iterative process that depends on many factors, it is difficult to estimate how much computer time is necessary to compute an optimal solution satisfying one of the termination criteria. The MAXTIME=, MAXITER=, and MAXFUNC= options can be used to restrict the amount of CPU time, the number of iterations, and the number of function calls in a single run of PROC NLP.

In each iteration k, the NRRIDG and LEVMAR techniques use symmetric Householder transformations to decompose the $n \times n$ Hessian (crossproduct Jacobian) matrix G,

$$G = V^T T V\ , \quad V \text{ orthogonal}, \quad T \text{ tridiagonal}$$

to compute the (Newton) search direction s:

$$s^{(k)} = -G^{(k-1)} g^{(k)}\ , \quad k = 1, 2, 3, \ldots$$

The QUADAS, TRUREG, NEWRAP, and HYQUAN techniques use the Cholesky decomposition to solve the same linear system while computing the search direction. The QUANEW, DBLDOG, CONGRA, and NMSIMP techniques do not need to invert or decompose a Hessian or crossproduct Jacobian matrix and thus require fewer computational resources then the first group of techniques.

The larger the problem, the more time is spent computing function values and derivatives. Therefore, many researchers compare optimization techniques by counting and comparing the respective numbers of function, gradient, and Hessian (crossproduct Jacobian) evaluations. You can save computer time and memory by specifying derivatives (using the GRADIENT, JACOBIAN, CRPJAC, or HESSIAN statement) since

you will typically produce a more efficient representation than the internal derivative compiler.

Finite-difference approximations of the derivatives are expensive since they require additional function or gradient calls.

- Forward-difference formulas:
  - First-order derivatives: n additional function calls are needed.
  - Second-order derivatives based on function calls only: for a dense Hessian, $n(n+3)/2$ additional function calls are needed.
  - Second-order derivatives based on gradient calls: n additional gradient calls are needed.
- Central-difference formulas:
  - First-order derivatives: $2n$ additional function calls are needed.
  - Second-order derivatives based on function calls only: for a dense Hessian, $2n(n+1)$ additional function calls are needed.
  - Second-order derivatives based on gradient: $2n$ additional gradient calls are needed.

Many applications need considerably more time for computing second-order derivatives (Hessian matrix) than for first-order derivatives (gradient). In such cases, a (dual) quasi-Newton or conjugate gradient technique is recommended, which does not require second-order derivatives.

The following table shows for each optimization technique which derivatives are needed (FOD: first-order derivatives; SOD: second-order derivatives), what kinds of constraints are supported (BC: boundary constraints; LIC: linear constraints), and the minimal memory (number of double floating point numbers) required. For various reasons, there are additionally about $7n + m$ double floating point numbers needed.

Quadratic Programming	FOD	SOD	BC	LIC	Memory
LICOMP	-	-	x	x	$18n + 3nn$
QUADAS	-	-	x	x	$1n + 2nn/2$
General Optimization	FOD	SOD	BC	LIC	Memory
TRUREG	x	x	x	x	$4n + 2nn/2$
NEWRAP	x	x	x	x	$2n + 2nn/2$
NRRIDG	x	x	x	x	$6n + nn/2$
QUANEW	x	-	x	x	$1n + nn/2$
DBLDOG	x	-	x	x	$7n + nn/2$
CONGRA	x	-	x	x	$3n$
NMSIMP	-	-	x	x	$4n + nn$
Least-Squares	FOD	SOD	BC	LIC	Memory
LEVMAR	x	-	x	x	$6n + nn/2$
HYQUAN	x	-	x	x	$2n + nn/2 + 3m$

Notes:

- Here, n denotes the number of parameters, nn the squared number of parameters, and $nn/2 := n(n+1)/2$.
- The value of m is the product of the number of functions specified in the MIN, MAX, or LSQ statement and the maximum number of observations in each BY group of a DATA= input data set. The following table also contains the number v of variables in the DATA= data set that are used in the program statements.
- For a diagonal Hessian matrix, the $nn/2$ term in QUADAS, TRUREG, NEWRAP, and NRRIDG is replaced by n.
- If the TRUREG, NRRIDG, or NEWRAP method is used to minimize a least-squares problem, the second derivatives are replaced by the crossproduct Jacobian matrix.
- The memory needed by the TECH=NONE specification depends on the output specifications (typically, it needs $3n+nn/2$ double floating point numbers and an additional mn if the Jacobian matrix is required).

The total amount of memory needed to run an optimization technique consists of the technique-specific memory listed in the preceding table, plus additional blocks of memory as shown in the following table.

	double	int	long	8byte
Basic Requirement	$7n+m$	n	$3n$	$n+m$
DATA= data set	v	-	-	v
JACOBIAN statement	$m(n+2)$	-	-	-
CRPJAC statement	$nn/2$	-	-	-
HESSIAN statement	$nn/2$	-	-	-
COV= option	$(2*)nn/2+n$	-	-	-
Scaling vector	n	-	-	-
BOUNDS statement	$2n$	n	-	-
Bounds in INEST=	$2n$	-	-	-
LINCON and TRUREG	$c(n+1)+nn+nn/2+4n$	$3c$	-	-
LINCON and other	$c(n+1)+nn+2nn/2+4n$	$3c$	-	-

Notes:

- For TECH=LICOMP, the total amount of memory needed for the linear or boundary constrained case is $18(n+c)+3(n+c)(n+c)$, where c is the number of constraints.
- The amount of memory needed to specify derivatives with a GRADIENT, JACOBIAN, CRPJAC, or HESSIAN statement (shown in this table) is small compared to that needed for using the internal function compiler to compute the derivatives. This is especially so for second-order derivatives.
- If the CONGRA technique is used, specifying the GRADCHECK=DETAIL option requires an additional $nn/2$ double floating point numbers to store the finite-difference Hessian matrix.

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the NLP procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

Examples: NLP Procedure

Example 4.1. Using the DATA= Option

This example illustrates the use of the DATA= option. The Bard function (refer to Moré, Garbow, and Hillstrom (1981)) is a least-squares problem with $n = 3$ parameters and $m = 15$ functions f_k:

$$f(x) = \frac{1}{2} \sum_{k=1}^{15} f_k^2(x), \quad x = (x_1, x_2, x_3)$$

where

$$f_k(x) = y_k - \left(x_1 + \frac{u_k}{v_k x_2 + w_k x_3} \right)$$

with $u_k = k$, $v_k = 16 - k$, $w_k = \min(u_k, v_k)$, and

$$y = (.14, .18, .22, .25, .29, .32, .35, .39, .37, .58, .73, .96, 1.34, 2.10, 4.39)$$

The minimum function value $f(x^*) = 4.107\text{E}{-3}$ is at the point $(0.08, 1.13, 2.34)$. The starting point $x^0 = (1, 1, 1)$ is used.

The following is the naive way of specifying the objective function.

```
proc nlp tech=levmar;
   lsq y1-y15;
   parms x1-x3 = 1;
   tmp1 = 15 * x2 + min(1,15) * x3;
   y1 = 0.14 - (x1 + 1 / tmp1);
   tmp1 = 14 * x2 + min(2,14) * x3;
   y2 = 0.18 - (x1 + 2 / tmp1);
   tmp1 = 13 * x2 + min(3,13) * x3;
   y3 = 0.22 - (x1 + 3 / tmp1);
   tmp1 = 12 * x2 + min(4,12) * x3;
   y4 = 0.25 - (x1 + 4 / tmp1);
   tmp1 = 11 * x2 + min(5,11) * x3;
   y5 = 0.29 - (x1 + 5 / tmp1);
   tmp1 = 10 * x2 + min(6,10) * x3;
   y6 = 0.32 - (x1 + 6 / tmp1);
   tmp1 = 9 * x2 + min(7,9) * x3;
   y7 = 0.35 - (x1 + 7 / tmp1);
   tmp1 = 8 * x2 + min(8,8) * x3;
   y8 = 0.39 - (x1 + 8 / tmp1);
   tmp1 = 7 * x2 + min(9,7) * x3;
   y9 = 0.37 - (x1 + 9 / tmp1);
   tmp1 = 6 * x2 + min(10,6) * x3;
   y10 = 0.58 - (x1 + 10 / tmp1);
   tmp1 = 5 * x2 + min(11,5) * x3;
   y11 = 0.73 - (x1 + 11 / tmp1);
   tmp1 = 4 * x2 + min(12,4) * x3;
   y12 = 0.96 - (x1 + 12 / tmp1);
   tmp1 = 3 * x2 + min(13,3) * x3;
   y13 = 1.34 - (x1 + 13 / tmp1);
   tmp1 = 2 * x2 + min(14,2) * x3;
   y14 = 2.10 - (x1 + 14 / tmp1);
   tmp1 = 1 * x2 + min(15,1) * x3;
   y15 = 4.39 - (x1 + 15 / tmp1);
run;
```

A more economical way to program this problem uses the DATA= option to input the 15 terms in $f(x)$.

```
data bard;
   input r @@;
      w1 = 16. - _n_;
      w2 = min(_n_ , 16. - _n_);
      datalines;
.14  .18  .22  .25  .29  .32  .35  .39
.37  .58  .73  .96 1.34 2.10 4.39
;

proc nlp data=bard tech=levmar;
   lsq y;
   parms x1-x3 = 1.;
   y = r - (x1 + _obs_ / (w1 * x2 + w2 * x3));
run;
```

Another way you can specify the objective function uses the ARRAY statement and an explicit do loop, as in the following code.

```
proc nlp tech=levmar;
   array r[15] .14  .18  .22  .25  .29  .32  .35  .39  .37  .58
               .73  .96 1.34 2.10 4.39 ;
   array y[15] y1-y15;
   lsq y1-y15;
   parms x1-x3 = 1.;
   do i = 1 to 15;
      w1 = 16. - i;
      w2 = min(i , w1);
      w3 = w1 * x2 + w2 * x3;
      y[i] = r[i] - (x1 + i / w3);
   end;
run;
```

Example 4.2. Using the INQUAD= Option

This example illustrates the INQUAD= option for specifying a quadratic programming problem:

$$\min \quad f(x) = \frac{1}{2}x^T Gx + g^T x + c, \quad \text{with} \quad G^T = G$$

Suppose that $c = -100$, $G = \text{diag}(.4, 4)$ and $2 \leq x_1 \leq 50$, $-50 \leq x_2 \leq 50$, and $10 \leq 10x_1 - x_2$.

You specify the constant c and the Hessian G in the data set QUAD1. Notice that the _TYPE_ variable contains the keywords that identify how the procedure should interpret the observations.

```
data quad1;
   input _type_ $ _name_ $ x1 x2;
   datalines;
const    .   -100  -100
```

```
quad   x1    0.4     0
quad   x2      0     4
;
```

You specify the QUAD1 data set with the INQUAD= option. Notice that the names of the variables in the QUAD1 data set and the _NAME_ variable match the names of the parameters in the PARMS statement.

```
proc nlp inquad=quad1 all;
   min ;
   parms x1 x2 = -1;
   bounds  2 <= x1 <= 50,
         -50 <= x2 <= 50;
   lincon 10 <= 10 * x1 - x2;
run;
```

Alternatively, you can use a sparse format for specifying the G matrix, eliminating the zeros. You use the special variables _ROW_, _COL_, and _VALUE_ to give the nonzero row and column names and value.

```
data quad2;
   input _type_ $ _row_ $ _col_ $ _value_;
   datalines;
const   .       .     -100
quad   x1     x1      0.4
quad   x2     x2        4
;
```

You can also include the constraints in the QUAD data set. Notice how the _TYPE_ variable contains keywords that identify how the procedure is to interpret the values in each observation.

```
data quad3;
   input _type_ $ _name_ $ x1    x2    _rhs_;
   datalines;
const       .      -100  -100    .
quad       x1     0.02     0    .
quad       x2     0.00     2    .
parms      .        -1    -1    .
lowerbd    .         2   -50    .
upperbd    .        50    50    .
ge         .        10    -1   10
;

proc nlp inquad=quad3;
   min ;
   parms x1 x2;
run;
```

Example 4.3. Using the INEST=Option

This example illustrates the use of the INEST= option for specifying a starting point and linear constraints. You name a data set with the INEST= option. The format of this data set is similar to the format of the QUAD data set described in the previous example.

Consider the Hock and Schittkowski (1981) Problem # 24:

$$\min f(x) = \frac{((x_1 - 3)^2 - 9)x_2^3}{27\sqrt{3}}$$

subject to:

$$\begin{aligned} 0 &\leq x_1, x_2 \\ 0 &\leq .57735x_1 - x_2 \\ 0 &\leq x_1 + 1.732x_2 \\ 6 &\geq x_1 + 1.732x_2 \end{aligned}$$

with minimum function value $f(x^*) = -1$ at $x^* = (3, \sqrt{3})$. The feasible starting point is $x^0 = (1, .5)$.

You can specify this model in PROC NLP as follows:

```
proc nlp tech=trureg outest=res;
   min y;
   parms x1 = 1,
         x2 = .5;
   bounds  0 <= x1-x2;
   lincon  .57735 * x1 -          x2 >=  0,
                   x1 + 1.732 * x2 >=  0,
                  -x1 - 1.732 * x2 >= -6;
   y = (((x1 - 3)**2 - 9.) * x2**3) / (27 * sqrt(3));
run;
```

Note that none of the data for this model are in a data set. Alternatively, you can save the starting point $(1, .5)$ and the linear constraints in a data set. Notice that the _TYPE_ variable contains keywords that identify how the procedure is to interpret each of the observations and that the parameters in the problems X1 and X2 are variables in the data set. The observation with _TYPE_=LOWERBD gives the lower bounds on the parameters. The observation with _TYPE_=GE gives the coefficients for the first constraint. Similarly, the subsequent observations contain specifications for the other constraints. Also notice that the special variable _RHS_ contains the right-hand-side values.

```
data betts1(type=est);
   input _type_ $ x1 x2 _rhs_;
   datalines;
```

```
parms     1         .5        .
lowerbd   0         0         .
ge        .57735    -1        .
ge        1         1.732     .
le        1         1.732     6
;
```

Now you can solve this problem with the following code. Notice that you specify the objective function and the parameters.

```
proc nlp inest=betts1 tech=trureg;
   min y;
   parms x1 x2;
   y = (((x1 - 3)**2 - 9) * x2**3) / (27 * sqrt(3));
run;
```

You can even include any constants used in the program statements in the INEST= data set. In the following code the variables A, B, C, and D contain some of the constants used in calculating the objective function Y.

```
data betts2(type=est);
   input _type_ $ x1    x2    _rhs_    a    b    c    d;
   datalines;
parms     1         .5        .      3    9   27    3
lowerbd   0         0         .      .    .    .    .
ge        .57735    -1        0      .    .    .    .
ge        1         1.732     0      .    .    .    .
le        1         1.732     6      .    .    .    .
;
```

Notice that in the program statement for calculating Y, the constants are replaced by the A, B, C, and D variables.

```
proc nlp inest=betts2 tech=trureg;
   min y;
   parms x1 x2;
   y = (((x1 - a)**2 - b) * x2**3) / (c * sqrt(d));
run;
```

Example 4.4. Restarting an Optimization

This example shows how you can restart an optimization problem using the OUTEST=, INEST=, OUTMODEL=, and MODEL= options and how to save output into an OUT= data set. The least-squares solution of the Rosenbrock function using the trust region method is used.

The following code solves the problem and saves the model in the MODEL data set and the solution in the EST and OUT1 data sets.

```
proc nlp tech=trureg outmodel=model outest=est out=out1;
   lsq y1 y2;
   parms x1 = -1.2 ,
         x2 =  1.;
   y1 = 10. * (x2 - x1 * x1);
   y2 = 1. - x1;
run;

proc print data=out1;
run;
```

The final parameter estimates $x^* = (1, 1)$ and the values of the functions f_1 =Y1 and f_2 =Y2 are written into an OUT= data set, shown in Output 4.4.1. Since OUTDER=0 is the default, the OUT= data set does not contain the Jacobian matrix.

Output 4.4.1. Solution in an OUT= Data Set

Obs	_OBS_	_TYPE_	y1	y2	x2	x1
1	1		0	0	1	1

Next, the procedure reads the optimal parameter estimates from the EST data set and the model from the MODEL data set. It does not do any optimization (TECH=NONE), but it saves the Jacobian matrix to the OUT=OUT2 data set because of the option OUTDER=1. It also displays the Jacobian matrix because of the option PJAC; the Jacobian matrix is shown in Output 4.4.2. Output 4.4.3 shows the contents of the OUT2 data set, which also contains the Jacobian matrix.

```
proc nlp tech=none model=model inest=est out=out2 outder=1 pjac;
   lsq y1 y2;
   parms x1 x2;
run;

proc print data=out2;
run;
```

Output 4.4.2. Jacobian Matrix Output

PROC NLP: Least Squares Minimization

Jacobian Matrix

x1	x2
-20	10
-1	0

Output 4.4.3. Jacobian Matrix in an OUT= Data Set

Obs	_OBS_	_TYPE_	y1	y2	_WRT_	x2	x1
1	1		0	0		1	1
2	1	ANALYTIC	10	0	x2	1	1
3	1	ANALYTIC	-20	-1	x1	1	1

Example 4.5. Approximate Standard Errors

The NLP procedure provides a variety of ways for estimating parameters in nonlinear statistical models and for obtaining approximate standard errors and covariance matrices for the estimators. These methods are illustrated by estimating the mean of a random sample from a normal distribution with mean μ and standard deviation σ. The simplicity of the example makes it easy to compare the results of different methods in NLP with the usual estimator, the sample mean.

The following data step is used:

```
data x;
   input x @@;
datalines;
1 3 4 5 7
;
```

The standard error of the mean, computed with $n - 1$ degrees of freedom, is 1. The usual maximum-likelihood approximation to the standard error of the mean, using a variance divisor of n rather than $n - 1$, is 0.894427.

The sample mean is a least-squares estimator, so it can be computed using an LSQ statement. Moreover, since this model is linear, the Hessian matrix and crossproduct Jacobian matrix are identical, and all three versions of the COV= option yield the same variance and standard error of the mean. Note that COV=j means that the crossproduct Jacobian is used. This is chosen because it requires the least computation.

```
proc nlp data=x cov=j pstderr pshort;
   lsq resid;
   parms mean=0;
   resid=x-mean;
run;
```

The results are the same as the usual estimates.

Output 4.5.1. Parameter Estimates

PROC NLP: Least Squares Minimization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Approx Std Err	t Value	Approx Pr > |t|
1	mean	4.000000	1.000000	4.000000	0.016130

Optimization Results
Parameter Estimates

Gradient Objective Function
0

Value of Objective Function = 10

PROC NLP can also compute maximum-likelihood estimates of μ and σ. In this case it is convenient to minimize the negative log likelihood. To get correct standard errors for maximum-likelihood estimators, the SIGSQ=1 option is required. The following program shows COV=1 but the output that follows has COV=2 and COV=3.

```
proc nlp data=x cov=1 sigsq=1 pstderr phes pcov pshort;
   min nloglik;
   parms mean=0, sigma=1;
   bounds 1e-12 < sigma;
   nloglik=.5*((x-mean)/sigma)**2 + log(sigma);
run;
```

The variance divisor is n instead of $n - 1$, so the standard error of the mean is 0.894427 instead of 1. The standard error of the mean is the same with all six types of covariance matrix, but the standard error of the standard deviation varies. The sampling distribution of the standard deviation depends on the higher moments of the population distribution, so different methods of estimation can produce markedly different estimates of the standard error of the standard deviation.

Output 4.5.2 shows the output when COV=1, Output 4.5.3 shows the output when COV=2, and Output 4.5.4 shows the output when COV=3.

Output 4.5.2. Solution for COV=1

```
                    PROC NLP: Nonlinear Minimization

                          Optimization Results
                           Parameter Estimates
                                      Approx                                Approx
N Parameter         Estimate         Std Err         t Value        Pr > |t|

1 mean              4.000000        0.894427        4.472136        0.006566
2 sigma             2.000000        0.458258        4.364358        0.007260

                          Optimization Results
                           Parameter Estimates
                                  Gradient
                                 Objective
                                  Function

                               1.33149E-10
                              -5.606415E-9

                Value of Objective Function = 5.9657359028


                             Hessian Matrix

                                    mean             sigma

                mean       1.2500000028      -1.33149E-10
                sigma      -1.33149E-10       2.500000014

                        Determinant = 3.1250000245

                   Matrix has Only Positive Eigenvalues


                    Covariance Matrix 1: M = (NOBS/d)
                           inv(G) JJ(f) inv(G)

                                    mean             sigma

                mean                 0.8      1.980107E-11
                sigma      1.980107E-11      0.2099999991

                             Factor sigm = 1


                        Determinant = 0.1679999993

                   Matrix has Only Positive Eigenvalues
```

Output 4.5.3. Solution for COV=2

```
                  PROC NLP: Nonlinear Minimization

                        Optimization Results
                         Parameter Estimates
                                    Approx                          Approx
N Parameter         Estimate         Std Err         t Value        Pr > |t|

1 mean              4.000000        0.894427        4.472136        0.006566
2 sigma             2.000000        0.632456        3.162278        0.025031

                        Optimization Results
                         Parameter Estimates
                               Gradient
                              Objective
                               Function

                            1.33149E-10
                           -5.606415E-9

               Value of Objective Function = 5.9657359028


                          Hessian Matrix

                                mean             sigma

               mean       1.2500000028      -1.33149E-10
               sigma      -1.33149E-10       2.500000014

                      Determinant = 3.1250000245

                 Matrix has Only Positive Eigenvalues


               Covariance Matrix 2: H = (NOBS/d) inv(G)

                                mean             sigma

               mean       0.7999999982      4.260766E-11
               sigma      4.260766E-11      0.3999999978

                          Factor sigm = 1


                      Determinant = 0.3199999975

                 Matrix has Only Positive Eigenvalues
```

Output 4.5.4. Solution for COV=3

```
                    PROC NLP: Nonlinear Minimization

                          Optimization Results
                           Parameter Estimates
                                      Approx                                Approx
N Parameter          Estimate         Std Err         t Value         Pr > |t|

1 mean               4.000000        0.509136        7.856442         0.000537
2 sigma              2.000000        0.419936        4.762634         0.005048

                          Optimization Results
                           Parameter Estimates
                                  Gradient
                                Objective
                                 Function

                              1.301733E-10
                              -5.940302E-9

                Value of Objective Function = 5.9657359028


                            Hessian Matrix

                                   mean             sigma

                mean       1.2500000028      -1.33149E-10
                sigma      -1.33149E-10       2.500000014

                        Determinant = 3.1250000245

                   Matrix has Only Positive Eigenvalues


                  Covariance Matrix 3: J = (1/d) inv(W)

                                   mean             sigma

                mean       0.2592197879      1.062283E-11
                sigma      1.062283E-11      0.1763460041

                            Factor sigm = 0.2


                        Determinant = 0.0457123738

                   Matrix has Only Positive Eigenvalues
```

Under normality, the maximum-likelihood estimators of μ and σ are independent, as indicated by the diagonal Hessian matrix in the previous example. Hence, the maximum-likelihood estimate of μ can be obtained by using any fixed value for σ, such as 1. However, if the fixed value of σ differs from the actual maximum-likelihood estimate (in this case 2), the model is misspecified and the standard errors obtained with COV=2 or COV=3 are incorrect. It is therefore necessary to use COV=1, which yields consistent estimates of the standard errors under a variety of forms of misspecification of the error distribution.

```
proc nlp data=x cov=1 sigsq=1 pstderr pcov pshort;
   min sqresid;
   parms mean=0;
   sqresid=.5*(x-mean)**2;
run;
```

This formulation produces the same standard error of the mean, 0.894427 (see Output 4.5.5).

Output 4.5.5. Solution for Fixed σ and COV=1

```
                    PROC NLP: Nonlinear Minimization

                          Optimization Results
                           Parameter Estimates
                                      Approx                                Approx
N Parameter          Estimate          Std Err          t Value          Pr > |t|

1 mean               4.000000         0.894427         4.472136         0.006566

                          Optimization Results
                           Parameter Estimates
                                 Gradient
                               Objective
                                Function

                                       0

                     Value of Objective Function = 10


                            Covariance Matrix
                         1: M = (NOBS/d) inv(G)
                              JJ(f) inv(G)

                                                mean

                         mean                    0.8

                            Factor sigm = 1
```

The maximum-likelihood formulation with fixed σ is actually a least-squares problem. The objective function, parameter estimates, and Hessian matrix are the same as those in the first example in this section using the LSQ statement. However, the Jacobian matrix is different, each row being multiplied by twice the residual. To treat this formulation as a least-squares problem, the SIGSQ=1 option can be omitted. But since the Jacobian is not the same as in the formulation using the LSQ statement, the COV=1 | M and COV=3 | J options, which use the Jacobian, do not yield correct standard errors. The correct standard error is obtained with COV=2 | H, which uses only the Hessian matrix:

```
proc nlp data=x cov=2 pstderr pcov pshort;
   min sqresid;
   parms mean=0;
   sqresid=.5*(x-mean)**2;
run;
```

The results are the same as in the first example.

Output 4.5.6. Solution for Fixed σ and COV=2

```
                    PROC NLP: Nonlinear Minimization

                          Optimization Results
                           Parameter Estimates
                                      Approx                          Approx
N Parameter          Estimate         Std Err         t Value        Pr > |t|

1 mean               4.000000        0.500000        8.000000        0.001324

                          Optimization Results
                           Parameter Estimates
                                  Gradient
                                Objective
                                 Function

                                       0

                    Value of Objective Function = 10


                          Covariance Matrix 2:
                          H = (NOBS/d) inv(G)

                                        mean

                       mean              0.25

                         Factor sigm = 1.25
```

In summary, to obtain appropriate standard errors for least-squares estimates, you can use the LSQ statement with any of the COV= options, or you can use the MIN statement with COV=2. To obtain appropriate standard errors for maximum-likelihood estimates, you can use the MIN statement with the negative log likelihood or the MAX statement with the log likelihood, and in either case you can use any of the COV= options provided that you specify SIGSQ=1. You can also use a log-likelihood function with a misspecified scale parameter provided that you use SIGSQ=1 and COV=1. For nonlinear models, all of these methods yield approximations based on asymptotic theory, and should therefore be interpreted cautiously.

Example 4.6. Maximum Likelihood Weibull Estimation

Two-Parameter Weibull Estimation

The following data are taken from Lawless (1982, p. 193) and represent the number of days it took rats painted with a carcinogen to develop carcinoma. The last two observations are censored data from a group of 19 rats:

```
data pike;
   input days cens @@;
   datalines;
143  0  164  0  188  0  188  0
190  0  192  0  206  0  209  0
213  0  216  0  220  0  227  0
230  0  234  0  246  0  265  0
304  0  216  1  244  1
;
```

Suppose that you want to show how to compute the maximum likelihood estimates of the scale parameter σ (α in Lawless), the shape parameter c (β in Lawless), and the location parameter θ (μ in Lawless). The observed likelihood function of the three-parameter Weibull transformation (Lawless 1982, p. 191) is

$$L(\theta,\sigma,c) = \frac{c^m}{\sigma^m} \prod_{i \in D} \left(\frac{t_i - \theta}{\sigma} \right)^{c-1} \prod_{i=1}^{p} \exp\left(- \left(\frac{t_i - \theta}{\sigma} \right)^c \right)$$

and the log likelihood is

$$l(\theta,\sigma,c) = m \log c - mc \log \sigma + (c-1) \sum_{i \in D} \log(t_i - \theta) - \sum_{i=1}^{p} \left(\frac{t_i - \theta}{\sigma} \right)^c$$

The log likelihood function can be evaluated only for $\sigma > 0$, $c > 0$, and $\theta < \min_i t_i$. In the estimation process, you must enforce these conditions using lower and upper boundary constraints. The three-parameter Weibull estimation can be numerically difficult, and it usually pays off to provide good initial estimates. Therefore, you first estimate σ and c of the two-parameter Weibull distribution for constant $\theta = 0$. You then use the optimal parameters $\hat{\sigma}$ and $\hat{c}$ as starting values for the three-parameter Weibull estimation.

Although the use of an INEST= data set is not really necessary for this simple example, it illustrates how it is used to specify starting values and lower boundary constraints:

```
data parm1(type=est);
   keep _type_ sig c theta;
   _type_='parms'; sig = .5;
      c = .5; theta = 0; output;
   _type_='lb'; sig = 1.0e-6;
      c = 1.0e-6; theta = .;  output;
run;
```

The following PROC NLP call specifies the maximization of the log likelihood function for the two-parameter Weibull estimation for constant $\theta = 0$:

```
proc nlp data=pike tech=tr inest=par1 outest=opar1
     outmodel=model cov=2 vardef=n pcov phes;
   max logf;
   parms sig c;
   profile sig c / alpha = .9 to .1 by -.1 .09 to .01 by -.01;

   x_th = days - theta;
   s    = - (x_th / sig)**c;
   if cens=0 then s + log(c) - c*log(sig) + (c-1)*log(x_th);
   logf = s;
run;
```

After a few iterations you obtain the solution given in Output 4.6.1.

Output 4.6.1. Optimization Results

PROC NLP: Nonlinear Maximization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Approx Std Err	t Value	Approx Pr > |t|
1	sig	234.318611	9.645908	24.292021	9.050475E-16
2	c	6.083147	1.068229	5.694611	0.000017269

Optimization Results
Parameter Estimates

	Gradient Objective Function
	1.3372183E-9
	-7.859311E-9

Value of Objective Function = -88.23273515

Since the gradient has only small elements and the Hessian (shown in Output 4.6.2) is negative definite (has only negative eigenvalues), the solution defines an isolated maximum point.

Output 4.6.2. Hessian Matrix at x^*

Hessian Matrix

	sig	c
sig	-0.011457556	0.0257527577
c	0.0257527577	-0.934221388

Determinant = 0.0100406894

Matrix has Only Negative Eigenvalues

The square roots of the diagonal elements of the approximate covariance matrix of parameter estimates are the approximate standard errors (ASE's). The covariance matrix is given in Output 4.6.3.

Output 4.6.3. Covariance Matrix

```
                Covariance Matrix 2:
                 H = (NOBS/d) inv(G)

                         sig                 c

sig          93.043549863       2.5648395794
c             2.5648395794       1.141112488

                   Factor sigm = 1


               Determinant = 99.594754608

             Matrix has 2 Positive Eigenvalue(s)
```

The confidence limits in Output 4.6.4 correspond to the α values in the PROFILE statement.

Output 4.6.4. Confidence Limits

PROC NLP: Nonlinear Maximization

Wald and PL Confidence Limits

N	Parameter	Estimate	Alpha	Profile Likelihood Confidence Limits	
1	sig	234.318611	0.900000	233.111324	235.532695
1	sig	.	0.800000	231.886549	236.772876
1	sig	.	0.700000	230.623280	238.063824
1	sig	.	0.600000	229.292797	239.436639
1	sig	.	0.500000	227.855829	240.935290
1	sig	.	0.400000	226.251597	242.629201
1	sig	.	0.300000	224.372260	244.643392
1	sig	.	0.200000	221.984557	247.278423
1	sig	.	0.100000	218.390824	251.394102
1	sig	.	0.090000	217.884162	251.987489
1	sig	.	0.080000	217.326988	252.645278
1	sig	.	0.070000	216.708814	253.383546
1	sig	.	0.060000	216.008815	254.228034
1	sig	.	0.050000	215.199301	255.215496
1	sig	.	0.040000	214.230116	256.411041
1	sig	.	0.030000	213.020874	257.935686
1	sig	.	0.020000	211.369067	260.066128
1	sig	.	0.010000	208.671091	263.687174
2	c	6.083147	0.900000	5.950029	6.217752
2	c	.	0.800000	5.815559	6.355576
2	c	.	0.700000	5.677909	6.499187
2	c	.	0.600000	5.534275	6.651789
2	c	.	0.500000	5.380952	6.817880
2	c	.	0.400000	5.212344	7.004485
2	c	.	0.300000	5.018784	7.225733
2	c	.	0.200000	4.776379	7.506166
2	c	.	0.100000	4.431310	7.931669
2	c	.	0.090000	4.382687	7.991457
2	c	.	0.080000	4.327815	8.056628
2	c	.	0.070000	4.270773	8.129238
2	c	.	0.060000	4.207130	8.211221
2	c	.	0.050000	4.134675	8.306218
2	c	.	0.040000	4.049531	8.418782
2	c	.	0.030000	3.945037	8.559677
2	c	.	0.020000	3.805759	8.749130
2	c	.	0.010000	3.588814	9.056751

Three-Parameter Weibull Estimation

You now prepare for the three-parameter Weibull estimation by using PROC UNIVARIATE to obtain the smallest data value for the upper boundary constraint for θ. For this small problem, you can do this much more simply by just using a value slightly smaller than the minimum data value 143.

```
/* Calculate upper bound for theta parameter */
proc univariate data=pike noprint;
   var days;
   output out=stats n=nobs min=minx range=range;
run;


data stats;
   set stats;
   keep _type_ theta;

   /* 1. write parms observation */
   theta = minx - .1 * range;
   if theta < 0 then theta = 0;
   _type_ = 'parms';
   output;

   /* 2. write ub observation */
   theta = minx * (1 - 1e-4);
   _type_ = 'ub';
   output;
run;
```

The data set PAR2 specifies the starting values and the lower and upper bounds for the three-parameter Weibull problem:

```
proc sort data=opar1;
   by _type_;
run;

data par2(type=est);
   merge opar1(drop=theta) stats;
   by _type_;
   keep _type_ sig c theta;
   if _type_ in ('parms' 'lowerbd' 'ub');
run;
```

The following PROC NLP call uses the MODEL= input data set containing the log likelihood function that was saved during the two-parameter Weibull estimation:

```
proc nlp data=pike tech=tr inest=par2 outest=opar2
      model=model cov=2 vardef=n pcov phes;
   max logf;
   parms sig c theta;
   profile sig c theta / alpha = .5 .1 .05 .01;
run;
```

After a few iterations, you obtain the solution given in Output 4.6.5.

Output 4.6.5. Optimization Results

PROC NLP: Nonlinear Maximization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Approx Std Err	t Value	Approx Pr > |t|
1	sig	108.382732	32.573396	3.327339	0.003540
2	c	2.711477	1.058759	2.560995	0.019108
3	theta	122.025942	28.692439	4.252895	0.000430

Optimization Results
Parameter Estimates

Gradient Objective Function
-8.91765E-12
-1.406349E-9
-1.93916E-10

Value of Objective Function = -87.32424712

From inspecting the first- and second-order derivatives at the optimal solution, you can verify that you have obtained an isolated maximum point. The Hessian matrix is shown in Output 4.6.6.

Output 4.6.6. Hessian Matrix

Hessian Matrix

	sig	c	theta
sig	-0.010639974	0.0453887849	-0.010033749
c	0.0453887849	-4.078687939	-0.083026333
theta	-0.010033749	-0.083026333	-0.014752091

Determinant = 0.0000502116

Matrix has Only Negative Eigenvalues

The square roots of the diagonal elements of the approximate covariance matrix of parameter estimates are the approximate standard errors. The covariance matrix is given in Output 4.6.7.

Output 4.6.7. Covariance Matrix

Covariance Matrix 2: H = (NOBS/d) inv(G)

	sig	c	theta
sig	1061.02613	29.926259613	−890.093361
c	29.926259613	1.1209710432	−26.66352284
theta	−890.093361	−26.66352284	823.2560786

Factor sigm = 1

Determinant = 19915.723162

Matrix has 3 Positive Eigenvalue(s)

The difference between the Wald and profile CLs for parameter PHI2 are remarkable, especially for the upper 95% and 99% limits, as shown in Output 4.6.8.

Output 4.6.8. Confidence Limits

Wald and PL Confidence Limits

N	Parameter	Estimate	Alpha	Profile Likelihood Confidence Limits	
1	sig	108.382732	0.500000	91.811562	141.564605
1	sig	.	0.100000	76.502373	.
1	sig	.	0.050000	72.215845	.
1	sig	.	0.010000	64.262384	.
2	c	2.711477	0.500000	2.139297	3.704052
2	c	.	0.100000	1.574162	9.250072
2	c	.	0.050000	1.424853	19.516166
2	c	.	0.010000	1.163096	19.540681
3	theta	122.025942	0.500000	91.027144	135.095454
3	theta	.	0.100000	.	141.833769
3	theta	.	0.050000	.	142.512603
3	theta	.	0.010000	.	142.967407

Wald and PL Confidence Limits

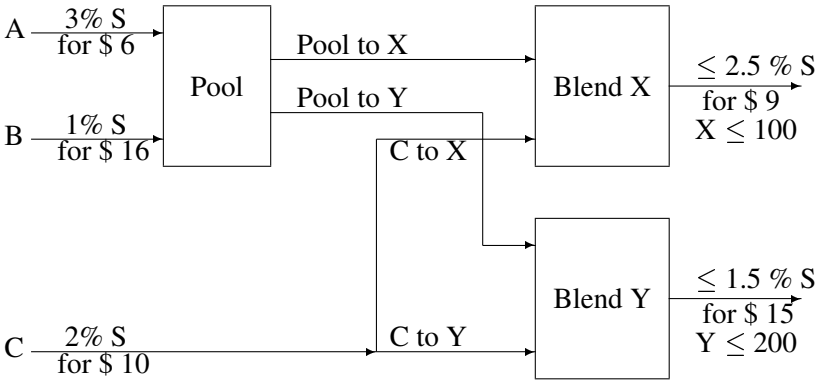
Wald Confidence Limits	
86.412310	130.353154
54.804263	161.961201
44.540049	172.225415
24.479224	192.286240
1.997355	3.425599
0.969973	4.452981
0.636347	4.786607
−0.015706	5.438660
102.673186	141.378698
74.831079	142.985700
65.789794	142.985700
48.119116	142.985700

Example 4.7. Simple Pooling Problem

The following optimization problem is discussed in Haverly (1978) and in Liebman et al. (1986, pp. 127–128). Two liquid chemicals, X and Y, are produced by the pooling and blending of three input liquid chemicals, A, B, and C. You know the sulfur impurity amounts of the input chemicals, and you have to respect upper limits of the sulfur impurity amounts of the output chemicals. The sulfur concentrations and the prices of the input and output chemicals are:

- Chemical A: Concentration = 3%, Price= \$6
- Chemical B: Concentration = 1%, Price= \$16
- Chemical C: Concentration = 2%, Price= \$10
- Chemical X: Concentration $\leq$ 2.5%, Price= \$9
- Chemical Y: Concentration $\leq$ 1.5%, Price= \$15

The problem is complicated by the fact that the two input chemicals A and B are available only as a mixture (they are either shipped together or stored together). Because the amounts of A and B are unknown, the sulfur concentration of the mixture is also unknown.



You know customers will buy no more than 100 units of X and 200 units of Y. The problem is determining how to operate the pooling and blending of the chemicals to maximize the profit. The objective function for the profit is

$$\begin{aligned} profit &= cost(x) \times amount(x) + cost(y) \times amount(y) \\ &- cost(a) \times amount(a) - cost(b) \times amount(b) - cost(c) \times amount(c) \end{aligned}$$

There are three groups of constraints:

1. The first group of constraint functions is the mass balance restrictions illustrated by the graph. These are four linear equality constraints:

   - $amount(a) + amount(b) = pool\_to\_x + pool\_to\_y$
   - $pool\_to\_x + c\_to\_x = amount(x)$
   - $pool\_to\_y + c\_to\_y = amount(y)$
   - $amount(c) = c\_to\_x + c\_to\_y$

2. You introduce a new variable, $pool\_s$, that represents the sulfur concentration of the pool. Using $pool\_s$ and the sulfur concentration of C (2%), you obtain two nonlinear inequality constraints for the sulfur concentrations of X and Y, one linear equality constraint for the sulfur balance, and lower and upper boundary restrictions for $pool\_s$:

   - $pool\_s \times pool\_to\_x + 2\, c\_to\_x \leq 2.5\, amount(x)$
   - $pool\_s \times pool\_to\_y + 2\, c\_to\_y \leq 1.5\, amount(y)$
   - $3\, amount(a) + 1\, amount(b) = pool\_s \times (amount(a) + amount(b))$
   - $1 \leq pool\_s \leq 3$

3. The last group assembles the remaining boundary constraints. First, you do not want to produce more than you can sell; and finally, all variables must be nonnegative:

   - $amount(x) \leq 100, \quad amount(y) \leq 200$
   - $amount(a), amount(b), amount(c), amount(x), amount(y) \geq 0$
   - $pool\_to\_x, pool\_to\_y, c\_to\_x, c\_to\_y \geq 0$

There exist several local optima to this problem that can be found by specifying different starting points. Using the starting point with all variables equal to 1 (specified with a PARMS statement), PROC NLP finds a solution with $profit = 400$:

```
proc nlp all;
   parms amountx amounty amounta amountb amountc
         pooltox pooltoy ctox ctoy pools = 1;
   bounds 0 <= amountx amounty amounta amountb amountc,
               amountx <= 100,
               amounty <= 200,
          0 <= pooltox pooltoy ctox ctoy,
          1 <= pools <= 3;
   lincon amounta + amountb = pooltox + pooltoy,
          pooltox + ctox = amountx,
          pooltoy + ctoy = amounty,
          ctox + ctoy    = amountc;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
   costa = 6; costb = 16; costc = 10;
   costx = 9; costy = 15;
   f = costx * amountx + costy * amounty
       - costa * amounta - costb * amountb - costc * amountc;
```

```
   nlc1 = 2.5 * amountx - pools * pooltox - 2. * ctox;
   nlc2 = 1.5 * amounty - pools * pooltoy - 2. * ctoy;
   nlc3 = 3 * amounta + amountb - pools * (amounta + amountb);
run;
```

The specified starting point was not feasible with respect to the linear equality constraints; therefore, a starting point is generated that satisfies linear and boundary constraints. Output 4.7.1 gives the starting parameter estimates.

Output 4.7.1. Starting Estimates

PROC NLP: Nonlinear Maximization

N	Parameter	Optimization Start Parameter Estimates Estimate	Gradient Objective Function	Gradient Lagrange Function	Lower Bound Constraint	Upper Bound Constraint
1	amountx	1.363636	9.000000	−0.843698	0	100.000000
2	amounty	1.363636	15.000000	−0.111882	0	200.000000
3	amounta	0.818182	−6.000000	−0.430733	0	.
4	amountb	0.818182	−16.000000	−0.542615	0	.
5	amountc	1.090909	−10.000000	0.017768	0	.
6	pooltox	0.818182	0	−0.669628	0	.
7	pooltoy	0.818182	0	−0.303720	0	.
8	ctox	0.545455	0	−0.174070	0	.
9	ctoy	0.545455	0	0.191838	0	.
10	pools	2.000000	0	0.068372	1.000000	3.000000

Value of Objective Function = 3.8181818182

Value of Lagrange Function = −2.866739915

The starting point satisfies the four equality constraints, as shown in Output 4.7.2. The nonlinear constraints are given in Output 4.7.3.

Output 4.7.2. Linear Constraints

```
                          Linear Constraints

1 -5.551E-16 : ACT        0  ==   +    1.0000 * amounta  +    1.0000 * amountb
   -    1.0000 * pooltox  -    1.0000 * pooltoy
2  -2.22E-16 : ACT        0  ==   -    1.0000 * amountx  +    1.0000 * pooltox
   +    1.0000 * ctox
3 1.1102E-16 : ACT        0  ==   -    1.0000 * amounty  +    1.0000 * pooltoy
   +    1.0000 * ctoy
4          0 : ACT        0  ==   -    1.0000 * amountc  +    1.0000 * ctox
   +    1.0000 * ctoy
```

Output 4.7.3. Nonlinear Constraints

```
                    Values of Nonlinear Constraints

                                          Lagrange
    Constraint          Value Residual Multiplier

[   5 ]    nlc3             0        0     4.9441 Active  NLEC
[   6 ]    nlc1_G      0.6818   0.6818           .
[   7 ]    nlc2_G     -0.6818  -0.6818    -9.8046 Violat. NLIC
```

Output 4.7.4 shows the settings of some important PROC NLP options.

Output 4.7.4. Options

```
                  PROC NLP: Nonlinear Maximization

        Minimum Iterations                                    0
        Maximum Iterations                                  200
        Maximum Function Calls                              500
        Iterations Reducing Constraint Violation             20
        ABSGCONV Gradient Criterion                     0.00001
        GCONV Gradient Criterion                           1E-8
        ABSFCONV Function Criterion                           0
        FCONV Function Criterion                  2.220446E-16
        FCONV2 Function Criterion                          1E-6
        FSIZE Parameter                                       0
        ABSXCONV Parameter Change Criterion                   0
        XCONV Parameter Change Criterion                      0
        XSIZE Parameter                                       0
        ABSCONV Function Criterion                1.340781E154
        Line Search Method                                    2
        Starting Alpha for Line Search                        1
        Line Search Precision LSPRECISION                   0.4
        DAMPSTEP Parameter for Line Search                    .
        FD Derivatives: Accurate Digits in Obj.F     15.653559775
        FD Derivatives: Accurate Digits in NLCon     15.653559775
        Singularity Tolerance (SINGULAR)                   1E-8
        Constraint Precision (LCEPS)                       1E-8
        Linearly Dependent Constraints (LCSING)            1E-8
        Releasing Active Constraints (LCDEACT)                .
```

The iteration history, given in Output 4.7.5, does not show any problems.

Output 4.7.5. Iteration History

PROC NLP: Nonlinear Maximization

Dual Quasi-Newton Optimization
Modified VMCWD Algorithm of Powell (1978, 1982)

Dual Broyden - Fletcher - Goldfarb - Shanno Update (DBFGS)
Lagrange Multiplier Update of Powell(1982)

Iter	Restarts	Function Calls	Objective Function	Maximum Constraint Violation	Predicted Function Reduction	Step Size	Maximum Gradient Element of the Lagrange Function
1	0	19	-1.42400	0.00962	6.9131	1.000	0.783
2′	0	20	2.77026	0.0166	5.3770	1.000	2.629
3	0	21	7.08706	0.1409	7.1965	1.000	9.452
4′	0	22	11.41264	0.0583	15.5771	1.000	23.390
5′	0	23	24.84630	1.78E-15	496.1	1.000	147.6
6	0	24	378.22995	147.4	3316.6	1.000	840.4
7′	0	25	307.56908	50.9348	607.9	1.000	27.143
8′	0	26	347.24557	1.8330	21.9896	1.000	28.483
9′	0	27	349.49342	0.00915	7.1838	1.000	28.291
10′	0	28	356.58483	0.1084	50.2539	1.000	27.480
11′	0	29	388.70516	2.4275	24.7951	1.000	21.114
12′	0	30	389.29922	0.0157	10.0489	1.000	18.648
13′	0	31	399.19185	0.7999	11.1897	1.000	0.416
14′	0	32	400.00000	0.0128	0.1535	1.000	0.00087
15′	0	33	400.00000	7.38E-11	2.43E-10	1.000	366E-12

Optimization Results

Iterations	15	Function Calls	34
Gradient Calls	18	Active Constraints	10
Objective Function	400	Maximum Constraint Violation	7.381118E-11
Maximum Projected Gradient	0	Value Lagrange Function	-400
Maximum Gradient of the Lagran Func	3.552714E-14	Slope of Search Direction	-2.43495E-10

FCONV2 convergence criterion satisfied.

The optimal solution in Output 4.7.6 shows that to obtain the maximum profit of \$400, you need only to produce the maximum 200 units of blending Y and no units of blending X.

Output 4.7.6. Optimization Solution

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Gradient Objective Function	Gradient Lagrange Function	Active Bound Constraint	
1	amountx	-1.40483E-11	9.000000	0	Lower BC	
2	amounty	200.000000	15.000000	8.881784E-16	Upper BC	
3	amounta	5.484612E-16	-6.000000	0	Lower BC	
4	amountb	100.000000	-16.000000	-5.32907E-15		
5	amountc	100.000000	-10.000000	-1.77636E-15		
6	pooltox	7.024625E-12	0	0	Lower BC	
7	pooltoy	100.000000	0	-1.42109E-14		
8	ctox	-2.10729E-11	0	0	Lower BC	LinDep
9	ctoy	100.000000	0	-1.77636E-15		
10	pools	1.000000	0	3.552714E-14	Lower BC	LinDep

Value of Objective Function = 400

Value of Lagrange Function = 400

The constraints are satisfied at the solution, as shown in Output 4.7.7

Output 4.7.7. Linear and Nonlinear Constraints at the Solution

Linear Constraints Evaluated at Solution

```
 1 ACT          0  =         0  +    1.0000 * amounta  +    1.0000 * amountb
   -    1.0000 * pooltox  -    1.0000 * pooltoy
 2 ACT 2.6603E-17  =         0  -    1.0000 * amountx  +    1.0000 * pooltox
   +    1.0000 * ctox
 3 ACT          0  =         0  -    1.0000 * amounty  +    1.0000 * pooltoy
   +    1.0000 * ctoy
 4 ACT          0  =         0  -    1.0000 * amountc  +    1.0000 * ctox
   +    1.0000 * ctoy
```

Values of Nonlinear Constraints

	Constraint	Value	Residual	Lagrange Multiplier			
[5]	nlc3	1.1E-15	1.1E-15	6.0000	Active	NLEC	
[6]	nlc1_G	4.18E-16	4.18E-16	.	Active	NLIC	LinDep
[7]	nlc2_G	0	0	-6.0000	Active	NLIC	

Linearly Dependent Active Boundary Constraints

Parameter	N	Kind
ctox	8	Lower BC
pools	10	Lower BC

Linearly Dependent Gradients of Active Nonlinear Constraints

Parameter	N
nlc3	6

The same problem can be specified in many different ways. For example, the following specification uses an INEST= data set containing the values of the starting point and of the constants COST, COSTB, COSTC, COSTX, COSTY, CA, CB, CC, and CD:

```
data init1(type=est);
   input _type_ $ amountx amounty amounta amountb
         amountc pooltox pooltoy ctox ctoy pools
         _rhs_ costa costb costc costx costy
         ca cb cc cd;
   datalines;
parms  1 1 1 1 1 1 1 1 1
       . 6 16 10 9 15  2.5 1.5 2. 3.
;


proc nlp inest=init1 all;
   parms amountx amounty amounta amountb amountc
         pooltox pooltoy ctox ctoy pools;
   bounds 0 <= amountx amounty amounta amountb amountc,
               amountx <= 100,
               amounty <= 200,
          0 <= pooltox pooltoy ctox ctoy,
          1 <= pools <= 3;
   lincon amounta + amountb = pooltox + pooltoy,
          pooltox + ctox = amountx,
          pooltoy + ctoy = amounty,
          ctox + ctoy    = amountc;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
   f = costx * amountx + costy * amounty
       - costa * amounta - costb * amountb - costc * amountc;
   nlc1 = ca * amountx - pools * pooltox - cc * ctox;
   nlc2 = cb * amounty - pools * pooltoy - cc * ctoy;
   nlc3 = cd * amounta + amountb - pools * (amounta + amountb);
run;
```

The third specification uses an INEST= data set containing the boundary and linear constraints in addition to the values of the starting point and of the constants. This specification also writes the model specification into an OUTMOD= data set:

```
data init2(type=est);
   input _type_ $ amountx amounty amounta amountb amountc
                  pooltox pooltoy ctox ctoy pools
                  _rhs_ costa costb costc costx costy;
   datalines;
parms      1   1  1  1  1    1   1   1  1  1
           .   6 16 10  9   15 2.5 1.5  2  3
lowerbd    0   0  0  0  0    0   0   0  0  1
           .   .  .  .  .    .   .   .  .  .
upperbd  100 200  .  .  .    .   .   .  .  3
           .   .  .  .  .    .   .   .  .  .
eq         .   .  1  1  .   -1  -1   .  .  .
           0   .  .  .  .    .   .   .  .  .
eq         1   .  .  .  .   -1   .  -1  .  .
           0   .  .  .  .    .   .   .  .  .
eq         .   1  .  .  .    .  -1   . -1  .
           0   .  .  .  .    .   .   .  .  .
eq         .   .  .  .  1    .   .  -1 -1  .
           0   .  .  .  .    .   .   .  .  .
;

proc nlp inest=init2 outmod=model all;
   parms amountx amounty amounta amountb amountc
         pooltox pooltoy ctox ctoy pools;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
   f = costx * amountx + costy * amounty
       - costa * amounta - costb * amountb - costc * amountc;
   nlc1 = 2.5 * amountx - pools * pooltox - 2. * ctox;
   nlc2 = 1.5 * amounty - pools * pooltoy - 2. * ctoy;
   nlc3 = 3 * amounta + amountb - pools * (amounta + amountb);
run;
```

The fourth specification not only reads the INEST=INIT2 data set, it also uses the model specification from the MODEL data set that was generated in the last specification. The PROC NLP call now contains only the defining variable statements:

```
proc nlp inest=init2 model=model all;
   parms amountx amounty amounta amountb amountc
         pooltox pooltoy ctox ctoy pools;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
run;
```

All four specifications start with the same starting point of all variables equal to 1 and generate the same results. However, there exist several local optima to this problem, as is pointed out in Liebman et al. (1986, p. 130).

```
proc nlp inest=init2 model=model all;
   parms amountx amounty amounta amountb amountc
```

```
        pooltox pooltoy ctox ctoy = 0,
        pools = 2;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
run;
```

This starting point with all variables equal to 0 is accepted as a local solution with $profit = 0$, which minimizes rather than maximizes the profit.

Example 4.8. Chemical Equilibrium

The following example is used in many test libraries for nonlinear programming and was taken originally from Bracken and McCormick (1968).

The problem is to determine the composition of a mixture of various chemicals satisfying its chemical equilibrium state. The second law of thermodynamics implies that a mixture of chemicals satisfies its chemical equilibrium state (at a constant temperature and pressure) when the free energy of the mixture is reduced to a minimum. Therefore the composition of the chemicals satisfying its chemical equilibrium state can be found by minimizing the function of the free energy of the mixture.

Notation:

m	number of chemical elements in the mixture
n	number of compounds in the mixture
x_j	number of moles for compound j, $j = 1, \ldots, n$
s	total number of moles in the mixture ($s = \sum_{i=1}^{n} x_j$)
a_{ij}	number of atoms of element i in a molecule of compound j
b_i	atomic weight of element i in the mixture

Constraints for the Mixture:

- The number of moles must be positive:

$$x_j > 0, \quad j = 1, \ldots, n$$

- There are m mass balance relationships,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$$

Objective Function: Total Free Energy of Mixture

$$f(x) = \sum_{j=1}^{n} x_j \left[c_j + \ln\left(\frac{x_j}{s}\right) \right]$$

with

$$c_j = \left(\frac{F^\circ}{RT}\right)_j + \ln P$$

where F°/RT is the model standard free energy function for the jth compound (found in tables) and P is the total pressure in atmospheres.

Minimization Problem:

Determine the parameters x_j that minimize the objective function $f(x)$ subject to the nonnegativity and linear balance constraints.

Numeric Example:

Determine the equilibrium composition of compound $\frac{1}{2}N_2H_4 + \frac{1}{2}O_2$ at temperature $T = 3500^\circ$K and pressure $P = 750$psi.

				a_{ij}		
				$i=1$	$i=2$	$i=3$
j	Compound	$(F^\circ/RT)_j$	c_j	H	N	O
1	H	-10.021	-6.089	1		
2	H_2	-21.096	-17.164	2		
3	H_2O	-37.986	-34.054	2		1
4	N	-9.846	-5.914		1	
5	N_2	-28.653	-24.721		2	
6	NH	-18.918	-14.986	1	1	
7	NO	-28.032	-24.100		1	1
8	O	-14.640	-10.708			1
9	O_2	-30.594	-26.662			2
10	OH	-26.111	-22.179	1		1

Example Specification:

```
proc nlp tech=tr pall;
   array c[10] -6.089 -17.164 -34.054  -5.914 -24.721
               -14.986 -24.100 -10.708 -26.662 -22.179;
   array x[10] x1-x10;
   min y;
   parms x1-x10 = .1;
   bounds 1.e-6 <= x1-x10;
   lincon 2. = x1 + 2. * x2 + 2. * x3 + x6 + x10,
          1. = x4 + 2. * x5 + x6 + x7,
          1. = x3 + x7  + x8 + 2. * x9 + x10;
```

```
      s = x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10;
      y = 0.;
      do j = 1 to 10;
         y = y + x[j] * (c[j] + log(x[j] / s));
      end;
   run;
```

Displayed Output:

The iteration history given in Output 4.8.1 does not show any problems.

Output 4.8.1. Iteration History

Trust Region Optimization

Without Parameter Scaling

Iter	Rest arts	Func Calls	Act Con	Objective Function	Obj Fun Change	Max Abs Gradient Element	Lambda	Trust Region Radius
1	0	2	3′	−47.33412	2.2790	6.0765	2.456	1.000
2	0	3	3′	−47.70043	0.3663	8.5592	0.908	0.418
3	0	4	3	−47.73074	0.0303	6.4942	0	0.359
4	0	5	3	−47.73275	0.00201	4.7606	0	0.118
5	0	6	3	−47.73554	0.00279	3.2125	0	0.0168
6	0	7	3	−47.74223	0.00669	1.9552	110.6	0.00271
7	0	8	3	−47.75048	0.00825	1.1157	102.9	0.00563
8	0	9	3	−47.75876	0.00828	0.4165	3.787	0.0116
9	0	10	3	−47.76101	0.00224	0.0716	0	0.0121
10	0	11	3	−47.76109	0.000083	0.00238	0	0.0111
11	0	12	3	−47.76109	9.609E−8	2.733E−6	0	0.00248

Optimization Results

Iterations	11	Function Calls	13
Hessian Calls	12	Active Constraints	3
Objective Function	−47.76109086	Max Abs Gradient Element	1.8637498E−6
Lambda	0	Actual Over Pred Change	0
Radius	0.0024776027		

GCONV convergence criterion satisfied.

Output 4.8.2 lists the optimal parameters with the gradient.

Output 4.8.2. Optimization Results

PROC NLP: Nonlinear Minimization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Gradient Objective Function
1	x1	0.040668	-9.785055
2	x2	0.147730	-19.570110
3	x3	0.783153	-34.792170
4	x4	0.001414	-12.968921
5	x5	0.485247	-25.937841
6	x6	0.000693	-22.753976
7	x7	0.027399	-28.190984
8	x8	0.017947	-15.222060
9	x9	0.037314	-30.444120
10	x10	0.096871	-25.007115

Value of Objective Function = -47.76109086

The three equality constraints are satisfied at the solution, as shown in Output 4.8.3.

Output 4.8.3. Linear Constraints at Solution

PROC NLP: Nonlinear Minimization

Linear Constraints Evaluated at Solution

```
1 ACT 6.9389E-17  =    2.0000  -    1.0000 * x1   -    2.0000 * x2   -
    2.0000 * x3   -    1.0000 * x6   -    1.0000 * x10
2 ACT -4.337E-16  =    1.0000  -    1.0000 * x4   -    2.0000 * x5   -
    1.0000 * x6   -    1.0000 * x7
3 ACT -4.163E-17  =    1.0000  -    1.0000 * x3   -    1.0000 * x7   -
    1.0000 * x8   -    2.0000 * x9   -    1.0000 * x10
```

The Lagrange multipliers are given in Output 4.8.4.

Output 4.8.4. Lagrange Multipliers

PROC NLP: Nonlinear Minimization

First Order Lagrange Multipliers

Active Constraint		Lagrange Multiplier
Linear EC	[1]	9.785055
Linear EC	[2]	12.968921
Linear EC	[3]	15.222060

The elements of the projected gradient must be small to satisfy a necessary first-order optimality condition. The projected gradient is given in Output 4.8.5

Output 4.8.5. Projected Gradient

PROC NLP: Nonlinear Minimization

Projected Gradient

Free Dimension	Projected Gradient
1	4.577009E-9
2	6.86833E-10
3	-7.283021E-9
4	-0.000001864
5	-0.000001434
6	-0.000001361
7	-0.000000294

The projected Hessian matrix shown in Output 4.8.6 is positive definite, satisfying the second-order optimality condition.

Output 4.8.6. Projected Hessian Matrix

Projected Hessian Matrix

	X1	X2	X3	X4
X1	20.903196985	-0.122067474	2.6480263467	3.3439156526
X2	-0.122067474	565.97299938	106.54631863	-83.7084843
X3	2.6480263467	106.54631863	1052.3567179	-115.230587
X4	3.3439156526	-83.7084843	-115.230587	37.529977667
X5	-1.373829641	-37.43971036	182.89278895	-4.621642366
X6	-1.491808185	-36.20703737	175.97949593	-4.574152161
X7	1.1462413516	-16.635529	-57.04158208	10.306551561

Projected Hessian Matrix

	X5	X6	X7
X1	-1.373829641	-1.491808185	1.1462413516
X2	-37.43971036	-36.20703737	-16.635529
X3	182.89278895	175.97949593	-57.04158208
X4	-4.621642366	-4.574152161	10.306551561
X5	79.326057844	22.960487404	-12.69831637
X6	22.960487404	66.669897023	-8.121228758
X7	-12.69831637	-8.121228758	14.690478023

The following PROC NLP call uses a specified analytical gradient and the Hessian matrix is computed by finite-difference approximations based on the analytic gradient:

```
proc nlp tech=tr fdhessian all;
   array c[10] -6.089 -17.164 -34.054  -5.914 -24.721
               -14.986 -24.100 -10.708 -26.662 -22.179;
   array x[10] x1-x10;
   array g[10] g1-g10;
   min y;
   parms x1-x10 = .1;
   bounds 1.e-6 <= x1-x10;
   lincon 2. = x1 + 2. * x2 + 2. * x3 + x6 + x10,
          1. = x4 + 2. * x5 + x6 + x7,
          1. = x3 + x7  + x8 + 2. * x9 + x10;
   s = x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10;
   y = 0.;
   do j = 1 to 10;
      y = y + x[j] * (c[j] + log(x[j] / s));
      g[j] = c[j] + log(x[j] / s);
   end;
run;
```

The results are almost identical to those of the previous run.

Example 4.9. Minimize Total Delay in a Network

The following example is taken from the user's guide of GINO (Liebman et al. 1986). A simple network of five roads (arcs) can be illustrated by the path diagram:

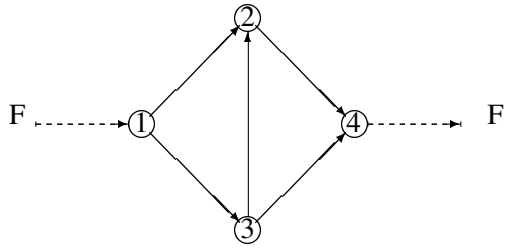


Figure 4.11. Simple Road Network

The five roads connect four intersections illustrated by numbered nodes. Each minute F vehicles enter and leave the network. Arc (i, j) refers to the road from intersection i to intersection j, and the parameter x_{ij} refers to the flow from i to j. The law that traffic flowing into each intersection j must also flow out is described by the linear equality constraint

$$\sum_i x_{ij} = \sum_i x_{ji}\,, \quad j = 1, \ldots, n$$

In general, roads also have an upper capacity, which is the number of vehicles which can be handled per minute. The upper limits c_{ij} can be enforced by boundary constraints

$$0 \leq x_{ij} \leq c_{ij}\,, \quad i, j = 1, \ldots, n$$

Finding the maximum flow through a network is equivalent to solving a simple linear optimization problem, and for large problems, PROC LP or PROC NETFLOW can

be used. The objective function is

$$\max \quad f = x_{24} + x_{34}$$

and the constraints are

$$\begin{aligned} x_{13} &= x_{32} + x_{34} \\ x_{12} + x_{32} &= x_{24} \\ x_{12} + x_{13} &= x_{24} + x_{34} \\ 0 \leq x_{12}, x_{32}, x_{34} &\leq 10 \\ 0 \leq x_{13}, x_{24} &\leq 30 \end{aligned}$$

The three linear equality constraints are linearly dependent. One of them is deleted automatically by the PROC NLP subroutines. Even though the default technique is used for this small example, any optimization subroutine can be used.

```
proc nlp all initial=.5;
   max y;
   parms x12 x13 x32 x24 x34;
   bounds x12 <= 10,
          x13 <= 30,
          x32 <= 10,
          x24 <= 30,
          x34 <= 10;
   /* what flows into an intersection must flow out */
   lincon x13 = x32 + x34,
          x12 + x32 = x24,
          x24 + x34 = x12 + x13;
   y = x24 + x34 + 0*x12 + 0*x13 + 0*x32;
run;
```

The iteration history is given in Output 4.9.1, and the optimal solution is given in Output 4.9.2.

Output 4.9.1. Iteration History

Newton-Raphson Ridge Optimization

Without Parameter Scaling

Iter	Rest arts	Func Calls	Act Con	Objective Function	Obj Fun Change	Max Abs Gradient Element	Ridge	Actual Over Pred Change
1*	0	2	4	20.25000	19.2500	0.5774	0.0313	0.860
2*	0	3	5	30.00000	9.7500	0	0.0313	1.683

Optimization Results

Iterations	2	Function Calls	4
Hessian Calls	3	Active Constraints	5
Objective Function	30	Max Abs Gradient Element	0
Ridge	0	Actual Over Pred Change	1.6834532374

All parameters are actively constrained. Optimization cannot proceed.

Output 4.9.2. Optimization Results

PROC NLP: Nonlinear Maximization

Optimization Results

Parameter Estimates

N	Parameter	Estimate	Gradient Objective Function	Active Bound Constraint
1	x12	10.000000	0	Upper BC
2	x13	20.000000	0	
3	x32	10.000000	0	Upper BC
4	x24	20.000000	1.000000	
5	x34	10.000000	1.000000	Upper BC

Value of Objective Function = 30

Finding a traffic pattern that minimizes the total delay to move F vehicles per minute from node 1 to node 4 introduces nonlinearities that, in turn, demand nonlinear optimization techniques. As traffic volume increases, speed decreases. Let t_{ij} be the travel time on arc (i, j) and assume that the following formulas describe the travel time as decreasing functions of the amount of traffic:

$$
\begin{aligned}
t_{12} &= 5 + 0.1x_{12}/(1 - x_{12}/10) \\
t_{13} &= x_{13}/(1 - x_{13}/30) \\
t_{32} &= 1 + x_{32}/(1 - x_{32}/10) \\
t_{24} &= x_{24}/(1 - x_{24}/30) \\
t_{34} &= 5 + 0.1x_{34}/(1 - x_{34}/10)
\end{aligned}
$$

These formulas use the road capacities (upper bounds), assuming $F = 5$ vehicles per minute have to be moved through the network. The objective function is now

$$\min \quad f = t_{12}x_{12} + t_{13}x_{13} + t_{32}x_{32} + t_{24}x_{24} + t_{34}x_{34}$$

and the constraints are

$$\begin{aligned} x_{13} &= x_{32} + x_{34} \\ x_{12} + x_{32} &= x_{24} \\ x_{24} + x_{34} &= F = 5 \\ 0 \leq x_{12}, x_{32}, x_{34} &\leq 10 \\ 0 \leq x_{13}, x_{24} &\leq 30 \end{aligned}$$

Again, the default algorithm is used:

```
proc nlp all initial=.5;
   min y;
   parms x12 x13 x32 x24 x34;
   bounds x12 x13 x32 x24 x34 >= 0;
   lincon x13 = x32 + x34,  /* flow in = flow out */
          x12 + x32 = x24,
          x24 + x34 = 5;    /* = f = desired flow */
   t12 = 5 + .1 * x12 / (1 - x12 / 10);
   t13 = x13 / (1 - x13 / 30);
   t32 = 1 + x32 / (1 - x32 / 10);
   t24 = x24 / (1 - x24 / 30);
   t34 = 5 + .1 * x34 / (1 - x34 / 10);
   y = t12*x12 + t13*x13 + t32*x32 + t24*x24 + t34*x34;
run;
```

The iteration history is given in Output 4.9.3, and the optimal solution is given in Output 4.9.4.

Output 4.9.3. Iteration History

Newton-Raphson Ridge Optimization

Without Parameter Scaling

Iter	Rest arts	Func Calls	Act Con	Objective Function	Obj Fun Change	Max Abs Gradient Element	Ridge	Actual Over Pred Change
1	0	2	4	40.30303	0.3433	1.23E-16	0	0.508

Optimization Results

Iterations	1	Function Calls	3
Hessian Calls	2	Active Constraints	4
Objective Function	40.303030303	Max Abs Gradient Element	1.233822E-16
Ridge	0	Actual Over Pred Change	0.5083585587

ABSGCONV convergence criterion satisfied.

Output 4.9.4. Optimization Results

PROC NLP: Nonlinear Minimization

Optimization Results
Parameter Estimates

N	Parameter	Estimate	Gradient Objective Function	Active Bound Constraint
1	x12	2.500000	5.777778	
2	x13	2.500000	5.702479	
3	x32	2.036945E-17	1.000000	Lower BC
4	x24	2.500000	5.702479	
5	x34	2.500000	5.777778	

Value of Objective Function = 40.303030303

The active constraints and corresponding Lagrange multiplier estimates (costs) are given in Output 4.9.5 and Output 4.9.6, respectively.

Output 4.9.5. Linear Constraints at Solution

PROC NLP: Nonlinear Minimization

Linear Constraints Evaluated at Solution

```
1 ACT -8.882E-16  =         0  +    1.0000 * x13  -    1.0000 * x32  -
    1.0000 * x34
2 ACT           0  =         0  +    1.0000 * x12  +    1.0000 * x32  -
    1.0000 * x24
3 ACT 1.7764E-15  =   -5.0000  +    1.0000 * x24  +    1.0000 * x34
```

Output 4.9.6. Lagrange Multipliers at Solution

First Order Lagrange Multipliers

Active Constraint		Lagrange Multiplier
Lower BC	x32	0.924702
Linear EC	[1]	5.702479
Linear EC	[2]	5.777778
Linear EC	[3]	11.480257

Output 4.9.7 shows that the projected gradient is very small, satisfying the first-order optimality criterion.

Output 4.9.7. Projected Gradient at Solution

Projected Gradient

Free Dimension	Projected Gradient
1	1.233822E-16

The projected Hessian matrix (shown in Output 4.9.8) is positive definite, satisfying the second-order optimality criterion.

Output 4.9.8. Projected Hessian at Solution

Projected Hessian Matrix	
	X1
X1	1.535309013

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Chapter 5
The NETFLOW Procedure

Chapter Contents

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Overview: NETFLOW Procedure

Introduction

Constrained network models can be used to describe a wide variety of real-world applications ranging from production, inventory, and distribution problems to financial applications. These problems can be solved with the NETFLOW procedure.

These models are conceptually easy since they are based on network diagrams that represent the problem pictorially. PROC NETFLOW accepts the network specification in a format that is particularly suited to networks. This not only simplifies problem description but also aids in the interpretation of the solution.

Certain algebraic features of networks are exploited by a specialized version of the simplex method so that solution times are reduced. Another optimization algorithm, the interior point algorithm, has been implemented in PROC NETFLOW and can be used as an alternative to the simplex algorithm to solve network problems.

Should PROC NETFLOW detect there are no arcs and nodes in the model's data, (that is, there is no network component), it assumes it is dealing with a linear programming (LP) problem. The interior point algorithm is automatically selected to perform the optimization.

Network Models

A network consists of a collection of nodes joined by a collection of arcs. The arcs connect nodes and convey flow of one or more commodities that are supplied at supply nodes and demanded at demand nodes in the network. Each arc has a cost per unit of flow, a flow capacity, and a lower flow bound associated with it. An important concept in network modeling is *conservation of flow*. Conservation of flow means that the total flow in arcs directed toward a node, plus the supply at the node, minus the demand at the node, equals the total flow in arcs directed away from the node.

A network and its associated data can be described in SAS data sets. PROC NETFLOW uses this description and finds the flow through each arc in the network that minimizes the total cost of flow, meets the demand at demand nodes using the supply at supply nodes so that the flow through each arc is on or between the arc's lower flow bound and its capacity, and satisfies the conservation of flow.

One class of network models is the production-inventory-distribution problem. The diagram in Figure 5.1 illustrates this problem. The subscripts on the Production, Inventory, and Sales nodes indicate the time period. Notice that if you replicate sections of the model, the notion of time can be included.

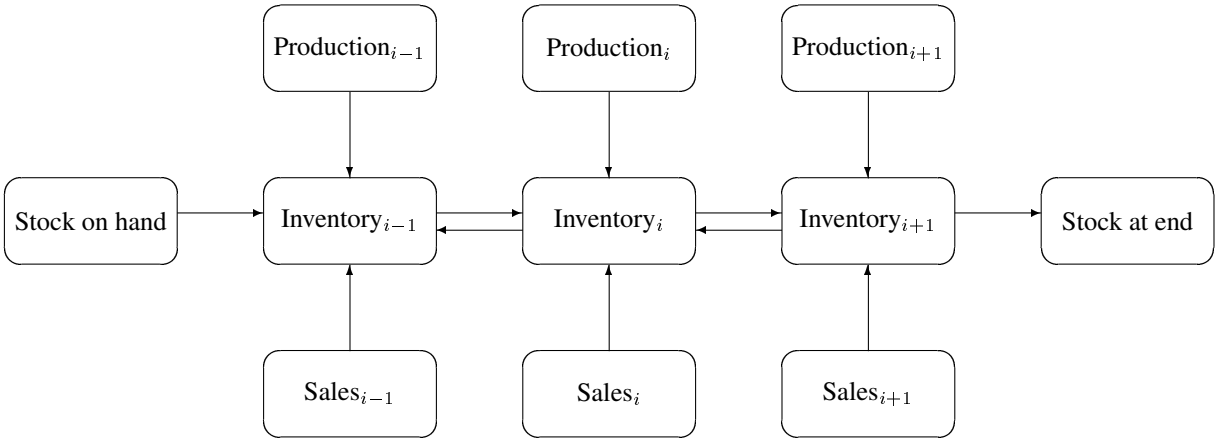


Figure 5.1. Production-Inventory-Distribution Problem

In this type of model, the nodes can represent a wide variety of facilities. Several examples are suppliers, spot markets, importers, farmers, manufacturers, factories, parts of a plant, production lines, waste disposal facilities, workstations, warehouses, coolstores, depots, wholesalers, export markets, ports, rail junctions, airports, road intersections, cities, regions, shops, customers, and consumers. The diversity of this selection demonstrates the richness of potential applications of this model.

Depending upon the interpretation of the nodes, the objectives of the modeling exercise can vary widely. Some common types of objectives are

- to reduce collection or purchase costs of raw materials
- to reduce inventory holding or backorder costs. Warehouses and other storage facilities sometimes have capacities, and there can be limits on the amount of goods that can be placed on backorder.
- to decide where facilities should be located and what the capacity of these should be. Network models have been used to help decide where factories, hospitals, ambulance and fire stations, oil and water wells, and schools should be sited.
- to determine the assignment of resources (machines, production capability, workforce) to tasks, schedules, classes, or files
- to determine the optimal distribution of goods or services. This usually means minimizing transportation costs, and reducing time in transit or distances covered.
- to find the shortest path from one location to another
- to ensure that demands (for example, production requirements, market demands, contractual obligations) are met
- to maximize profits from the sale of products or the charge for services
- to maximize production by identifying bottlenecks

Some specific applications are

- car distribution models. These help determine which models and numbers of cars should be manufactured in which factories and where to distribute cars from these factories to zones in the United States in order to meet customer demand at least cost.
- models in the timber industry. These help determine when to plant and mill forests, schedule production of pulp, paper and wood products, and distribute products for sale or export.
- military applications. The nodes can be theatres, bases, ammunition dumps, logistical suppliers, or radar installations. Some models are used to find the best ways to mobilize personnel and supplies and to evacuate the wounded in the least amount of time.
- communications applications. The nodes can be telephone exchanges, transmission lines, satellite links, and consumers. In a model of an electrical grid, the nodes can be transformers, powerstations, watersheds, reservoirs, dams, and consumers. Of concern might be the effect of high loads or outages.

Side Constraints

Often all the details of a problem cannot be specified in a network model alone. In many of these cases, these details can be represented by the addition of side constraints to the model. Side constraints are a linear function of arc variables (variables containing flow through an arc) and nonarc variables (variables that are not part of the network). This enhancement to the basic network model allows for very general problems. In fact, any linear program can be represented with network models having these types of side constraints. The examples that follow help to clarify the notion of side constraints.

PROC NETFLOW enables you to specify side constraints. The data for a side constraint consist of coefficients of arcs and coefficients of nonarc variables, a constraint type (that is, $\leq$, $=$, or $\geq$) and a right-hand-side value (rhs). A nonarc variable has a name, an objective function coefficient analogous to an arc cost, an upper bound analogous to an arc capacity, and a lower bound analogous to an arc lower flow bound. PROC NETFLOW finds the flow through the network and the values of any nonarc variables that minimize the total cost of the solution. Flow conservation is met, flow through each arc is on or between the arc's lower flow bound and capacity, the value of each nonarc variable is on or between the nonarc's lower and upper bounds, and the side constraints are satisfied. Note that, since many linear programs have large embedded networks, PROC NETFLOW is an attractive alternative to the LP procedure in many cases.

In order for arcs to be specified in side constraints, they must be named. By default, PROC NETFLOW names arcs using the names of the nodes at the head and tail of the arc. An arc is named with its tail node name followed by an underscore and its head node name. For example, an arc from node *from* to node *to* is called *from_to*.

Proportionality Constraints

Side constraints in network models fall into several categories that have special structure. They are frequently used when the flow through an arc must be proportional to the flow through another arc. Such constraints are called *proportionality constraints* and are useful in models where production is subject to refining or modification into different materials. The amount of each output, or any waste, evaporation, or reduction can be specified as a proportion of input.

Typically the arcs near the supply nodes carry raw materials and the arcs near the demand nodes carry refined products. For example, in a model of the milling industry, the flow through some arcs may represent quantities of wheat. After the wheat is processed, the flow through other arcs might be flour. For others it might be bran. The side constraints model the relationship between the amount of flour or bran produced as a proportion of the amount of wheat milled. Some of the wheat can end up as neither flour, bran, nor any useful product, so this waste is drained away via arcs to a waste node.

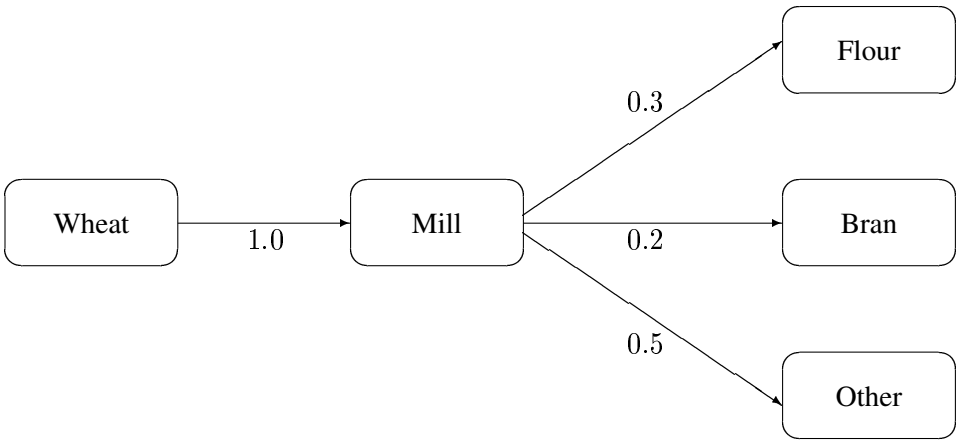


Figure 5.2. Proportionality Constraints

Consider the network fragment in Figure 5.2. The arc Wheat_Mill conveys the wheat milled. The cost of flow on this arc is the milling cost. The capacity of this arc is the capacity of the mill. The lower flow bound on this arc is the minimum quantity that must be milled for the mill to operate economically. The constraints

$$0.3\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Flour} = 0.0$$
$$0.2\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Bran} = 0.0$$

force every unit of wheat that is milled to produce 0.3 units of flour and 0.2 units of bran. Note that it is not necessary to specify the constraint

$$0.5\,\mathsf{Wheat\_Mill} - \mathsf{Mill\_Other} = 0.0$$

since flow conservation implies that any flow that does not traverse through Mill_Flour or Mill_Bran must be conveyed through Mill_Other. And, computationally, it is better if this constraint is not specified, since there is one less side constraint

and fewer problems with numerical precision. Notice that the sum of the proportions must equal 1.0 exactly; otherwise, flow conservation is violated.

Blending Constraints

Blending or quality constraints can also influence the recipes or proportions of ingredients that are mixed. For example, different raw materials can have different properties. In an application of the oil industry, the amount of products that are obtained could be different for each type of crude oil. Furthermore, fuel might have a minimum octane requirement or limited sulphur or lead content, so that a blending of crudes is needed to produce the product.

The network fragment in Figure 5.3 shows an example of this.

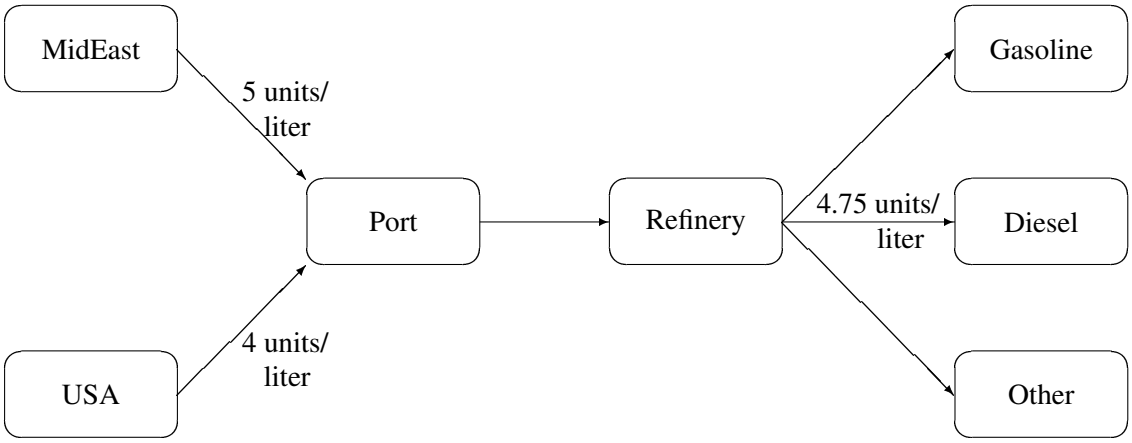


Figure 5.3. Blending Constraints

The arcs MidEast_Port and USA_Port convey crude oil from the two sources. The arc Port_Refinery represents refining while the arcs Refinery_Gasoline and Refinery_Diesel carry the gas and diesel produced. The proportionality constraints

$$0.4\,\mathsf{Port\_Refinery} - \mathsf{Refinery\_Gasoline} = 0.0$$
$$0.2\,\mathsf{Port\_Refinery} - \mathsf{Refinery\_Diesel} = 0.0$$

capture the restrictions for producing gasoline and diesel from crude. Suppose that, if only crude from the Middle East is used, the resulting diesel would contain 5 units of sulphur per liter. If only crude from the USA is used, the resulting diesel would contain 4 units of sulphur per liter. Diesel can have at most 4.75 units of sulphur per liter. Some crude from the USA must be used if Middle East crude is used in order to meet the 4.75 sulphur per liter limit. The side constraint to model this requirement is

$$5\,\mathsf{MidEast\_Port} + 4\,\mathsf{USA\_Port} - 4.75\,\mathsf{Port\_Refinery} \leq 0.0$$

Since $\mathsf{Port\_Refinery} = \mathsf{MidEast\_Port} + \mathsf{USA\_Port}$, flow conservation allows this constraint to be simplified to

$$1\,\mathsf{MidEast\_Port} - 3\,\mathsf{USA\_Port} \leq 0.0$$

If, for example, 120 units of crude from the Middle East is used, then at least 40 units of crude from the USA must be used. The preceding constraint is simplified because you assume that the sulphur concentration of diesel is proportional to the sulphur concentration of the crude mix. If this is not the case, the relation

$$0.2\,\mathsf{Port\_Refinery} = \mathsf{Refinery\_Diesel}$$

is used to obtain

$$5\,\mathsf{MidEast\_Port} + 4\,\mathsf{USA\_Port} - 4.75\,(1.0/0.2\ \ \mathsf{Refinery\_Diesel}) \leq 0.0$$

which equals

$$5\,\mathsf{MidEast\_Port} + 4\,\mathsf{USA\_Port} - 23.75\,\mathsf{Refinery\_Diesel} \leq 0.0$$

An example similar to this Oil Industry problem is solved in the section "Introductory Example" on page 447.

Multicommodity Problems

Side constraints are also used in models in which there are capacities on transportation or some other shared resource, or there are limits on overall production or demand in multicommodity, multidivisional or multiperiod problems. Each commodity, division or period can have a separate network coupled to one main system by the side constraints. Side constraints are used to combine the outputs of subdivisions of a problem (either commodities, outputs in distinct time periods, or different process streams) to meet overall demands or to limit overall production or expenditures. This method is more desirable than doing separate *local* optimizations for individual commodity, process, or time networks and then trying to establish relationships between each when determining an overall policy if the *global* constraint is not satisfied. Of course, to make models more realistic, side constraints may be necessary in the local problems.

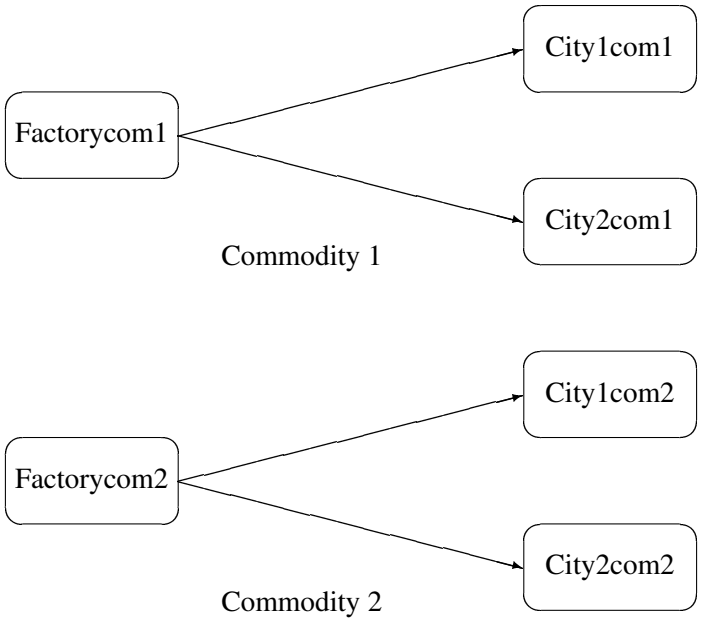


Figure 5.4. Multicommodity Problem

Figure 5.4 shows two network fragments. They represent identical production and distribution sites of two different commodities. Suffix *com1* represents commodity 1 and suffix *com2* represents commodity 2. The nodes Factorycom1 and Factorycom2 model the same factory, and nodes City1com1 and City1com2 model the same location, city 1. Similarly, City2com1 and City2com2 are the same location, city 2. Suppose that commodity 1 occupies 2 cubic meters, commodity 2 occupies 3 cubic meters, the truck dispatched to city 1 has a capacity of 200 cubic meters, and the truck dispatched to city 2 has a capacity of 250 cubic meters. How much of each commodity can be loaded onto each truck? The side constraints for this case are

$$2\,\text{Factorycom1\_City1com1} + 3\,\text{Factorycom2\_City1com2} \leq 200$$
$$2\,\text{Factorycom1\_City2com1} + 3\,\text{Factorycom2\_City2com2} \leq 250$$

Large Modeling Strategy

In many cases, the flow through an arc might actually represent the flow or movement of a commodity from place to place or from time period to time period. However, sometimes an arc is included in the network as a method of capturing some aspect of the problem that you would not normally think of as part of a network model. For example, in a multiprocess, multiproduct model (Figure 5.5), there might be subnetworks for each process and each product. The subnetworks can be joined together by a set of arcs that have flows that represent the amount of product j produced by process i. To model an upper limit constraint on the total amount of product j that can be produced, direct all arcs carrying product j to a single node and from there through a single arc. The capacity of this arc is the upper limit of product j production. It is preferable to model this structure in the network rather than to include it in the side constraints because the efficiency of the optimizer is affected less by a reasonable increase in the size of the network.

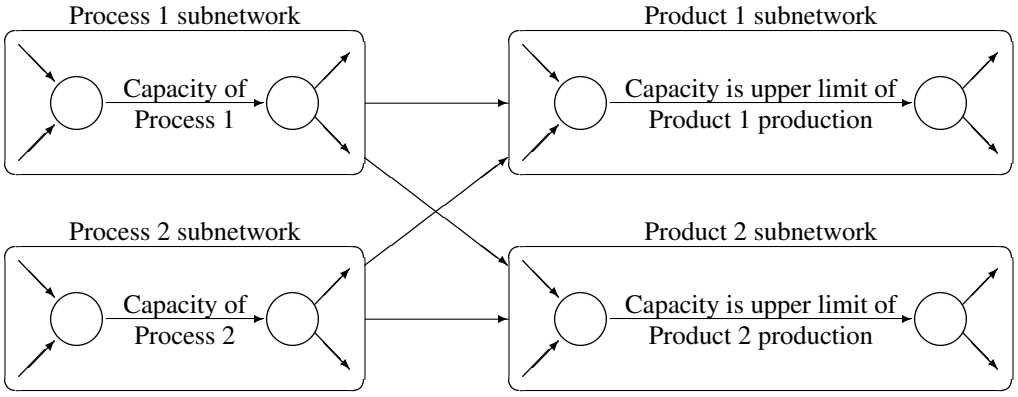


Figure 5.5. Multiprocess, Multiproduct Example

It is often a good strategy when starting a project to use a small network formulation and then use that model as a framework upon which to add detail. For example, in the multiprocess, multiproduct model, you might start with the network depicted in Figure 5.5. Then, for example, the process subnetwork can be enhanced to include the distribution of products. Other phases of the operation could be included by adding more subnetworks. Initially, these subnetworks can be single nodes, but in subsequent studies they can be expanded to include greater detail.

The NETFLOW procedure accepts the side constraints in the same dense and sparse formats that the LP procedure provides. Although PROC LP can solve network problems, the NETFLOW procedure generally solves network flow problems more efficiently than PROC LP.

Advantages of Network Models over LP Models

Many linear programming problems have large embedded network structures. Such problems often result when modeling manufacturing processes, transportation or distribution networks, or resource allocation, or when deciding where to locate facilities. Often, some commodity is to be moved from place to place, so the more natural formulation in many applications is that of a constrained network rather than a linear program.

Using a network diagram to visualize a problem makes it possible to capture the important relationships in an easily understood picture form. The network diagram aids the communication between model builder and model user, making it easier to comprehend how the model is structured, how it can be changed, and how results can be interpreted.

If a network structure is embedded in a linear program, the problem is a network programming problem with side constraints (NPSC). When the network part of the problem is large compared to the nonnetwork part, especially if the number of side constraints is small, it is worthwhile to exploit this structure in the solution process. This is what PROC NETFLOW does. It uses a variant of the revised primal simplex algorithm that exploits the network structure to reduce solution time.

Mathematical Description of NPSC

If a network programming problem with side constraints has n nodes, a arcs, g nonarc variables, and k side constraints, then the formal statement of the problem solved by PROC NETFLOW is

$$\begin{array}{ll} \text{minimize} & c^T x + d^T z \\ \text{subject to} & Fx = b \\ & Hx + Qz \geq, =, \leq r \\ & l \leq x \leq u \\ & m \leq z \leq v \end{array}$$

where

- c is the $a \times 1$ arc variable objective function coefficient vector (the cost vector)
- x is the $a \times 1$ arc variable value vector (the flow vector)
- d is the $g \times 1$ nonarc variable objective function coefficient vector
- z is the $g \times 1$ nonarc variable value vector
- F is the $n \times a$ node-arc incidence matrix of the network, where

$$F_{i,j} = \begin{cases} -1, & \text{if arc } j \text{ is directed from node } i \\ 1, & \text{if arc } j \text{ is directed toward node } i \\ 0, & \text{otherwise} \end{cases}$$

- b is the $n \times 1$ node supply/demand vector, where

$$b_i = \begin{cases} s, & \text{if node } i \text{ has supply capability of } s \text{ units of flow} \\ -d, & \text{if node } i \text{ has demand of } d \text{ units of flow} \\ 0, & \text{if node } i \text{ is a trans-shipment node} \end{cases}$$

- H is the $k \times a$ side constraint coefficient matrix for arc variables, where $H_{i,j}$ is the coefficient of arc j in the ith side constraint
- Q is the $k \times g$ side constraint coefficient matrix for nonarc variables, where $Q_{i,j}$ is the coefficient of nonarc j in the ith side constraint
- r is the $k \times 1$ side constraint right-hand-side vector
- l is the $a \times 1$ arc lower flow bound vector
- u is the $a \times 1$ arc capacity vector
- m is the $g \times 1$ nonarc variable lower bound vector
- v is the $g \times 1$ nonarc variable upper bound vector

Flow Conservation Constraints

The constraints $Fx = b$ are referred to as the nodal flow conservation constraints. These constraints algebraically state that the sum of the flow through arcs directed toward a node plus that node's supply, if any, equals the sum of the flow through arcs directed away from that node plus that node's demand, if any. The flow conservation constraints are implicit in the network model and should not be specified explicitly in side constraint data when using PROC NETFLOW. The constrained problems most amenable to being solved by the NETFLOW procedure are those that, after the removal of the flow conservation constraints, have very few constraints. PROC NETFLOW is superior to linear programming optimizers when the network part of the problem is significantly larger than the nonnetwork part.

The NETFLOW procedure can also be used to solve an unconstrained network problem, that is, one in which H, Q, d, r, and z do not exist.

Nonarc Variables

If the constrained problem to be solved has no nonarc variables, then Q, d, and z do not exist. However, nonarc variables can be used to simplify side constraints. For example, if a sum of flows appears in many constraints, it may be worthwhile to equate this expression with a nonarc variable and use this in the other constraints. By assigning a nonarc variable a nonzero objective function, it is then possible to incur a cost for using resources above some lowest feasible limit. Similarly, a profit (a negative objective function coefficient value) can be made if all available resources are not used.

In some models, nonarc variables are used in constraints to absorb excess resources or supply needed resources. Then, either the excess resource can be used or the needed resource can be supplied to another component of the model.

For example, consider a multicommodity problem of making television sets that have either 19- or 25-inch screens. In their manufacture, 3 and 4 chips, respectively, are used. Production occurs at 2 factories during March and April. The supplier of chips can supply only 2600 chips to factory 1 and 3750 chips to factory 2 each month. The names of arcs are in the form Prod*n_s_m* , where *n* is the factory number, *s* is the screen size, and *m* is the month. For example, Prod1_25_Apr is the arc that conveys the number of 25-inch TVs produced in factory 1 during April. You might have to determine similar systematic naming schemes for your application.

As described, the constraints are

$$
\begin{aligned}
3\,\text{Prod1\_19\_Mar} + 4\,\text{Prod1\_25\_Mar} &\leq 2600 \\
3\,\text{Prod2\_19\_Mar} + 4\,\text{Prod2\_25\_Mar} &\leq 3750 \\
3\,\text{Prod1\_19\_Apr} + 4\,\text{Prod1\_25\_Apr} &\leq 2600 \\
3\,\text{Prod2\_19\_Apr} + 4\,\text{Prod2\_25\_Apr} &\leq 3750
\end{aligned}
$$

If there are chips that could be obtained for use in March but not used for production in March, why not keep these unused chips until April? Furthermore, if the March

excess chips at factory 1 could be used either at factory 1 or factory 2 in April, the model becomes

$$
\begin{aligned}
3\,\mathsf{Prod1\_19\_Mar} + 4\,\mathsf{Prod1\_25\_Mar} + \mathsf{F1\_Unused\_Mar} &= 2600 \\
3\,\mathsf{Prod2\_19\_Mar} + 4\,\mathsf{Prod2\_25\_Mar} + \mathsf{F2\_Unused\_Mar} &= 3750 \\
3\,\mathsf{Prod1\_19\_Apr} + 4\,\mathsf{Prod1\_25\_Apr} - \mathsf{F1\_Kept\_Since\_Mar} &= 2600 \\
3\,\mathsf{Prod2\_19\_Apr} + 4\,\mathsf{Prod2\_25\_Apr} - \mathsf{F2\_Kept\_Since\_Mar} &= 3750 \\
\mathsf{F1\_Unused\_Mar} + \mathsf{F2\_Unused\_Mar} \text{ (continued)} & \\
- \mathsf{F1\_Kept\_Since\_Mar} - \mathsf{F2\_Kept\_Since\_Mar} &\geq 0.0
\end{aligned}
$$

where F1_Kept_Since_Mar is the number of chips used during April at factory 1 that were obtained in March at either factory 1 or factory 2 and F2_Kept_Since_Mar is the number of chips used during April at factory 2 that were obtained in March. The last constraint ensures that the number of chips used during April that were obtained in March does not exceed the number of chips not used in March. There may be a cost to hold chips in inventory. This can be modeled having a positive objective function coefficient for the nonarc variables F1_Kept_Since_Mar and F2_Kept_Since_Mar. Moreover, nonarc variable upper bounds represent an upper limit on the number of chips that can be held in inventory between March and April.

See Example 5.4 through Example 5.8 for a series of examples that use this TV problem. The use of nonarc variables as described previously is illustrated.

Warm Starts

If you have a problem that has already been partially solved and is to be solved further to obtain a better, optimal solution, information describing the solution now available may be used as an initial solution. This is called *warm starting* the optimization, and the supplied solution data are called the *warm start*.

Some data can be changed between the time when a warm start is created and when it is used as a warm start for a subsequent PROC NETFLOW run. Elements in the arc variable cost vector, the nonarc variable objective function coefficient vector, and sometimes capacities, upper value bounds, and side constraint data can be changed between PROC NETFLOW calls. See the section "Warm Starts" on page 542. Also, see Example 5.4 through Example 5.8 (the TV problem) for a series of examples that show the use of warm starts.

Getting Started: NETFLOW Procedure

To solve network programming problems with side constraints using PROC NETFLOW, you save a representation of the network and the side constraints in three SAS data sets. These data sets are then passed to PROC NETFLOW for solution. There are various forms that a problem's data can take. You can use any one or a combination of several of these forms.

The NODEDATA= data set contains the names of the supply and demand nodes and the supply or demand associated with each. These are the elements in the column vector b in problem (NPSC).

The ARCDATA= data set contains information about the variables of the problem. Usually these are arcs, but there can be data related to nonarc variables in the ARCDATA= data set as well.

An arc is identified by the names of its tail node (where it originates) and head node (where it is directed). Each observation can be used to identify an arc in the network and, optionally, the cost per flow unit across the arc, the arc's capacity, lower flow bound, and name. These data are associated with the matrix F and the vectors c, l, and u in problem (NPSC).

Note: Although F is a node-arc incidence matrix, it is specified in the ARCDATA= data set by arc definitions.

In addition, the ARCDATA= data set can be used to specify information about nonarc variables, including objective function coefficients, lower and upper value bounds, and names. These data are the elements of the vectors d, m, and v in problem (NPSC). Data for an arc or nonarc variable can be given in more than one observation.

Supply and demand data also can be specified in the ARCDATA= data set. In such a case, the NODEDATA= data set may not be needed.

The CONDATA= data set describes the side constraints and their right-hand sides. These data are elements of the matrices H and Q and the vector r. Constraint types are also specified in the CONDATA= data set. You can include in this data set upper bound values or capacities, lower flow or value bounds, and costs or objective function coefficients. It is possible to give all information about some or all nonarc variables in the CONDATA= data set.

An arc is identified in this data set by its name. If you specify an arc's name in the ARCDATA= data set, then this name is used to associate data in the CONDATA= data set with that arc. Each arc also has a default name that is the name of the tail and head node of the arc concatenated together and separated by an underscore character; tail_head, for example.

If you use the dense side constraint input format (described in the section "CONDATA= Data Set" on page 515) and want to use the default arc names, these arc names are names of SAS variables in the VAR list of the CONDATA= data set.

If you use the sparse side constraint input format (see the section "CONDATA= Data Set" on page 515) and want to use the default arc names, these arc names are values of the COLUMN list SAS variable of the CONDATA= data set.

The execution of PROC NETFLOW has three stages. In the preliminary (zeroth) stage, the data are read from the NODEDATA= data set, the ARCDATA= data set, and the CONDATA= data set. Error checking is performed, and an initial basic feasible solution is found. If an unconstrained solution warm start is being used, then an initial basic feasible solution is obtained by reading additional data containing that information in the NODEDATA= data set and the ARCDATA= data set. In this case, only constraint data and nonarc variable data are read from the CONDATA= data set.

In the first stage, an optimal solution to the network flow problem neglecting any side constraints is found. The primal and dual solutions for this relaxed problem can be saved in the ARCOUT= data set and the NODEOUT= data set, respectively. These data sets are named in the PROC NETFLOW, RESET, and SAVE statements.

In the second stage, an optimal solution to the network flow problem with side constraints is found. The primal and dual solutions for this side constrained problem are saved in the CONOUT= data set and the DUALOUT= data set, respectively. These data sets are also named in the PROC NETFLOW, RESET, and SAVE statements.

If a constrained solution warm start is being used, PROC NETFLOW does not perform the zeroth and first stages. This warm start can be obtained by reading basis data containing additional information in the NODEDATA= data set (also called the DUALIN= data set) and the ARCDATA= data set.

If warm starts are to be used in future optimizations, the FUTURE1 and FUTURE2 options must be used in addition to specifying names for the data sets that contain the primal and dual solutions in stages one and two. Then, most of the information necessary for restarting problems is available in the output data sets containing the primal and dual solutions of both the relaxed and side constrained network programs.

Introductory Example

Consider the following trans-shipment problem for an oil company. Crude oil is shipped to refineries where it is processed into gasoline and diesel fuel. The gasoline and diesel fuel are then distributed to service stations. At each stage, there are shipping, processing, and distribution costs. Also, there are lower flow bounds and capacities.

In addition, there are two sets of side constraints. The first set is that two times the crude from the Middle East cannot exceed the throughput of a refinery plus 15 units. (The phrase "plus 15 units" that finishes the last sentence is used to enable some side constraints in this example to have a nonzero rhs.) The second set of constraints are necessary to model the situation that one unit of crude mix processed at a refinery yields three-fourths of a unit of gasoline and one-fourth of a unit of diesel fuel.

Because there are two products that are not independent in the way in which they flow through the network, a network programming problem with side constraints is an appropriate model for this example (see Figure 5.6). The side constraints are used

to model the limitations on the amount of Middle Eastern crude that can be processed by each refinery and the conversion proportions of crude to gasoline and diesel fuel.

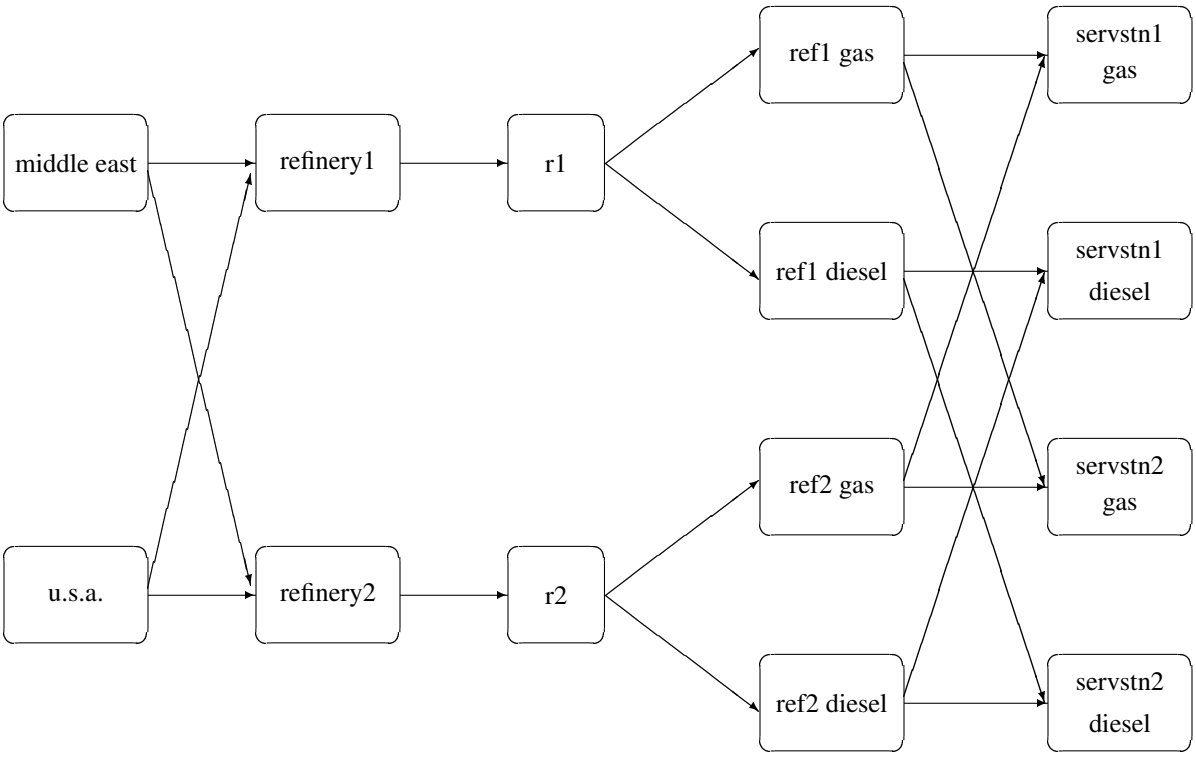


Figure 5.6. Oil Industry Example

To solve this problem with PROC NETFLOW, save a representation of the model in three SAS data sets. In the NODEDATA= data set, you name the supply and demand nodes and give the associated supplies and demands. To distinguish demand nodes from supply nodes, specify demands as negative quantities. For the oil example, the NODEDATA= data set can be saved as follows:

```
title  'Oil Industry Example';
title3 'Setting Up Nodedata = Noded For Proc Netflow';
data noded;
   input   _node_&$15. _sd_;
   datalines;
middle east       100
u.s.a.             80
servstn1 gas      -95
servstn1 diesel   -30
servstn2 gas      -40
servstn2 diesel   -15
;
```

The ARCDATA= data set contains the rest of the information about the network. Each observation in the data set identifies an arc in the network and gives the cost per

flow unit across the arc, the capacities of the arc, the lower bound on flow across the arc, and the name of the arc.

```
title3 'Setting Up Arcdata = Arcd1 For Proc Netflow';
data arcd1;
   input _from_&$11. _to_&$15. _cost_ _capac_ _lo_ _name_ $;
   datalines;
middle east     refinery 1        63     95   20    m_e_ref1
middle east     refinery 2        81     80   10    m_e_ref2
u.s.a.          refinery 1        55      .    .    .
u.s.a.          refinery 2        49      .    .    .
refinery 1      r1               200    175   50    thruput1
refinery 2      r2               220    100   35    thruput2
r1              ref1 gas           .    140    .    r1_gas
r1              ref1 diesel        .     75    .    .
r2              ref2 gas           .    100    .    r2_gas
r2              ref2 diesel        .     75    .    .
ref1 gas        servstn1 gas      15     70    .    .
ref1 gas        servstn2 gas      22     60    .    .
ref1 diesel     servstn1 diesel   18      .    .    .
ref1 diesel     servstn2 diesel   17      .    .    .
ref2 gas        servstn1 gas      17     35    5    .
ref2 gas        servstn2 gas      31      .    .    .
ref2 diesel     servstn1 diesel   36      .    .    .
ref2 diesel     servstn2 diesel   23      .    .    .
;
```

Finally, the CONDATA= data set contains the side constraints for the model.

```
title3 'Setting Up Condata = Cond1 For Proc Netflow';
data cond1;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
         _type_ $ _rhs_;
   datalines;
-2   .   1 .   . . >= -15
 . -2   . .   1 . GE -15
 .   . -3 4   . . EQ   0
 .   .   . . -3 4  =   0
;
```

Note that the SAS variable names in the CONDATA= data set are the names of arcs given in the ARCDATA= data set. These are the arcs that have nonzero constraint coefficients in side constraints. For example, the proportionality constraint that specifies that one unit of crude at each refinery yields three-fourths of a unit of gasoline and one-fourth of a unit of diesel fuel is given for REFINERY 1 in the third observation and for REFINERY 2 in the last observation. The third observation requires that each unit of flow on arc THRUPUT1 equals three-fourths of a unit of flow on arc R1_GAS. Because all crude processed at REFINERY 1 flows through THRUPUT1 and all gasoline produced at REFINERY 1 flows through R1_GAS, the constraint models the situation. It proceeds similarly for REFINERY 2 in the last observation.

To find the minimum cost flow through the network that satisfies the supplies, demands, and side constraints, invoke PROC NETFLOW as follows:

```
proc netflow
   nodedata=noded        /* the supply and demand data */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
```

The following messages, which appear on the SAS log, summarize the model as read by PROC NETFLOW and note the progress toward a solution:

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 180 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 8 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 50600 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of arc and nonarc variable side constraint
      coefficients= 8 .
NOTE: Number of iterations, optimizing with constraints= 4 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= 50875 .
NOTE: The data set WORK.SOLUTION has 18 observations and 14
      variables.
```

Unlike PROC LP, which displays the solution and other information as output, PROC NETFLOW saves the optimum in output SAS data sets that you specify. For this example, the solution is saved in the SOLUTION data set. It can be displayed with the PRINT procedure as

```
proc print data=solution;
   var _from_ _to_ _cost_ _capac_ _lo_ _name_
       _supply_ _demand_ _flow_ _fcost_ _rcost_;
   sum _fcost_;
   title3 'Constrained Optimum';
   run;
```

Constrained Optimum

Obs	_from_	_to_	_cost_	_capac_	_lo_	_name_
1	refinery 1	r1	200	175	50	thruput1
2	refinery 2	r2	220	100	35	thruput2
3	r1	ref1 diesel	0	75	0	
4	r1	ref1 gas	0	140	0	r1_gas
5	r2	ref2 diesel	0	75	0	
6	r2	ref2 gas	0	100	0	r2_gas
7	middle east	refinery 1	63	95	20	m_e_ref1
8	u.s.a.	refinery 1	55	99999999	0	
9	middle east	refinery 2	81	80	10	m_e_ref2
10	u.s.a.	refinery 2	49	99999999	0	
11	ref1 diesel	servstn1 diesel	18	99999999	0	
12	ref2 diesel	servstn1 diesel	36	99999999	0	
13	ref1 gas	servstn1 gas	15	70	0	
14	ref2 gas	servstn1 gas	17	35	5	
15	ref1 diesel	servstn2 diesel	17	99999999	0	
16	ref2 diesel	servstn2 diesel	23	99999999	0	
17	ref1 gas	servstn2 gas	22	60	0	
18	ref2 gas	servstn2 gas	31	99999999	0	

Obs	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_	_RCOST_
1	.	.	145.00	29000.00	.
2	.	.	35.00	7700.00	29
3	.	.	36.25	0.00	.
4	.	.	108.75	0.00	.
5	.	.	8.75	0.00	.
6	.	.	26.25	0.00	.
7	100	.	80.00	5040.00	.
8	80	.	65.00	3575.00	.
9	100	.	20.00	1620.00	.
10	80	.	15.00	735.00	.
11	.	30	30.00	540.00	.
12	.	30	0.00	0.00	12
13	.	95	68.75	1031.25	.
14	.	95	26.25	446.25	.
15	.	15	6.25	106.25	.
16	.	15	8.75	201.25	.
17	.	40	40.00	880.00	.
18	.	40	0.00	0.00	7
				========	
				50875.00	

Figure 5.7. CONOUT=SOLUTION

Notice that, in CONOUT=SOLUTION (Figure 5.7), the optimal flow through each arc in the network is given in the variable named _FLOW_, and the cost of flow through each arc is given in the variable _FCOST_.

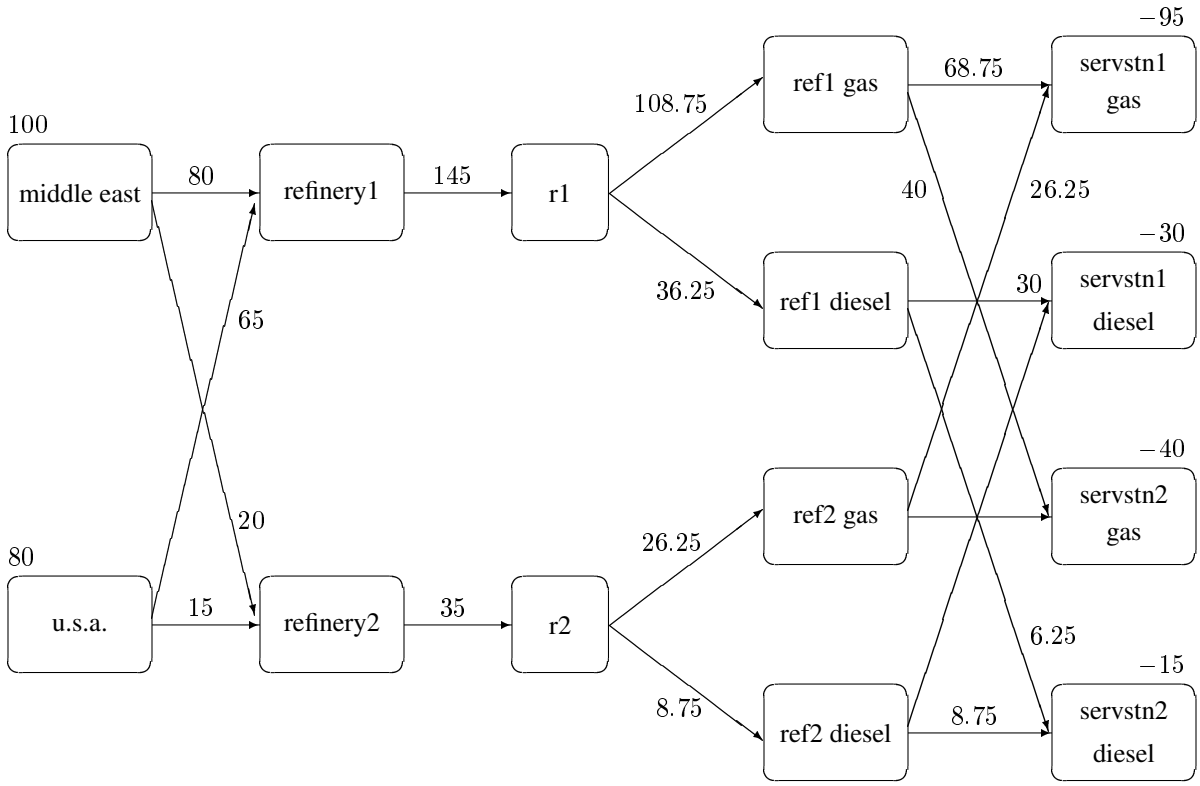


Figure 5.8. Oil Industry Solution

Syntax: NETFLOW Procedure

Below are statements used in PROC NETFLOW, listed in alphabetical order as they appear in the text that follows.

PROC NETFLOW *options* ;
CAPACITY *variable* ;
COEF *variables* ;
COLUMN *variable* ;
CONOPT;
COST *variable* ;
DEMAND *variable* ;
HEADNODE *variable* ;
ID *variables* ;
LO *variable* ;
NAME *variable* ;
NODE *variable* ;
PIVOT;
PRINT *options* ;
QUIT;
RESET *options* ;
RHS *variables* ;

ROW *variables* ;
RUN;
SAVE *options* ;
SHOW *options* ;
SUPDEM *variable* ;
SUPPLY *variable* ;
TAILNODE *variable* ;
TYPE *variable* ;
VAR *variables* ;

Functional Summary

The following table outlines the options available for the NETFLOW procedure classified by function.

Table 5.1. Functional Summary

Description	Statement	Option
Input Data Set Options:		
arcs input data set	PROC NETFLOW	ARCDATA=
nodes input data set	PROC NETFLOW	NODEDATA=
constraint input data set	PROC NETFLOW	CONDATA=
Output Data Set Options:		
unconstrained primal solution data set	PROC NETFLOW	ARCOUT=
unconstrained dual solution data set	PROC NETFLOW	NODEOUT=
constrained primal solution data set	PROC NETFLOW	CONOUT=
constrained dual solution data set	PROC NETFLOW	DUALOUT=
Data Set Read Options:		
CONDATA has sparse data format	PROC NETFLOW	SPARSECONDATA
default constraint type	PROC NETFLOW	DEFCONTYPE=
special COLUMN variable value	PROC NETFLOW	TYPEOBS=
special COLUMN variable value	PROC NETFLOW	RHSOBS=
used to interpret arc and nonarc variable names	PROC NETFLOW	NAMECTRL=
no new nonarc variables	PROC NETFLOW	SAME_NONARC_DATA
no nonarc data in ARCDATA	PROC NETFLOW	ARCS_ONLY_ARCDATA
data for an arc found once in ARCDATA	PROC NETFLOW	ARC_SINGLE_OBS
data for a constraint found once in CONDATA	PROC NETFLOW	CON_SINGLE_OBS
data for a coefficient found once in CONDATA	PROC NETFLOW	NON_REPLIC=
data is grouped, exploited during data read	PROC NETFLOW	GROUPED=
Problem Size Specification Options:		
approximate number of nodes	PROC NETFLOW	NNODES=
approximate number of arcs	PROC NETFLOW	NARCS=
approximate number of nonarc variables	PROC NETFLOW	NNAS=
approximate number of coefficients	PROC NETFLOW	NCOEFS=

Description	Statement	Option
approximate number of constraints	PROC NETFLOW	NCONS=
Network Options:		
default arc cost	PROC NETFLOW	DEFCOST=
default arc capacity	PROC NETFLOW	DEFCAPACITY=
default arc lower flow bound	PROC NETFLOW	DEFMINFLOW=
network's only supply node	PROC NETFLOW	SOURCE=
SOURCE's supply capability	PROC NETFLOW	SUPPLY=
network's only demand node	PROC NETFLOW	SINK=
SINK's demand	PROC NETFLOW	DEMAND=
convey excess supply/demand through network	PROC NETFLOW	THRUNET
find maximal flow between SOURCE and SINK	PROC NETFLOW	MAXFLOW
cost of bypass arc for MAXFLOW problem	PROC NETFLOW	BYPASSDIVIDE=
find shortest path from SOURCE to SINK	PROC NETFLOW	SHORTPATH
specify generalized networks	PROC NETFLOW	GENNET
specify excess demand or supply	PROC NETFLOW	EXCESS=
Memory Control Options:		
issue memory usage messages to SAS log	PROC NETFLOW	MEMREP
number of bytes to use for main memory	PROC NETFLOW	BYTES=
proportion of memory for arrays	PROC NETFLOW	COREFACTOR=
memory allocated for LU factors	PROC NETFLOW	DWIA=
linked list for updated column	PROC NETFLOW	SPARSEP2
use 2-dimensional array for basis matrix	PROC NETFLOW	INVD_2D
maximum bytes for a single array	PROC NETFLOW	MAXARRAYBYTES=
Simplex Options:		
use big-M instead of two-phase method, stage 1	RESET	BIGM1
use Big-M instead of two-phase method, stage 2	RESET	BIGM2
anti-cycling option	RESET	CYCLEMULT1=
interchange first nonkey with leaving key arc	RESET	INTFIRST
controls working basis matrix inversions	RESET	INVFREQ=
maximum number of L row operations allowed before refactorization	RESET	MAXL=
maximum number of LU factor column updates	RESET	MAXLUUPDATES=
anti-cycling option	RESET	MINBLOCK1=
use first eligible leaving variable, stage 1	RESET	LRATIO1
use first eligible leaving variable, stage 2	RESET	LRATIO2
negates INTFIRST	RESET	NOINTFIRST
negates LRATIO1	RESET	NOLRATIO1
negates LRATIO2	RESET	NOLRATIO2
negates PERTURB1	RESET	NOPERTURB1
anti-cycling option	RESET	PERTURB1
controls working basis matrix refactorization	RESET	REFACTFREQ=
use two-phase instead of big-M method, stage 1	RESET	TWOPHASE1

Description	Statement	Option
use two-phase instead of big-M method, stage 2	RESET	TWOPHASE2
pivot element selection parameter	RESET	U=
zero tolerance, stage 1	RESET	ZERO1=
zero tolerance, stage 2	RESET	ZERO2=
zero tolerance, real number comparisons	RESET	ZEROTOL=
Pricing Options:		
frequency of dual value calculation	RESET	DUALFREQ=
pricing strategy, stage 1	RESET	PRICETYPE1=
pricing strategy, stage 2	RESET	PRICETYPE2=
used when P1SCAN=PARTIAL	RESET	P1NPARTIAL=
controls search for entering candidate, stage 1	RESET	P1SCAN=
used when P2SCAN=PARTIAL	RESET	P2NPARTIAL=
controls search for entering candidate, stage 2	RESET	P2SCAN=
initial queue size, stage 1	RESET	QSIZE1=
initial queue size, stage 2	RESET	QSIZE2=
used when Q1FILLSCAN=PARTIAL	RESET	Q1FILLNPARTIAL=
controls scan when filling queue, stage 1	RESET	Q1FILLSCAN=
used when Q2FILLSCAN=PARTIAL	RESET	Q2FILLNPARTIAL=
controls scan when filling queue, stage 2	RESET	Q2FILLSCAN=
queue size reduction factor, stage 1	RESET	REDUCEQSIZE1=
queue size reduction factor, stage 2	RESET	REDUCEQSIZE2=
frequency of refreshing queue, stage 1	RESET	REFRESHQ1=
frequency of refreshing queue, stage 2	RESET	REFRESHQ2=
Optimization Termination Options:		
pause after stage 1; don't start stage 2	RESET	ENDPAUSE1
pause when feasible, stage 1	RESET	FEASIBLEPAUSE1
pause when feasible, stage 2	RESET	FEASIBLEPAUSE2
maximum number of iterations, stage 1	RESET	MAXIT1=
maximum number of iterations, stage 2	RESET	MAXIT2=
negates ENDPAUSE1	RESET	NOENDPAUSE1
negates FEASIBLEPAUSE1	RESET	NOFEASIBLEPAUSE1
negates FEASIBLEPAUSE2	RESET	NOFEASIBLEPAUSE2
pause every PAUSE1 iterations, stage 1	RESET	PAUSE1=
pause every PAUSE2 iterations, stage 2	RESET	PAUSE2=
Interior Point Algorithm Options:		
use interior point algorithm	PROC NETFLOW	INTPOINT
factorization method	RESET	FACT_METHOD=
allowed amount of dual infeasibility	RESET	TOLDINF=
allowed amount of primal infeasibility	RESET	TOLPINF=
allowed total amount of dual infeasibility	RESET	TOLTOTDINF=
allowed total amount of primal infeasibility	RESET	TOLTOTPINF=
cut-off tolerance for Cholesky factorization	RESET	CHOLTINYTOL=

Description	Statement	Option
density threshold for Cholesky processing	RESET	DENSETHR=
step-length multiplier	RESET	PDSTEPMULT=
preprocessing type	RESET	PRSLTYPE=
print optimization progress on SAS log	RESET	PRINTLEVEL2=
Interior Point Stopping Criteria Options:		
maximum number of interior point iterations	RESET	MAXITERB=
primal-dual (duality) gap tolerance	RESET	PDGAPTOL=
stop because of complementarity	RESET	STOP_C=
stop because of duality gap	RESET	STOP_DG=
stop because of $infeas_b$	RESET	STOP_IB=
stop because of $infeas_c$	RESET	STOP_IC=
stop because of $infeas_d$	RESET	STOP_ID=
stop because of complementarity	RESET	AND_STOP_C=
stop because of duality gap	RESET	AND_STOP_DG=
stop because of $infeas_b$	RESET	AND_STOP_IB=
stop because of $infeas_c$	RESET	AND_STOP_IC=
stop because of $infeas_d$	RESET	AND_STOP_ID=
stop because of complementarity	RESET	KEEPGOING_C=
stop because of duality gap	RESET	KEEPGOING_DG=
stop because of $infeas_b$	RESET	KEEPGOING_IB=
stop because of $infeas_c$	RESET	KEEPGOING_IC=
stop because of $infeas_d$	RESET	KEEPGOING_ID=
stop because of complementarity	RESET	AND_KEEPGOING_C=
stop because of duality gap	RESET	AND_KEEPGOING_DG=
stop because of $infeas_b$	RESET	AND_KEEPGOING_IB=
stop because of $infeas_c$	RESET	AND_KEEPGOING_IC=
stop because of $infeas_d$	RESET	AND_KEEPGOING_ID=
PRINT Statement Options:		
display everything	PRINT	PROBLEM
display arc information	PRINT	ARCS
display nonarc variable information	PRINT	NONARCS
display variable information	PRINT	VARIABLES
display constraint information	PRINT	CONSTRAINTS
display information for some arcs	PRINT	SOME_ARCS
display information for some nonarc variables	PRINT	SOME_NONARCS
display information for some variables	PRINT	SOME_VARIABLES
display information for some constraints	PRINT	SOME_CONS
display information for some constraints associated with some arcs	PRINT	CON_ARCS
display information for some constraints associated with some nonarc variables	PRINT	CON_NONARCS
display information for some constraints associated with some variables	PRINT	CON_VARIABLES

Description	Statement	Option
PRINT Statement Qualifiers:		
produce a short report	PRINT	/ SHORT
produce a long report	PRINT	/ LONG
display arcs/variables with zero flow/value	PRINT	/ ZERO
display arcs/variables with nonzero flow/value	PRINT	/ NONZERO
display basic arcs/variables	PRINT	/ BASIC
display nonbasic arcs/variables	PRINT	/ NONBASIC
SHOW Statement Options:		
show problem, optimization status	SHOW	STATUS
show network model parameters	SHOW	NETSTMT
show data sets that have been or will be created	SHOW	DATASETS
show options that pause optimization	SHOW	PAUSE
show simplex algorithm options	SHOW	SIMPLEX
show pricing strategy options	SHOW	PRICING
show miscellaneous options	SHOW	MISC
SHOW Statement Qualifiers:		
display information only on relevant options	SHOW	/ RELEVANT
display options for current stage only	SHOW	/ STAGE
Miscellaneous Options:		
infinity value	PROC NETFLOW	INFINITY=
scale constraint row, nonarc variable column coefficients, or both	PROC NETFLOW	SCALE=
maximization instead of minimization	PROC NETFLOW	MAXIMIZE
use warm start solution	PROC NETFLOW	WARM
all-artificial starting solution	PROC NETFLOW	ALLART
output complete basis information to ARCOUT= and NODEOUT= data sets	RESET	FUTURE1
output complete basis information to CONOUT= and DUALOUT= data sets	RESET	FUTURE2
turn off infeasibility or optimality flags	RESET	MOREOPT
negates FUTURE1	RESET	NOFUTURE1
negates FUTURE2	RESET	NOFUTURE2
negates SCRATCH	RESET	NOSCRATCH
negates ZTOL1	RESET	NOZTOL1
negates ZTOL2	RESET	NOZTOL2
write optimization time to SAS log	RESET	OPTIM_TIMER
no stage 1 optimization; do stage 2 optimization	RESET	SCRATCH
suppress similar SAS log messages	RESET	VERBOSE=
use zero tolerance, stage 1	RESET	ZTOL1
use zero tolerance, stage 2	RESET	ZTOL2

Interactivity

PROC NETFLOW can be used interactively. You begin by giving the PROC NETFLOW statement, and you must specify the ARCDATA= data set. The CONDATA= data set must also be specified if the problem has side constraints. If necessary, specify the NODEDATA= data set.

The variable lists should be given next. If you have variables in the input data sets that have special names (for example, a variable in the ARCDATA= data set named _TAIL_ that has tail nodes of arcs as values), it may not be necessary to have many or any variable lists.

The CONOPT, PIVOT, PRINT, QUIT, SAVE, SHOW, RESET, and RUN statements follow and can be listed in any order. The CONOPT and QUIT statements can be used only once. The others can be used as many times as needed.

Use the RESET or SAVE statement to change the names of the output data sets. With RESET, you can also indicate the reasons why optimization should stop (for example, you can indicate the maximum number of stage 1 or stage 2 iterations that can be performed). PROC NETFLOW then has a chance to either execute the next statement, or, if the next statement is one that PROC NETFLOW does not recognize (the next PROC or DATA step in the SAS session), do any allowed optimization and finish. If no new statement has been submitted, you are prompted for one. Some options of the RESET statement enable you to control aspects of the primal simplex algorithm. Specifying certain values for these options can reduce the time it takes to solve a problem. Note that any of the RESET options can be specified in the PROC NETFLOW statement.

The RUN statement starts or resumes optimization. The PIVOT statement makes PROC NETFLOW perform one simplex iteration. The QUIT statement immediately stops PROC NETFLOW. The CONOPT statement forces PROC NETFLOW to consider constraints when it next performs optimization. The SAVE statement has options that enable you to name output data sets; information about the current solution is put in these output data sets. Use the SHOW statement if you want to examine the values of options of other statements. Information about the amount of optimization that has been done and the STATUS of the current solution can also be displayed using the SHOW statement.

The PRINT statement instructs PROC NETFLOW to display parts of the problem. PRINT ARCS produces information on all arcs. PRINT SOME_ARCS limits this output to a subset of arcs. There are similar PRINT statements for nonarc variables and constraints:

```
print nonarcs;
print some_nonarcs;
print constraints;
print some_cons;
```

PRINT CON_ARCS enables you to limit constraint information that is obtained to members of a set of arcs that have nonzero constraint coefficients in a set of con-

straints. PRINT CON_NONARCS is the corresponding statement for nonarc variables.

For example, an interactive PROC NETFLOW run might look something like this:

```
proc netflow arcdata=data set
      other options;
   variable list specifications; /* if necessary */
   reset options;
   print options;    /* look at problem                    */
   run;              /* do some optimization               */
   /* suppose that optimization stopped for  */
   /* some reason or you manually stopped it */
   print options;    /* look at the current solution       */
   save options;     /* keep current solution              */
   show options;     /* look at settings                   */
   reset options;    /* change some settings, those that */
                     /* caused optimization to stop      */
   run;              /* do more optimization               */
   print options;    /* look at the optimal solution       */
   save options;     /* keep optimal solution              */
```

If you are interested only in finding the optimal solution, have used SAS variables that have special names in the input data sets, and want to use default settings for everything, then the following statement is all you need:

PROC NETFLOW ARCDATA= *data set* **;**

PROC NETFLOW Statement

PROC NETFLOW *options* **;**

This statement invokes the procedure. The following options and the options listed with the RESET statement can appear in the PROC NETFLOW statement.

Data Set Options

This section briefly describes all the input and output data sets used by PROC NETFLOW. The ARCDATA= data set, NODEDATA= data set, and CONDATA= data set can contain SAS variables that have special names, for instance _CAPAC_, _COST_, and _HEAD_. PROC NETFLOW looks for such variables if you do not give explicit variable list specifications. If a SAS variable with a special name is found and that SAS variable is not in another variable list specification, PROC NETFLOW determines that values of the SAS variable are to be interpreted in a special way. By using SAS variables that have special names, you may not need to have any variable list specifications.

ARCDATA=*SAS-data-set*
names the data set that contains arc and, optionally, nonarc variable information and nodal supply/demand data. The ARCDATA= data set must be specified in all PROC NETFLOW statements.

ARCOUT=*SAS-data-set*
AOUT=*SAS-data-set*
names the output data set that receives all arc and nonarc variable data, including flows or values, and other information concerning the unconstrained optimal solution. The supply and demand information can also be found in the ARCOUT= data set. Once optimization that considers side constraints starts, you are not able to obtain an ARCOUT= data set. Instead, use the CONOUT= data set to get the current solution. See the section "ARCOUT= and CONOUT= Data Sets" on page 524 for more information.

CONDATA=*SAS-data-set*
names the data set that contains the side constraint data. The data set can also contain other data such as arc costs, capacities, lower flow bounds, nonarc variable upper and lower bounds, and objective function coefficients. PROC NETFLOW needs a CONDATA= data set to solve a constrained problem or a linear programming problem. See the section "CONDATA= Data Set" on page 515 for more information.

CONOUT=*SAS-data-set*
COUT=*SAS-data-set*
names the output data set that receives an optimal primal solution to the problem obtained by performing optimization that considers the side constraints. See the section "ARCOUT= and CONOUT= Data Sets" on page 524 for more information.

DUALOUT=*SAS-data-set*
DOUT=*SAS-data-set*
names the output data set that receives an optimal dual solution to the problem obtained by performing optimization that considers the side constraints. See the section "NODEOUT= and DUALOUT= Data Sets" on page 525 for more information.

NODEDATA=*SAS-data-set*
DUALIN=*SAS-data-set*
names the data set that contains the node supply and demand specifications. You do not need observations in the NODEDATA= data set for trans-shipment nodes. (Trans-shipment nodes neither supply nor demand flow.) All nodes are assumed to be trans-shipment nodes unless supply or demand data indicate otherwise. It is acceptable for some arcs to be directed toward supply nodes or away from demand nodes.

The use of the NODEDATA= data set is optional in the PROC NETFLOW statement provided that, if the NODEDATA= data set is not used, supply and demand details are specified by other means. Other means include using the MAXFLOW or SHORTPATH option, SUPPLY or DEMAND list variables (or both) in the ARCDATA= data set, and the SOURCE=, SUPPLY=, SINK=, or DEMAND= option in the PROC NETFLOW statement.

NODEOUT=*SAS-data-set*
names the output data set that receives all information about nodes (supply and demand and nodal dual variable values) and other information concerning the optimal solution found by the optimizer when neglecting side constraints. Once optimization that considers side constraints starts, you are not able to obtain a NODEOUT= data set. Instead, use the DUALOUT= data set to get the current solution dual informa-

tion. See the section "NODEOUT= and DUALOUT= Data Sets" on page 525 for a more complete description.

General Options

The following is a list of options you can use with PROC NETFLOW. The options are listed in alphabetical order.

ALLART
indicates that PROC NETFLOW uses an all artificial initial solution (Kennington and Helgason 1980, p. 68) instead of the default *good path* method for determining an initial solution (Kennington and Helgason 1980, p. 245). The ALLART initial solution is generally not as good; more iterations are usually required before the optimal solution is obtained. However, because less time is used when setting up an ALLART start, it can offset the added expenditure of CPU time in later computations.

ARCS_ONLY_ARCDATA
indicates that data for only arcs are in the ARCDATA= data set. When PROC NETFLOW reads the data in ARCDATA= data set, memory would not be wasted to receive data for nonarc variables. The read might then be performed faster. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

ARC_SINGLE_OBS
indicates that for all arcs and nonarc variables, data for each arc or nonarc variable is found in only one observation of the ARCDATA= data set. When reading the data in the ARCDATA= data set, PROC NETFLOW knows that the data in an observation is for an arc or a nonarc variable that has not had data previously read that needs to be checked for consistency. The read might then be performed faster.

If you specify ARC_SINGLE_OBS, PROC NETFLOW automatically works as if GROUPED=ARCDATA is also specified.

See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

BYPASSDIVIDE=*b*
BYPASSDIV=*b*
BPD=*b*
should be used only when the MAXFLOW option has been specified; that is, PROC NETFLOW is solving a maximal flow problem. PROC NETFLOW prepares to solve maximal flow problems by setting up a bypass arc. This arc is directed from the SOURCE to the SINK and will eventually convey flow equal to INFINITY minus the maximal flow through the network. The cost of the bypass arc must be expensive enough to drive flow through the network, rather than through the bypass arc. However, the cost of the bypass arc must be less than the cost of artificial variables (otherwise these might have nonzero optimal value and a false infeasibility error will result). Also, the cost of the bypass arc must be greater than the eventual total cost of the maximal flow, which can be nonzero if some network arcs have nonzero costs. The cost of the bypass is set to the value of the INFINITY= option. Valid values for the BYPASSDIVIDE= option must be greater than or equal to 1.1.

If there are no nonzero costs of arcs in the MAXFLOW problem, the cost of the bypass arc is set to 1.0 (-1.0 if maximizing) if you do not specify the BYPASSDIVIDE= option. The reduced costs in the ARCOUT= data set and the CONOUT= data set will correctly reflect the value that would be added to the maximal flow if the capacity of the arc is increased by one unit. If there are nonzero costs, or if you specify the BYPASSDIVIDE= option, the reduced costs may be contaminated by the cost of the bypass arc and no economic interpretation can be given to reduced cost values. The default value for the BYPASSDIVIDE= option (in the presence of nonzero arc costs) is 100.0.

BYTES=*b*

indicates the size of the main working memory (in bytes) that PROC NETFLOW will allocate. The default value for the BYTES= option is near to the number of bytes of the largest contiguous memory that can be allocated for this purpose. The working memory is used to store all the arrays and buffers used by PROC NETFLOW. If this memory has a size smaller than what is required to store all arrays and buffers, PROC NETFLOW uses various schemes that page information between memory and disk.

PROC NETFLOW uses more memory than the main working memory. The additional memory requirements cannot be determined at the time when the main working memory is allocated. For example, every time an output data set is created, some additional memory is required. Do not specify a value for the BYTES= option equal to the size of available memory.

CON_SINGLE_OBS

improves how the CONDATA= data set is read. How it works depends on whether the CONDATA has a dense or sparse format.

If CONDATA has the dense format, specifying CON_SINGLE_OBS indicates that, for each constraint, data can be found in only one observation of CONDATA.

If CONDATA has a sparse format, and data for each arc and nonarc variable can be found in only one observation of CONDATA, then specify the CON_SINGLE_OBS option. If there are n SAS variables in the ROW and COEF list, then each arc or nonarc can have at most n constraint coefficients in the model. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

COREFACTOR=*c*
CF=*c*

enables you to specify the maximum proportion of memory to be used by the arrays frequently accessed by PROC NETFLOW. PROC NETFLOW strives to maintain all information required during optimization in core. If the amount of available memory is not great enough to store the arrays completely in core, either initially or as memory requirements grow, PROC NETFLOW can change the memory management scheme it uses. Large problems can still be solved. When necessary, PROC NETFLOW transfers data from random access memory (RAM) or core that can be accessed quickly but is of limited size to slower access large capacity disk memory. This is called *paging*.

Some of the arrays and buffers used during constrained optimization either vary in size, are not required as frequently as other arrays, or are not required throughout

the simplex iteration. Let *a* be the amount of memory in bytes required to store frequently accessed arrays of nonvarying size. Specify the MEMREP option in the PROC NETFLOW statement to get the value for *a* and a report of memory usage. If the size of the main working memory BYTES=*b* multiplied by COREFACTOR=*c* is greater than *a*, PROC NETFLOW keeps the frequently accessed arrays of nonvarying size resident in core throughout the optimization. If the other arrays cannot fit into core, they are paged in and out of the remaining part of the main working memory.

If *b* multiplied by *c* is less than *a*, PROC NETFLOW uses a different memory scheme. The working memory is used to store only the arrays needed in the part of the algorithm being executed. If necessary, these arrays are read from disk into the main working area. Paging, if required, is done for all these arrays, and sometimes information is written back to disk at the end of that part of the algorithm. This memory scheme is not as fast as the other memory schemes. However, problems can be solved with memory that is too small to store every array.

PROC NETFLOW is capable of solving very large problems in a modest amount of available memory. However, as more time is spent doing input/output operations, the speed of PROC NETFLOW decreases. It is important to choose the value of the COREFACTOR= option carefully. If the value is too small, the memory scheme that needs to be used might not be as efficient as another that could have been used had a larger value been specified. If the value is too large, too much of the main working memory is occupied by the frequently accessed, nonvarying sized arrays, leaving too little for the other arrays. The amount of input/output operations for these other arrays can be so high that another memory scheme might have been used more beneficially.

The valid values of COREFACTOR=*c* are between 0.0 and 0.95, inclusive. The default value for *c* is 0.75 when there are over 200 side constraints, and 0.9 when there is only one side constraint. When the problem has between 2 and 200 constraints, the value of *c* lies between the two points (1, 0.9) and (201, 0.75).

DEFCAPACITY=*c*
DC=*c*

requests that the default arc capacity and the default nonarc variable value upper bound be *c*. If this option is not specified, then DEFCAPACITY= INFINITY.

DEFCONTYPE=*c*
DEFTYPE=*c*
DCT=*c*

specifies the default constraint type. This default constraint type is either *less than or equal to* or is the type indicated by DEFCONTYPE=*c*. Valid values for this option are

LE, le, <=	for *less than or equal to*
EQ, eq, =	for *equal to*
GE, ge, >=	for *greater than or equal to*

The values do not need to be enclosed in quotes.

DEFCOST=*c*

requests that the default arc cost and the default nonarc variable objective function coefficient be *c*. If this option is not specified, then DEFCOST=0.0.

DEFMINFLOW=*m*

DMF=*m*

requests that the default lower flow bound through arcs and the default lower value bound of nonarc variables be *m*. If a value is not specified, then DEFMINFLOW=0.0.

DEMAND=*d*

specifies the demand at the SINK node specified by the SINK= option. The DEMAND= option should be used only if the SINK= option is given in the PROC NETFLOW statement and neither the SHORTPATH option nor the MAXFLOW option is specified. If you are solving a minimum cost network problem and the SINK= option is used to identify the sink node, but the DEMAND= option is not specified, then the demand at the sink node is made equal to the network's total supply.

DWIA=*i*

controls the initial amount of memory to be allocated to store the **LU** factors of the working basis matrix. DWIA stands for D_W *initial allocation* and *i* is the number of nonzeros and matrix row operations in the **LU** factors that can be stored in this memory. Due to fill-in in the **U** factor and the growth in the number of row operations, it is often necessary to move information about elements of a particular row or column to another location in the memory allocated for the **LU** factors. This process leaves some memory temporarily unoccupied. Therefore, DWIA=*i* must be greater than the memory required to store only the **LU** factors.

Occasionally, it is necessary to compress the **U** factor so that it again occupies contiguous memory. Specifying too large a value for DWIA means that more memory is required by PROC NETFLOW. This might cause more expensive memory mechanisms to be used than if a smaller but adequate value had been specified for DWIA=. Specifying too small a value for the DWIA= option can make time-consuming compressions more numerous. The default value for the DWIA= option is eight times the number of side constraints.

EXCESS=*option*

enables you to specify how to handle excess supply or demand in a network, if it exists.

For pure networks EXCESS=ARCS and EXCESS=SLACKS are valid options. By default EXCESS=ARCS is used. Note that if you specify EXCESS=SLACKS, then the interior point solver is used and you need to specify the output data set using the CONOUT= data set. For more details see the section "Using the New EXCESS= Option in Pure Networks: NETFLOW Procedure" on page 595.

For generalized networks you can either specify EXCESS=DEMAND or EXCESS=SUPPLY to indicate that the network has excess demand or excess supply, respectively. For more details see the section "Using the New EXCESS= Option in Generalized Networks: NETFLOW Procedure" on page 601.

GENNET

This option is necessary if you need to solve a generalized network flow problem and there are no arc multipliers specified in the ARCDATA= data set.

GROUPED=*grouped*

PROC NETFLOW can take a much shorter time to read data if the data have been grouped prior to the PROC NETFLOW call. This enables PROC NETFLOW to conclude that, for instance, a new NAME list variable value seen in an ARCDATA= data set grouped by the values of the NAME list variable before PROC NETFLOW was called is new. PROC NETFLOW does not need to check that the NAME has been read in a previous observation. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

- GROUPED=ARCDATA indicates that the ARCDATA= data set has been grouped by values of the NAME list variable. If _NAME_ is the name of the NAME list variable, you could use PROC SORT DATA=ARCDATA; BY _NAME_; prior to calling PROC NETFLOW. Technically, you do not have to sort the data, only ensure that all similar values of the NAME list variable are grouped together. If you specify the ARCS_ONLY_ARCDATA option, PROC NETFLOW automatically works as if GROUPED=ARCDATA is also specified.
- GROUPED=CONDATA indicates that the CONDATA= data set has been grouped.

  If the CONDATA= data set has a dense format, GROUPED=CONDATA indicates that the CONDATA= data set has been grouped by values of the ROW list variable. If _ROW_ is the name of the ROW list variable, you could use PROC SORT DATA=CONDATA; BY _ROW_; prior to calling PROC NETFLOW. Technically, you do not have to sort the data, only ensure that all similar values of the ROW list variable are grouped together. If you specify the CON_SINGLE_OBS option, or if there is no ROW list variable, PROC NETFLOW automatically works as if GROUPED=CONDATA has been specified.

  If the CONDATA= data set has the sparse format, GROUPED=CONDATA indicates that the CONDATA= data set has been grouped by values of the COLUMN list variable. If _COL_ is the name of the COLUMN list variable, you could use PROC SORT DATA=CONDATA; BY _COL_; prior to calling PROC NETFLOW. Technically, you do not have to sort the data, only ensure that all similar values of the COLUMN list variable are grouped together.
- GROUPED=BOTH indicates that both GROUPED=ARCDATA and GROUPED=CONDATA are TRUE.
- GROUPED=NONE indicates that the data sets have not been grouped, that is, neither GROUPED=ARCDATA nor GROUPED=CONDATA is TRUE. This is the default, but it is much better if GROUPED=ARCDATA, or GROUPED=CONDATA, or GROUPED=BOTH.

A data set like

```
... _XXXXX_ ....
      bbb
      bbb
      aaa
      ccc
      ccc
```

is a candidate for the GROUPED= option. Similar values are grouped together. When PROC NETFLOW is reading the ith observation, either the value of the _XXXXX_ variable is the same as the $(i-1)$st (that is, the previous observation's) _XXXXX_ value, or it is a new _XXXXX_ value not seen in any previous observation. This also means that if the ith _XXXXX_ value is different from the $(i-1)$st _XXXXX_ value, the value of the $(i-1)$st _XXXXX_ variable will not be seen in any observations $i, i+1, \ldots$.

INFINITY=*i*
INF=*i*

is the largest number used by PROC NETFLOW in computations. A number too small can adversely affect the solution process. You should avoid specifying an enormous value for the INFINITY= option because numerical roundoff errors can result. If a value is not specified, then INFINITY=99999999. The INFINITY= option cannot be assigned a value less than 9999.

INTPOINT

indicates that the interior point algorithm is to be used. The INTPOINT option must be specified if you want the interior point algorithm to be used for solving network problems, otherwise the simplex algorithm is used instead. For linear programming problems (problems with no network component), PROC NETFLOW must use the interior point algorithm, so you need not specify the INTPOINT option.

INVD_2D

controls the way in which the inverse of the working basis matrix is stored. How this matrix is stored affects computations as well as how the working basis or its inverse is updated. The working basis matrix is defined in the section "Details: NETFLOW Procedure" on page 515. If INVD_2D is specified, the working basis matrix inverse is stored as a matrix. Typically, this memory scheme is best when there are few side constraints or when the working basis is dense.

If INVD_2D is not specified, lower (**L**) and upper (**U**) factors of the working basis matrix are used. **U** is an upper triangular matrix and **L** is a lower triangular matrix corresponding to a sequence of elementary matrix row operations. The sparsity-exploiting variant of the Bartels-Golub decomposition is used to update the **LU** factors. This scheme works well when the side constraint coefficient matrix is sparse or when many side constraints are nonbinding.

MAXARRAYBYTES=*m*

specifies the maximum number of bytes an individual array can occupy. This option is of most use when solving large problems and the amount of available memory is

insufficient to store all arrays at once. Specifying the MAXARRAYBYTES= option ensures that arrays that need a large amount of memory do not consume too much memory at the expense of other arrays.

There is one array that contains information about nodes and the network basis spanning tree description. This tree description enables computations involving the network part of the basis to be performed very quickly and is the reason why PROC NETFLOW is more suited to solving constrained network problems than PROC LP. It is beneficial that this array be stored in core when possible, otherwise this array must be paged, slowing down the computations. Try not to specify a MAXARRAYBYTES=*m* value smaller than the amount of memory needed to store the main node array. You are told what this memory amount is on the SAS log if you specify the MEMREP option in the PROC NETFLOW statement.

MAXFLOW
MF

specifies that PROC NETFLOW solve a maximum flow problem. In this case, the PROC NETFLOW procedure finds the maximum flow from the node specified by the SOURCE= option to the node specified by the SINK= option. PROC NETFLOW automatically assigns an INFINITY= option supply to the SOURCE= option node and the SINK= option is assigned the INFINITY= option demand. In this way, the MAXFLOW option sets up a maximum flow problem as an equivalent minimum cost problem.

You can use the MAXFLOW option when solving any flow problem (not necessarily a maximum flow problem) when the network has one supply node (with infinite supply) and one demand node (with infinite demand). The MAXFLOW option can be used in conjunction with all other options (except SHORTPATH, SUPPLY=, and DEMAND=) and capabilities of PROC NETFLOW.

MAXIMIZE
MAX

specifies that PROC NETFLOW find the maximum cost flow through the network. If both the MAXIMIZE and the SHORTPATH options are specified, the solution obtained is the longest path between the SOURCE= and SINK= nodes. Similarly, MAXIMIZE and MAXFLOW together cause PROC NETFLOW to find the minimum flow between these two nodes; this is zero if there are no nonzero lower flow bounds.

MEMREP

indicates that information on the memory usage and paging schemes (if necessary) is reported by PROC NETFLOW on the SAS log. As optimization proceeds, you are informed of any changes in the memory requirements and schemes used by PROC NETFLOW.

NAMECTRL=*i*

is used to interpret arc and nonarc variable names in the CONDATA= data set.

In the ARCDATA= data set, an arc is identified by its tail and head node. In the CONDATA= data set, arcs are identified by names. You can give a name to an arc by

having a NAME list specification that indicates a SAS variable in the ARCDATA= data set that has names of arcs as values.

PROC NETFLOW requires arcs that have information about them in the CONDATA= data set to have names, but arcs that do not have information about them in the CONDATA= data set can also have names. Unlike a nonarc variable whose name uniquely identifies it, an arc can have several different names. An arc has a default name in the form *tail_head*, that is, the name of the arc's tail node followed by an underscore and the name of the arc's head node.

In the CONDATA= data set, if the dense data format is used, (described in the section "CONDATA= Data Set" on page 515) a name of an arc or a nonarc variable is the *name* of a SAS variable listed in the VAR list specification. If the sparse data format of the CONDATA= data set is used, a name of an arc or a nonarc variable is a *value* of the SAS variable listed in the COLUMN list specification.

The NAMECTRL= option is used when a name of an arc or nonarc variable in the CONDATA= data set (either a VAR list SAS variable name or value of the COLUMN list SAS variable) is in the form *tail_head* and there exists an arc with these end nodes. If *tail_head* has not already been tagged as belonging to an arc or nonarc variable in the ARCDATA= data set, PROC NETFLOW needs to know whether *tail_head* is the name of the arc or the name of a nonarc variable.

If you specify NAMECTRL=1, a name that is not defined in the ARCDATA= data set is assumed to be the name of a nonarc variable. NAMECTRL=2 treats *tail_head* as the name of the arc with these endnodes, provided no other name is used to associate data in the CONDATA= data set with this arc. If the arc does have other names that appear in the CONDATA= data set, *tail_head* is assumed to be the name of a nonarc variable. If you specify NAMECTRL=3, *tail_head* is assumed to be a name of the arc with these end nodes, whether the arc has other names or not. The default value of NAMECTRL is 3. Note that if you use the dense side constraint input format, the default arc name *tail_head* is not recognized (regardless of the NAMECTRL value) unless the head node and tail node names contain no lowercase letters.

If the dense format is used for the CONDATA= data set, the SAS System converts SAS variable names in a SAS program to uppercase. The VAR list variable names are uppercased. Because of this, PROC NETFLOW automatically uppercases names of arcs and nonarc variables (the values of the NAME list variable) in the ARCDATA= data set. The names of arcs and nonarc variables (the values of the NAME list variable) appear uppercased in the ARCOUT= data set and the CONOUT= data set, and in the PRINT statement output.

Also, if the dense format is used for the CONDATA= data set, be careful with default arc names (names in the form tailnode_headnode). Node names (values in the TAILNODE and HEADNODE list variables) in the ARCDATA= data set are not uppercased by PROC NETFLOW. Consider the following code:

```
data arcdata;
   input _from_ $ _to_ $ _name $ ;
   datalines;
from to1 .
```

```
from to2 arc2
TAIL TO3 .
;

data densecon;
   input from_to1 from_to2 arc2 tail_to3;
   datalines;
2 3 3 5
;

proc netflow
   arcdata=arcdata condata=densecon;
   run;
```

The SAS System does not uppercase character string values. PROC NETFLOW never uppercases node names, so the arcs in observations 1, 2, and 3 in the preceding ARCDATA= data set have the default names "from_to1", "from_to2", and "TAIL_TO3", respectively. When the dense format of the CONDATA= data set is used, PROC NETFLOW does uppercase values of the NAME list variable, so the name of the arc in the second observation of the ARCDATA= data set is "ARC2". Thus, the second arc has two names: its default "from_to2" and the other that was specified "ARC2".

As the SAS System does uppercase program code, you must think of the input statement

```
input from_to1 from_to2 arc2 tail_to3;
```

as really being

```
INPUT FROM_TO1 FROM_TO2 ARC2 TAIL_TO3;
```

The SAS variables named "FROM_TO1" and "FROM_TO2" are *not* associated with any of the arcs in the preceding ARCDATA= data set. The values "FROM_TO1" and "FROM_TO2" are different from all of the arc names "from_to1", "from_to2", "TAIL_TO3", and "ARC2". "FROM_TO1" and "FROM_TO2" could end up being the names of two nonarc variables. It is sometimes useful to specify PRINT NONARCS; before commencing optimization to ensure that the model is correct (has the right set of nonarc variables).

The SAS variable named "ARC2" is the name of the second arc in the ARCDATA= data set, even though the name specified in the ARCDATA= data set looks like "arc2". The SAS variable named "TAIL_TO3" is the default name of the third arc in the ARCDATA= data set.

NARCS=*n*

specifies the approximate number of arcs. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

NCOEFS=*n*

specifies the approximate number of constraint coefficients. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

NCONS=*n*
specifies the approximate number of constraints. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

NNAS=*n*
specifies the approximate number of nonarc variables. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

NNODES=*n*
specifies the approximate number of nodes. See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

NON_REPLIC=*non_replic*
prevents PROC NETFLOW from doing unnecessary checks of data previously read.

- NON_REPLIC=COEFS indicates that each constraint coefficient is specified *once* in the CONDATA= data set.
- NON_REPLIC=NONE indicates that constraint coefficients can be specified more than once in the CONDATA= data set. NON_REPLIC=NONE is the default.

See the section "How to Make the Data Read of PROC NETFLOW More Efficient" on page 546.

RHSOBS=*charstr*
specifies the keyword that identifies a right-hand-side observation when using the sparse format for data in the CONDATA= data set. The keyword is expected as a value of the SAS variable in the CONDATA= data set named in the COLUMN list specification. The default value of the RHSOBS= option is _RHS_ or _rhs_. If *charstr* is not a valid SAS variable name, enclose it in single quotes.

SAME_NONARC_DATA
SND
If all nonarc variable data are given in the ARCDATA= data set, or if the problem has no nonarc variables, the unconstrained warm start can be read more quickly if the option SAME_NONARC_DATA is specified. SAME_NONARC_DATA indicates that any nonconstraint nonarc variable data in the CONDATA= data set is to be ignored. Only side constraint data in the CONDATA= data set are read.

If you use an unconstrained warm start and SAME_NONARC_DATA is not specified, any nonarc variable objective function coefficient, upper bound, or lower bound can be changed. Any nonarc variable data in the CONDATA= data set overrides (without warning messages) corresponding data in the ARCDATA= data set. You can possibly introduce new nonarc variables to the problem, that is, nonarc variables that were not in the problem when the warm start was generated.

SAME_NONARC_DATA should be specified if nonarc variable data in the CONDATA= data set are to be deliberately ignored. Consider

```
proc netflow options arcdata=arc0 nodedata=node0
          condata=con0
```

```
            /* this data set has nonarc variable        */
            /* objective function coefficient data      */
         future1 arcout=arc1 nodeout=node1;
   run;

data arc2;
   reset arc1;   /* this data set has nonarc variable obs */
   if _cost_<50.0 then _cost_=_cost_*1.25;
               /* some objective coefficients of nonarc */
               /* variable might be changed             */
proc netflow options
         warm arcdata=arc2 nodedata=node1
         condata=con0 same_nonarc_data
            /* This data set has old nonarc variable    */
            /* obj, fn. coefficients. same_nonarc_data  */
            /* indicates that the "new" coefs in the    */
            /* arcdata=arc2 are to be used.             */
   run;
```

SCALE=*scale*

indicates that the side constraints are to be scaled. Scaling is useful when some coefficients of a constraint or nonarc variable are either much larger or much smaller than other coefficients. Scaling might make all coefficients have values that have a smaller range, and this can make computations more stable numerically. Try the SCALE= option if PROC NETFLOW is unable to solve a problem because of numerical instability. Specify

- SCALE=ROW, SCALE=CON, or SCALE=CONSTRAINT if the largest absolute value of coefficients in each constraint is about 1.0
- SCALE=COL, SCALE=COLUMN, or SCALE=NONARC if nonarc variable columns are scaled so that the absolute value of the largest constraint coefficient of a nonarc variable is near to 1
- SCALE=BOTH if the largest absolute value of coefficients in each constraint, and the absolute value of the largest constraint coefficient of a nonarc variable is near to 1. This is the default.
- SCALE=NONE if no scaling should be done

SHORTPATH
SP

specifies that PROC NETFLOW solve a shortest path problem. The NETFLOW procedure finds the shortest path between the nodes specified in the SOURCE= option and the SINK= option. The costs of arcs are their *lengths*. PROC NETFLOW automatically assigns a supply of one flow unit to the SOURCE= node, and the SINK= node is assigned to have a one flow unit demand. In this way, the SHORTPATH option sets up a shortest path problem as an equivalent minimum cost problem.

If a network has one supply node (with supply of one unit) and one demand node (with demand of one unit), you could specify the SHORTPATH option, with the SOURCE= and SINK= nodes, even if the problem is not a shortest path problem.

You then should not provide any supply or demand data in the NODEDATA= data set or the ARCDATA= data set.

SINK=*sinkname*
SINKNODE=*sinkname*
identifies the demand node. The SINK= option is useful when you specify the MAXFLOW option or the SHORTPATH option and need to specify toward which node the shortest path or maximum flow is directed. The SINK= option also can be used when a minimum cost problem has only one demand node. Rather than having this information in the ARCDATA= data set or the NODEDATA= data set, use the SINK= option with an accompanying DEMAND= specification for this node. The SINK= option must be the name of a head node of at least one arc; thus, it must have a character value. If the value of the SINK= option is not a valid SAS character variable name, it must be enclosed in single quotes and can contain embedded blanks.

SOURCE=*sourcename*
SOURCENODE=*sourcename*
identifies a supply node. The SOURCE= option is useful when you specify the MAXFLOW or the SHORTPATH option and need to specify from which node the shortest path or maximum flow originates. The SOURCE= option also can be used when a minimum cost problem has only one supply node. Rather than having this information in the ARCDATA= data set or the NODEDATA= data set, use the SOURCE= option with an accompanying SUPPLY= amount of supply at this node. The SOURCE= option must be the name of a tail node of at least one arc; thus, it must have a character value. If the value of the SOURCE= option is not a valid SAS character variable name, it must be enclosed in single quotes and can contain embedded blanks.

SPARSECONDATA
SCDATA
indicates that the CONDATA= data set has data in the sparse data format. Otherwise, it is assumed that the data are in the dense format.

Note: If the SPARSECONDATA option is not specified, and you are running SAS software Version 6 or you have specified options validvarname=v6;, all NAME list variable values in the ARCDATA= data set are uppercased. See the section "Case Sensitivity" on page 527.

SPARSEP2
SP2
indicates that the new column of the working basis matrix that replaces another column be held in a linked list. If the SPARSEP2 option is not specified, a one-dimensional array is used to store this column's information, that can contain elements that are 0.0 and use more memory than the linked list. The linked list mechanism requires more work if the column has numerous nonzero elements in many iterations. Otherwise, it is superior. Sometimes, specifying SPARSEP2 is beneficial when the side constrained coefficient matrix is very sparse or when some paging is necessary.

SUPPLY=*s*
specifies the supply at the source node specified by the SOURCE= option. The SUPPLY= option should be used only if the SOURCE= option is given in the PROC NETFLOW statement and neither the SHORTPATH option nor the MAXFLOW option is specified. If you are solving a minimum cost network problem and the SOURCE= option is used to identify the source node and the SUPPLY= option is not specified, then by default the supply at the source node is made equal to the network's total demand.

THRUNET
tells PROC NETFLOW to force through the network any excess supply (the amount by which total supply exceeds total demand) or any excess demand (the amount by which total demand exceeds total supply) as is required. If a network problem has unequal total supply and total demand and the THRUNET option is not specified, PROC NETFLOW drains away the excess supply or excess demand in an optimal manner. The consequences of specifying or not specifying THRUNET are discussed in the section "Balancing Total Supply and Total Demand" on page 541.

TYPEOBS=*charstr*
specifies the keyword that identifies a type observation when using the sparse format for data in the CONDATA= data set. The keyword is expected as a value of the SAS variable in the CONDATA= data set named in the COLUMN list specification. The default value of the TYPEOBS= option is _TYPE_ or _type_. If *charstr* is not a valid SAS variable name, enclose it in single quotes.

WARM
indicates that the NODEDATA= data set or the DUALIN= data set and the ARCDATA= data set contain extra information of a warm start to be used by PROC NETFLOW. See the section "Warm Starts" on page 542.

CAPACITY Statement

CAPACITY *variable* **;**

CAPAC *variable* **;**

UPPERBD *variable* **;**

The CAPACITY statement identifies the SAS variable in the ARCDATA= data set that contains the maximum feasible flow or capacity of the network arcs. If an observation contains nonarc variable information, the CAPACITY list variable is the upper value bound for the nonarc variable named in the NAME list variable in that observation. The CAPACITY list variable must have numeric values. It is not necessary to have a CAPACITY statement if the name of the SAS variable is _CAPAC_, _UPPER_, _UPPERBD, or _HI_.

COEF Statement

COEF *variables* **;**

The COEF list is used with the sparse input format of the CONDATA= data set. The COEF list can contain more than one SAS variable, each of which must have numeric values. If the COEF statement is not specified, the CONDATA= data set is searched and SAS variables with names beginning with _COE are used. The number of SAS variables in the COEF list must be no greater than the number of SAS variables in the ROW list.

The values of the COEF list variables in an observation can be interpreted differently than these variables' values in other observations. The values can be coefficients in the side constraints, costs and objective function coefficients, bound data, constraint type data, or rhs data. If the COLUMN list variable has a value that is a name of an arc or nonarc variable, the ith COEF list variable is associated with the constraint or special row name named in the ith ROW list variable. Otherwise, the COEF list variables indicate type values, rhs values, or missing values.

COLUMN Statement

COLUMN *variable* **;**

The COLUMN list is used with the sparse input format of side constraints. This list consists of one SAS variable in the CONDATA= data set that has as values the names of arc variables, nonarc variables, or missing values. Some, if not all of these values, also can be values of the NAME list variables of the ARCDATA= data set. The COLUMN list variable can have other special values (refer to the TYPEOBS= and RHSOBS= options). If the COLUMN list is not specified after the PROC NETFLOW statement, the CONDATA= data set is searched and a SAS variable named _COLUMN_ is used. The COLUMN list variable must have character values.

CONOPT Statement

CONOPT ;

The CONOPT statement has no options. It is equivalent to specifying RESET SCRATCH;. The CONOPT statement should be used before stage 2 optimization commences. It indicates that the optimization performed next should consider the side constraints.

Usually, the optimal unconstrained network solution is used as a starting solution for constrained optimization. Finding the unconstrained optimum usually reduces the amount of stage 2 optimization. Furthermore, the unconstrained optimum is almost always "closer" to the constrained optimum than the initial basic solution determined before any optimization is performed. However, as the optimum is approached during stage 1 optimization, the flow change candidates become scarcer and a solution good enough to start stage 2 optimization may already be at hand. You should then specify the CONOPT statement.

COST Statement

COST *variable* **;**

OBJFN *variable* **;**

The COST statement identifies the SAS variable in the ARCDATA= data set that contains the per unit flow cost through an arc. If an observation contains nonarc variable information, the value of the COST list variable is the objective function coefficient of the nonarc variable named in the NAME list variable in that observation. The COST list variable must have numeric values. It is not necessary to specify a COST statement if the name of the SAS variable is _COST_ or _LENGTH_.

DEMAND Statement

DEMAND *variable* **;**

The DEMAND statement identifies the SAS variable in the ARCDATA= data set that contains the demand at the node named in the corresponding HEADNODE list variable. The DEMAND list variable must have numeric values. It is not necessary to have a DEMAND statement if the name of this SAS variable is _DEMAND_.

HEADNODE Statement

HEADNODE *variable* **;**

HEAD *variable* **;**

TONODE *variable* **;**

TO *variable* **;**

The HEADNODE statement specifies the SAS variable that must be present in the ARCDATA= data set that contains the names of nodes toward which arcs are directed. It is not necessary to have a HEADNODE statement if the name of the SAS variable is _HEAD_ or _TO_. The HEADNODE variable must have character values.

ID Statement

ID *variables* **;**

The ID statement specifies SAS variables containing values for pre- and post-optimal processing and analysis. These variables are not processed by PROC NETFLOW but are read by the procedure and written in the ARCOUT= and CONOUT= data sets and the output of PRINT statements. For example, imagine a network used to model a distribution system. The SAS variables listed on the ID statement can contain information on type of vehicle, transportation mode, condition of road, time to complete journey, name of driver, or other ancillary information useful for report writing or describing facets of the operation that do not have bearing on the optimization. The ID variables can be character, numeric, or both.

If no ID list is specified, the procedure forms an ID list of all SAS variables not included in any other implicit or explicit list specification. If the ID list is specified,

any SAS variables in the ARCDATA= data set not in any list are dropped and do not appear in the ARCOUT= or CONOUT= data sets, or in the PRINT statement output.

LO Statement

LO *variable* **;**

LOWERBD *variable* **;**

MINFLOW *variable* **;**

The LO statement identifies the SAS variable in the ARCDATA= data set that contains the minimum feasible flow or lower flow bound for arcs in the network. If an observation contains nonarc variable information, the LO list variable has the value of the lower bound for the nonarc variable named in the NAME list variable. The LO list variables must have numeric values. It is not necessary to have a LO statement if the name of this SAS variable is _LOWER_, _LO_, _LOWERBD, or _MINFLOW.

MULT Statement

MULT *variables* **;**

MULTIPLIER *variables* **;**

The MULT statement identifies the SAS variable in the ARCDATA= data set associated with the values of arc multipliers in the network. These values must be positive real numbers. It is not necessary to have a MULT statement if the name of this SAS variable is _MULT_.

NAME Statement

NAME *variable* **;**

ARCNAME *variable* **;**

VARNAME *variable* **;**

Each arc and nonarc variable that has data in the CONDATA= data set must have a unique name. This variable is identified in the ARCDATA= data set. The NAME list variable must have character values (see the NAMECTRL= option in the PROC NETFLOW statement for more information). It is not necessary to have a NAME statement if the name of this SAS variable is _NAME_.

NODE Statement

NODE *variable* **;**

The NODE list variable, which must be present in the NODEDATA= data set, has names of nodes as values. These values must also be values of the TAILNODE list variable, the HEADNODE list variable, or both. If this list is not explicitly specified, the NODEDATA= data set is searched for a SAS variable with the name _NODE_. The NODE list variable must have character values.

PIVOT Statement

PIVOT ;

The PIVOT statement has no options. It indicates that one simplex iteration is to be performed. The PIVOT statement forces a simplex iteration to be performed in spite of the continued presence of any reasons or solution conditions that caused optimization to be halted. For example, if the number of iterations performed exceeds the value of the MAXIT1= or MAXIT2= option and you issue a PIVOT statement, the iteration is performed even though the MAXIT1= or MAXIT2= value has not yet been changed using a RESET statement.

PRINT Statement

PRINT *options / qualifiers* **;**

The options available with the PRINT statement of PROC NETFLOW are summarized by purpose in the following table.

Table 5.2. Functional Summary, PRINT Statement

Description	Statement	Option
PRINT Statement Options:		
display everything	PRINT	PROBLEM
display arc information	PRINT	ARCS
display nonarc variable information	PRINT	NONARCS
display variable information	PRINT	VARIABLES
display constraint information	PRINT	CONSTRAINTS
display information for some arcs	PRINT	SOME_ARCS
display information for some nonarc variables	PRINT	SOME_NONARCS
display information for some variables	PRINT	SOME_VARIABLES
display information for some constraints	PRINT	SOME_CONS
display information for some constraints associated with some arcs	PRINT	CON_ARCS
display information for some constraints associated with some nonarc variables	PRINT	CON_NONARCS
display information for some constraints associated with some variables	PRINT	CON_VARIABLES
PRINT Statement Qualifiers:		
produce a short report	PRINT	/ SHORT
produce a long report	PRINT	/ LONG
display arcs/variables with zero flow/value	PRINT	/ ZERO
display arcs/variables with nonzero flow/value	PRINT	/ NONZERO
display basic arcs/variables	PRINT	/ BASIC
display nonbasic arcs/variables	PRINT	/ NONBASIC

The PRINT statement enables you to examine part or all of the problem. You can limit the amount of information displayed when a PRINT statement is processed by specifying PRINT statement options. The name of the PRINT option indicates what part of the problem is to be examined. If no options are specified, or PRINT PROBLEM is specified, information about the entire problem is produced.

The amount of displayed information can be limited further by following any PRINT statement options with a slash (/) and one or more of the qualifiers SHORT or LONG, ZERO or NONZERO, BASIC or NONBASIC.

Some of the PRINT statement options require you to specify a list of some type of entity, thereby enabling you to indicate what entities are of interest. The entities of interest are the ones you want to display. These entities might be tail node names, head node names, nonarc variable names, or constraint names. The entity list is made up of one or more of the following constructs. Each construct can add none, one, or more entities to the set of entities to be displayed.

- _ALL_
  Display all entities in the required list.
- entity
  Display the named entity that is interesting.
- entity1 - entity2 (one hyphen)
  entity1 -- entity2 (two hyphens)
  entity1 - CHARACTER - entity2 or
  entity1 - CHAR - entity2
  Both entity1 and entity2 have names made up of the same character string prefix followed by a numeric suffix. The suffixes of both entity1 and entity2 have the same number of numerals but can have different values. A specification of entity1 - entity2 indicates that all entities with the same prefix and suffixes with values on or between the suffixes of entity1 and entity2 are to be put in the set of entities to be displayed. The numeric suffix of both entity1 and entity2 can be followed by a character string. For example, _OBS07_ - _OBS13_ is a valid construct of the forms entity1 - entity2.
- part_of_entity_name:
  Display all entities in the required list that have names beginning with the character string preceding the colon.

The following options can appear in the PRINT statement:

ARCS
indicates that you want to have displayed information about all arcs.

SOME_ARCS (*taillist,headlist*)
is similar to the statement PRINT ARCS except that, instead of displaying information about all arcs, only arcs directed from nodes in a specified set of tail nodes to nodes in a specified set of head nodes are included. The nodes or node constructs belonging to the *taillist* list are separated by blanks. The nodes or node constructs

belonging to the *headlist* list are also separated by blanks. The lists are separated by a comma.

NONARCS
VARIABLES
indicates that information is to be displayed about all nonarc variables.

SOME_NONARCS (*nonarclist***)**
SOME_VARIABLES (*variablelist)*
is similar to the PRINT NONARCS statement except that, instead of displaying information about all nonarc variables, only those belonging to a specified set of nonarc variables have information displayed. The nonarc variables or nonarc variable constructs belonging to the *nonarclist* list are separated by blanks.

CONSTRAINTS
indicates that you want to have displayed information about all constraint coefficients.

SOME_CONS (*conlist***)**
displays information for coefficients in a specified set of constraints. The constraints or constraint constructs belonging to the *conlist* list are separated by blanks.

CON_ARCS (*taillist, headlist***)**
is similar to the PRINT SOME_CONS (*conlist*) statement except that, instead of displaying information about all coefficients in specified constraints, information about only those coefficients that are associated with arcs directed from a set of specified tail nodes toward a set of specified head nodes is displayed. The constraints or constraint constructs belonging to the *conlist* list are separated by blanks; so too are the nodes or node constructs belonging to the *taillist* list and the nodes or node constructs belonging to the *headlist* list. The lists are separated by commas.

CON_NONARCS (*conlist, nonarclist***)**
CON_VARIABLES (*conlist, variablelist***)**
is similar to the PRINT SOME_CONS (*conlist*) statement except that, instead of displaying information about all coefficients in specified constraints, information about only those coefficients that are associated with nonarc variables in a specified set is displayed. The constraints or constraint constructs belonging to the *conlist* list are separated by blanks. The nonarc variables or nonarc variable constructs belonging to the *nonarclist* list are separated by blanks. The lists are separated by a comma.

PROBLEM
is equivalent to the statement PRINT ARCS NONARCS CONSTRAINTS.

Following a slash (/), the qualifiers SHORT or LONG, ZERO or NONZERO, BASIC or NONBASIC can appear in any PRINT statement. These qualifiers are described below.

- BASIC
  Only rows that are associated with arcs or nonarc variables that are basic are displayed. The _STATUS_ column values are KEY_ARC BASIC or NONKEY ARC BASIC for arcs, and NONKEY_BASIC for nonarc variables.

- LONG
  All table columns are displayed (the default when no qualifier is used).
- NONBASIC
  Only rows that are associated with arcs or nonarc variables that are nonbasic are displayed. The _STATUS_ column values are LOWERBD NONBASIC or UPPERBD NONBASIC.
- NONZERO
  Only rows that have nonzero _FLOW_ column values (nonzero arc flows, nonzero nonarc variable values) are displayed.
- SHORT
  The table columns are _N_, _FROM_, _TO_, _COST_, _CAPAC_, _LO_, _NAME_, and _FLOW_, or the names of the SAS variables specified in the corresponding variable lists (TAILNODE, HEADNODE, COST, CAPACITY, LO, and NAME lists). _COEF_ or the name of the SAS variable in the COEF list specification will head a column when the SHORT qualifier is used in PRINT CONSTRAINTS, SOME_CONS, CON_ARCS, or CON_NONARCS.
- ZERO
  Only rows that have zero _FLOW_ column values (zero arc flows, zero nonarc variable values) are displayed.

The default qualifiers are BASIC, NONBASIC, ZERO, NONZERO, and LONG.

Displaying Information On All Constraints

In the oil refinery problem, if you had entered

```
print constraints;
```

after the RUN statement, the output in Figure 5.9 would have been produced.

Displaying Information About Selected Arcs

In the oil refinery problem, if you had entered

```
print some_arcs(refin:,_all_)/short;
```

after the RUN statement, the output in Figure 5.10 would have been produced.

Oil Industry Example

Setting Up Condata = Cond1 For Proc Netflow

The NETFLOW Procedure

_N_	_CON_	_type_	_rhs_	_name_	_from_	_to_
1	_OBS1_	GE	-15	m_e_ref1	middle east	refinery 1
2	_OBS1_	GE	-15	thruput1	refinery 1	r1
3	_OBS2_	GE	-15	m_e_ref2	middle east	refinery 2
4	_OBS2_	GE	-15	thruput2	refinery 2	r2
5	_OBS3_	EQ	0	thruput1	refinery 1	r1
6	_OBS3_	EQ	0	r1_gas	r1	ref1 gas
7	_OBS4_	EQ	0	thruput2	refinery 2	r2
8	_OBS4_	EQ	0	r2_gas	r2	ref2 gas

_N_	_cost_	_capac_	_lo_	_SUPPLY_	_DEMAND_	_FLOW_	_COEF_
1	63	95	20	100	.	80	-2
2	200	175	50	.	.	145	1
3	81	80	10	100	.	20	-2
4	220	100	35	.	.	35	1
5	200	175	50	.	.	145	-3
6	0	140	0	.	.	108.75	4
7	220	100	35	.	.	35	-3
8	0	100	0	.	.	26.25	4

_N_	_FCOST_	_RCOST_	_STATUS_	
1	5040	.	KEY_ARC	BASIC
2	29000	.	KEY_ARC	BASIC
3	1620	.	NONKEY ARC	BASIC
4	7700	29	LOWERBD	NONBASIC
5	29000	.	KEY_ARC	BASIC
6	0	.	KEY_ARC	BASIC
7	7700	29	LOWERBD	NONBASIC
8	0	.	KEY_ARC	BASIC

Figure 5.9. PRINT CONSTRAINTS

The NETFLOW Procedure

_N_	_from_	_to_	_cost_	_capac_	_lo_	_name_
1	refinery 1	r1	200	175	50	thruput1
2	refinery 2	r2	220	100	35	thruput2

_N_	_FLOW_
1	145
2	35

Figure 5.10. PRINT SOME_ARCS

Displaying Information About Selected Constraints

In the oil refinery problem, if you had entered

```
print some_cons(_obs3_-_obs4_)/nonzero short;
```

after the RUN statement, the output in Figure 5.11 would have been produced.

The NETFLOW Procedure

_N_	_CON_	_type_	_rhs_	_name_	_from_	_to_
1	_OBS3_	EQ	0	thruput1	refinery 1	r1
2	_OBS3_	EQ	0	r1_gas	r1	ref1 gas
3	_OBS4_	EQ	0	thruput2	refinery 2	r2
4	_OBS4_	EQ	0	r2_gas	r2	ref2 gas

_N_	_cost_	_capac_	_lo_	_FLOW_	_COEF_
1	200	175	50	145	-3
2	0	140	0	108.75	4
3	220	100	35	35	-3
4	0	100	0	26.25	4

Figure 5.11. PRINT SOME_CONS

If you had entered

```
print con_arcs(_all_,r1 r2,_all_)/short;
```

after the RUN statement, the output in Figure 5.12 would have been produced. Constraint information about arcs directed from selected tail nodes is displayed.

The NETFLOW Procedure

_N_	_CON_	_type_	_rhs_	_name_	_from_	_to_
1	_OBS3_	EQ	0	r1_gas	r1	ref1 gas
2	_OBS4_	EQ	0	r2_gas	r2	ref2 gas

_N_	_cost_	_capac_	_lo_	_FLOW_	_COEF_
1	0	140	0	108.75	4
2	0	100	0	26.25	4

Figure 5.12. PRINT CON_ARCS

Cautions

This subsection has two parts; the first part is applicable if you are running Version 7 or later of the SAS System, and the second part is applicable if you are running Version 6. You can get later versions to "act" like Version 6 by specifying

```
options validvarname=v6;
```

For Version 7 onward, PROC NETFLOW strictly respects case sensitivity. The PRINT statements of PROC NETFLOW that require lists of entities will work properly *only* if the entities have the same case as in the input data sets. Entities that contain blanks must be enclosed in single or double quotes. For example,

```
print some_arcs (_all_,"Ref1 Gas");
```

In this example, a head node of an arc in the model is "Ref1 Gas" (without the quotes). If you omit the quotes, PROC NETFLOW issues a message on the SAS log:

```
WARNING: The node Ref1 in the list of head nodes in the PRINT
         SOME_ARCS or PRINT CON_ARCS is not a node in the
         problem. This statement will be ignored.
```

If you had specified

```
print some_arcs (_all_,"ref1 Gas");
```

(note the small r), you would have been warned

```
WARNING: The node ref1 Gas in the list of head nodes in the PRINT
         SOME_ARCS or PRINT CON_ARCS is not a node in the
         problem. This statement will be ignored.
```

If you are running Version 6, or if you are running a later version and you have specified

```
options validvarname=v6;
```

when information is parsed to procedures, the SAS System converts the text that makes up statements into uppercase. The PRINT statements of PROC NETFLOW that require lists of entities will work properly *only* if the entities are uppercase in the input data sets. If you do not want this uppercasing to occur, you must enclose the entity in single or double quotes.

```
print some_arcs('lowercase tail','lowercase head');
print some_cons('factory07'-'factory12');
print some_cons('_factory07_'-'_factory12_');
print some_nonarcs("CO2 content":);
```

Entities that contain blanks must be enclosed in single or double quotes.

QUIT Statement

QUIT ;

The QUIT statement indicates that PROC NETFLOW is to be terminated immediately. The solution is not saved in the current output data sets. The QUIT statement has no options.

RESET Statement

RESET *options* **;**

SET *options* **;**

The RESET statement is used to change options after PROC NETFLOW has started execution. Any of the following options can appear in the PROC NETFLOW statement.

Another name for the RESET statement is SET. You can use RESET when you are resetting options and SET when you are setting options for the first time.

The following options fall roughly into five categories:

- output data set specifications
- options that indicate conditions under which optimization is to be halted temporarily, giving you an opportunity to use PROC NETFLOW interactively
- options that control aspects of the operation of the network primal simplex optimization
- options that control the pricing strategies of the network simplex optimizer
- miscellaneous options

If you want to examine the setting of any options, use the SHOW statement. If you are interested in looking at only those options that fall into a particular category, the SHOW statement has options that enable you to do this.

The execution of PROC NETFLOW has three stages. In stage zero the problem data are read from the NODEDATA=, ARCDATA=, and CONDATA= data sets. If a warm start is not available, an initial basic feasible solution is found. Some options of the PROC NETFLOW statement control what occurs in stage zero. By the time the first RESET statement is processed, stage zero has already been completed.

In the first stage, an optimal solution to the network flow problem neglecting any side constraints is found. The primal and dual solutions for this relaxed problem can be saved in the ARCOUT= data set and the NODEOUT= data set, respectively.

In the second stage, the side constraints are examined and some initializations occur. Some preliminary work is also needed to commence optimization that considers the constraints. An optimal solution to the network flow problem with side constraints is found. The primal and dual solutions for this side-constrained problem are saved in the CONOUT= data set and the DUALOUT= data set, respectively.

Many options in the RESET statement have the same name except that they have as a suffix the numeral 1 or 2. Such options have much the same purpose, but option1 controls what occurs during the first stage when optimizing the network neglecting any side constraints and option2 controls what occurs in the second stage when PROC NETFLOW is performing constrained optimization.

Some options can be turned off by the option prefixed by the word *NO*. For example, FEASIBLEPAUSE1 may have been specified in a RESET statement and in a later RESET statement, you can specify NOFEASIBLEPAUSE1. In a later RESET statement, you can respecify FEASIBLEPAUSE1 and, in this way, toggle this option.

The options available with the RESET statement are summarized by purpose in the following table.

Table 5.3. Functional Summary, RESET Statement

Description	Statement	Option
Output Data Set Options:		
unconstrained primal solution data set	RESET	ARCOUT=
unconstrained dual solution data set	RESET	NODEOUT=
constrained primal solution data set	RESET	CONOUT=
constrained dual solution data set	RESET	DUALOUT=
Simplex Options:		
use big-M instead of two-phase method, stage 1	RESET	BIGM1
use Big-M instead of two-phase method, stage 2	RESET	BIGM2
anti-cycling option	RESET	CYCLEMULT1=
interchange first nonkey with leaving key arc	RESET	INTFIRST
controls working basis matrix inversions	RESET	INVFREQ=
maximum number of L row operations allowed before refactorization	RESET	MAXL=
maximum number of LU factor column updates	RESET	MAXLUUPDATES=
anti-cycling option	RESET	MINBLOCK1=
use first eligible leaving variable, stage 1	RESET	LRATIO1
use first eligible leaving variable, stage 2	RESET	LRATIO2
negates INTFIRST	RESET	NOINTFIRST
negates LRATIO1	RESET	NOLRATIO1
negates LRATIO2	RESET	NOLRATIO2
negates PERTURB1	RESET	NOPERTURB1
anti-cycling option	RESET	PERTURB1
controls working basis matrix refactorization	RESET	REFACTFREQ=
use two-phase instead of big-M method, stage 1	RESET	TWOPHASE1
use two-phase instead of big-M method, stage 2	RESET	TWOPHASE2
pivot element selection parameter	RESET	U=
zero tolerance, stage 1	RESET	ZERO1=
zero tolerance, stage 2	RESET	ZERO2=
zero tolerance, real number comparisons	RESET	ZEROTOL=

Description	Statement	Option
Pricing Options:		
frequency of dual value calculation	RESET	DUALFREQ=
pricing strategy, stage 1	RESET	PRICETYPE1=
pricing strategy, stage 2	RESET	PRICETYPE2=
used when P1SCAN=PARTIAL	RESET	P1NPARTIAL=
controls search for entering candidate, stage 1	RESET	P1SCAN=
used when P2SCAN=PARTIAL	RESET	P2NPARTIAL=
controls search for entering candidate, stage 2	RESET	P2SCAN=
initial queue size, stage 1	RESET	QSIZE1=
initial queue size, stage 2	RESET	QSIZE2=
used when Q1FILLSCAN=PARTIAL	RESET	Q1FILLNPARTIAL=
controls scan when filling queue, stage 1	RESET	Q1FILLSCAN=
used when Q2FILLSCAN=PARTIAL	RESET	Q2FILLNPARTIAL=
controls scan when filling queue, stage 2	RESET	Q2FILLSCAN=
queue size reduction factor, stage 1	RESET	REDUCEQSIZE1=
queue size reduction factor, stage 2	RESET	REDUCEQSIZE2=
frequency of refreshing queue, stage 1	RESET	REFRESHQ1=
frequency of refreshing queue, stage 2	RESET	REFRESHQ2=
Optimization Termination Options:		
pause after stage 1; don't start stage 2	RESET	ENDPAUSE1
pause when feasible, stage 1	RESET	FEASIBLEPAUSE1
pause when feasible, stage 2	RESET	FEASIBLEPAUSE2
maximum number of iterations, stage 1	RESET	MAXIT1=
maximum number of iterations, stage 2	RESET	MAXIT2=
negates ENDPAUSE1	RESET	NOENDPAUSE1
negates FEASIBLEPAUSE1	RESET	NOFEASIBLEPAUSE1
negates FEASIBLEPAUSE2	RESET	NOFEASIBLEPAUSE2
pause every PAUSE1 iterations, stage 1	RESET	PAUSE1=
pause every PAUSE2 iterations, stage 2	RESET	PAUSE2=
Interior Point Algorithm Options:		
factorization method	RESET	FACT_METHOD=
allowed amount of dual infeasibility	RESET	TOLDINF=
allowed amount of primal infeasibility	RESET	TOLPINF=
allowed total amount of dual infeasibility	RESET	TOLTOTDINF=
allowed total amount of primal infeasibility	RESET	TOLTOTPINF=
cut-off tolerance for Cholesky factorization	RESET	CHOLTINYTOL=
density threshold for Cholesky processing	RESET	DENSETHR=
step-length multiplier	RESET	PDSTEPMULT=
preprocessing type	RESET	PRSLTYPE=
print optimization progress on SAS log	RESET	PRINTLEVEL2=
Interior Point Stopping Criteria Options:		
maximum number of interior point iterations	RESET	MAXITERB=

Description	Statement	Option
primal-dual (duality) gap tolerance	RESET	PDGAPTOL=
stop because of complementarity	RESET	STOP_C=
stop because of duality gap	RESET	STOP_DG=
stop because of $infeas_b$	RESET	STOP_IB=
stop because of $infeas_c$	RESET	STOP_IC=
stop because of $infeas_d$	RESET	STOP_ID=
stop because of complementarity	RESET	AND_STOP_C=
stop because of duality gap	RESET	AND_STOP_DG=
stop because of $infeas_b$	RESET	AND_STOP_IB=
stop because of $infeas_c$	RESET	AND_STOP_IC=
stop because of $infeas_d$	RESET	AND_STOP_ID=
stop because of complementarity	RESET	KEEPGOING_C=
stop because of duality gap	RESET	KEEPGOING_DG=
stop because of $infeas_b$	RESET	KEEPGOING_IB=
stop because of $infeas_c$	RESET	KEEPGOING_IC=
stop because of $infeas_d$	RESET	KEEPGOING_ID=
stop because of complementarity	RESET	AND_KEEPGOING_C=
stop because of duality gap	RESET	AND_KEEPGOING_DG=
stop because of $infeas_b$	RESET	AND_KEEPGOING_IB=
stop because of $infeas_c$	RESET	AND_KEEPGOING_IC=
stop because of $infeas_d$	RESET	AND_KEEPGOING_ID=
Miscellaneous Options:		
output complete basis information to ARCOUT= and NODEOUT= data sets	RESET	FUTURE1
output complete basis information to CONOUT= and DUALOUT= data sets	RESET	FUTURE2
turn off infeasibility or optimality flags	RESET	MOREOPT
negates FUTURE1	RESET	NOFUTURE1
negates FUTURE2	RESET	NOFUTURE2
negates SCRATCH	RESET	NOSCRATCH
negates ZTOL1	RESET	NOZTOL1
negates ZTOL2	RESET	NOZTOL2
write optimization time to SAS log	RESET	OPTIM_TIMER
no stage 1 optimization; do stage 2 optimization	RESET	SCRATCH
suppress similar SAS log messages	RESET	VERBOSE=
use zero tolerance, stage 1	RESET	ZTOL1
use zero tolerance, stage 2	RESET	ZTOL2

Output Data Set Specifications

In a RESET statement, you can specify an ARCOUT= data set, a NODEOUT= data set, a CONOUT= data set, or a DUALOUT= data set. You are advised to specify these output data sets early because if you make a syntax error when using PROC

NETFLOW interactively or, for some other reason, PROC NETFLOW encounters or does something unexpected, these data sets will contain information about the solution that was reached. If you had specified the FUTURE1 or FUTURE2 option in a RESET statement, PROC NETFLOW may be able to resume optimization in a subsequent run.

You can turn off these current output data set specifications by specifying ARCOUT=NULL, NODEOUT=NULL, CONOUT=NULL, or DUALOUT=NULL.

If PROC NETFLOW is outputting observations to an output data set and you want this to stop, press the keys used to stop SAS procedures. PROC NETFLOW waits, if necessary, and then executes the next statement.

ARCOUT=*SAS-data-set*
AOUT=*SAS-data-set*

names the output data set that receives all information concerning arc and nonarc variables, including flows and other information concerning the current solution and the supply and demand information. The current solution is the latest solution found by the optimizer when the optimization neglecting side constraints is halted or the unconstrained optimum is reached.

You can specify an ARCOUT= data set in any RESET statement before the unconstrained optimum is found (even at commencement). Once the unconstrained optimum has been reached, use the SAVE statement to produce observations in an ARCOUT= data set. Once optimization that considers constraints starts, you will be unable to obtain an ARCOUT= data set. Instead, use a CONOUT= data set to get the current solution. See the section "ARCOUT= and CONOUT= Data Sets" on page 524 for more information.

CONOUT=*SAS-data-set*
COUT=*SAS-data-set*

names the output data set that contains the primal solution obtained after optimization considering side constraints reaches the optimal solution. You can specify a CONOUT= data set in any RESET statement before the constrained optimum is found (even at commencement or while optimizing neglecting constraints). Once the constrained optimum has been reached, or during stage 2 optimization, use the SAVE statement to produce observations in a CONOUT= data set. See the section "ARCOUT= and CONOUT= Data Sets" on page 524 for more information.

DUALOUT=*SAS-data-set*
DOUT=*SAS-data-set*

names the output data set that contains the dual solution obtained after doing optimization that considers side constraints reaches the optimal solution. You can specify a DUALOUT= data set in any RESET statement before the constrained optimum is found (even at commencement or while optimizing neglecting constraints). Once the constrained optimum has been reached, or during stage 2 optimization, use the SAVE statement to produce observations in a DUALOUT= data set. See the section "NODEOUT= and DUALOUT= Data Sets" on page 525 for more information.

NODEOUT=*SAS-data-set*
NOUT=*SAS-data-set*
names the output data set that receives all information about nodes (supply/demand and nodal dual variable values) and other information concerning the unconstrained optimal solution.

You can specify a NODEOUT= data set in any RESET statement before the unconstrained optimum is found (even at commencement). Once the unconstrained optimum has been reached, or during stage 1 optimization, use the SAVE statement to produce observations in a NODEOUT= data set. Once optimization that considers constraints starts, you will not be able to obtain a NODEOUT= data set. Instead use a DUALOUT= data set to get the current solution. See the section "NODEOUT= and DUALOUT= Data Sets" on page 525 for more information.

Options to Halt Optimization

The following options indicate conditions when optimization is to be halted. You then have a chance to use PROC NETFLOW interactively. If the NETFLOW procedure is optimizing and you want optimization to halt immediately, press the CTRL-BREAK key combination used to stop SAS procedures. Doing this is equivalent to PROC NETFLOW finding that some prespecified condition of the current solution under which optimization should stop has occurred.

If optimization does halt, you may need to change the conditions for when optimization should stop again. For example, if the number of iterations exceeded MAXIT2, use the RESET statement to specify a larger value for the MAXIT2= option before the next RUN statement. Otherwise, PROC NETFLOW will immediately find that the number of iterations still exceeds MAXIT2 and halt without doing any additional optimization.

ENDPAUSE1
indicates that PROC NETFLOW will pause after the unconstrained optimal solution has been obtained and information about this solution has been output to the current ARCOUT= data set, NODEOUT= data set, or both. The procedure then executes the next statement, or waits if no subsequent statement has been specified.

FEASIBLEPAUSE1
FP1
indicates that unconstrained optimization should stop once a feasible solution is reached. PROC NETFLOW checks for feasibility every 10 iterations. A solution is feasible if there are no artificial arcs having nonzero flow assigned to be conveyed through them. The presence of artificial arcs with nonzero flows means that the current solution does not satisfy all the nodal flow conservation constraints implicit in network problems.

MAXIT1=*m*
specifies the maximum number of primal simplex iterations PROC NETFLOW is to perform in stage 1. The default value for the MAXIT1= option is 1000. If MAXIT1=*m* iterations are performed and you want to continue unconstrained optimization, reset MAXIT1= to a number larger than the number of iterations already performed and issue another RUN statement.

NOENDPAUSE1
NOEP1
negates the ENDPAUSE1 option.

NOFEASIBLEPAUSE1
NOFP1
negates the FEASIBLEPAUSE1 option.

PAUSE1=*p*
indicates that PROC NETFLOW will halt unconstrained optimization and pause when the remainder of the number of stage 1 iterations divided by the value of the PAUSE1= option is zero. If present, the next statement is executed; if not, the procedure waits for the next statement to be specified. The default value for PAUSE1= is 999999.

FEASIBLEPAUSE2
FP2
NOFEASIBLEPAUSE2
NOFP2
PAUSE2=*p*
MAXIT2=*m*
are the stage 2 constrained optimization counterparts of the options described previously and having as a suffix the numeral 1.

Options Controlling the Network Simplex Optimization

BIGM1
NOTWOPHASE1
TWOPHASE1
NOBIGM1
BIGM1 indicates that the "big-M" approach to optimization is used. Artificial variables are treated like real arcs, slacks, surpluses and nonarc variables. Artificials have very expensive costs. BIGM1 is the default.

TWOPHASE1 indicates that the two-phase approach is used instead of the big-M approach. At first, artificial variables are the only variables to have nonzero objective function coefficients. An artificial variable's objective function coefficient is temporarily set to 1 and PROC NETFLOW minimizes. When all artificial variables have zero value, PROC NETFLOW has found a feasible solution, and phase 2 commences. Arcs and nonarc variables have their real costs and objective function coefficients.

Before all artificial variables are driven to have zero value, you can toggle between the big-M and the two-phase approaches by specifying BIGM1 or TWOPHASE1 in a RESET statement. The option NOTWOPHASE1 is synonymous with BIGM1, and NOBIGM1 is synonymous with TWOPHASE1.

CYCLEMULT1=*c*
MINBLOCK1=*m*
NOPERTURB1
PERTURB1

In an effort to reduce the number of iterations performed when the problem is highly degenerate, PROC NETFLOW has in stage 1 optimization adopted an algorithm outlined in Ryan and Osborne (1988).

If the number of consecutive degenerate pivots (those with no progress toward the optimum) performed equals the value of the CYCLEMULT1= option times the number of nodes, the arcs that were "blocking" (can leave the basis) are added to a list. In subsequent iterations, of the arcs that now can leave the basis, the one chosen to leave is an arc on the list of arcs that could have left in the previous iteration. In other words, perference is given to arcs that "block" many iterations. After several iterations, the list is cleared.

If the number of blocking arcs is less than the value of the MINBLOCK1= option, a list is not kept. Otherwise, if PERTURB1 is specified, the arc flows are perturbed by a random quantity, so that arcs on the list that block subsequent iterations are chosen to leave the basis randomly. Although perturbation often pays off, it is computationally expensive. Periodically, PROC NETFLOW has to clear out the lists and un-perturb the solution. You can specify NOPERTURB1 to prevent perturbation.

Defaults are CYCLEMULT1=0.15, MINBLOCK1=2, and NOPERTURB1.

LRATIO1

specifies the type of ratio test to use in determining which arc leaves the basis in stage 1. In some iterations, more than one arc is eligible to leave the basis. Of those arcs that can leave the basis, the leaving arc is the first encountered by the algorithm if the LRATIO1 option is specified. Specifying the LRATIO1 option can decrease the chance of cycling but can increase solution times. The alternative to the LRATIO1 option is the NOLRATIO1 option, which is the default.

LRATIO2

specifies the type of ratio test to use in determining what leaves the basis in stage 2. In some iterations, more than one arc, constraint slack, surplus, or nonarc variable is eligible to leave the basis. If the LRATIO2 option is specified, the leaving arc, constraint slack, surplus, or nonarc variable is the one that is eligible to leave the basis first encountered by the algorithm. Specifying the LRATIO2 option can decrease the chance of cycling but can increase solution times. The alternative to the LRATIO2 option is the NOLRATIO2 option, which is the default.

NOLRATIO1

specifies the type of ratio test to use in determining which arc leaves the basis in stage 1. If the NOLRATIO1 option is specified, of those arcs that can leave the basis, the leaving arc has the minimum (maximum) cost if the leaving arc is to be nonbasic with flow capacity equal to its capacity (lower flow bound). If more than one possible leaving arc has the minimum (maximum) cost, the first such arc encountered is chosen. Specifying the NOLRATIO1 option can decrease solution times, but can increase the chance of cycling. The alternative to the NOLRATIO1 option is the LRATIO1 option. The NOLRATIO1 option is the default.

NOLRATIO2

specifies the type of ratio test to use in determining which arc leaves the basis in stage 2. If the NOLRATIO2 option is specified, the leaving arc, constraint slack, surplus, or nonarc variable is the one eligible to leave the basis with the minimum (maximum) cost or objective function coefficient if the leaving arc, constraint slack or nonarc variable is to be nonbasic with flow or value equal to its capacity or upper value bound (lower flow or value bound), respectively. If several possible leaving arcs, constraint slacks, surpluses, or nonarc variables have the minimum (maximum) cost or objective function coefficient, then the first encountered is chosen. Specifying the NOLRATIO2 option can decrease solution times, but can increase the chance of cycling. The alternative to the NOLRATIO2 option is the LRATIO2 option. The NOLRATIO2 option is the default.

Options Applicable to Constrained Optimization

The INVFREQ= option is relevant only if INVD_2D is specified in the PROC NETFLOW statement; that is, the inverse of the working basis matrix is being stored and processed as a two-dimensional array. The REFACTFREQ=, U=, MAXLUUPDATES=, and MAXL= options are relevant if the INVD_2D option is not specified in the PROC NETFLOW statement; that is, if the working basis matrix is **LU** factored.

BIGM2
NOTWOPHASE2
TWOPHASE2
NOBIGM2

are the stage 2 constrained optimization counterparts of the options BIGM1, NOTWOPHASE1, TWOPHASE1, and NOBIGM1.

The TWOPHASE2 option is often better than the BIGM2 option when the problem has many side constraints.

INVFREQ=*n*

recalculates the working basis matrix inverse whenever *n* iterations have been performed where *n* is the value of the INVFREQ= option. Although a relatively expensive task, it is prudent to do as roundoff errors accumulate, especially affecting the elements of this matrix inverse. The default is INVFREQ=50. The INVFREQ= option should be used only if the INVD_2D option is specified in the PROC NETFLOW statement.

INTFIRST

In some iterations, it is found that what must leave the basis is an arc that is part of the spanning tree representation of the network part of the basis (called a *key* arc). It is necessary to interchange another basic arc not part of the tree (called a *nonkey* arc) with the tree arc that leaves to permit the basis update to be performed efficiently. Specifying the INTFIRST option indicates that of the nonkey arcs eligible to be swapped with the leaving key arc, the one chosen to do so is the first encountered by the algorithm. If the INTFIRST option is not specified, all such arcs are examined and the one with the best cost is chosen.

The terms *key* and *nonkey* are used because the algorithm used by PROC NETFLOW for network optimization considering side constraints (GUB-based, Primal Partitioning, or Factorization) is a variant of an algorithm originally developed to solve linear programming problems with generalized upper bounding constraints. The terms *key* and *nonkey* were coined then. The STATUS SAS variable in the ARCOUT= and CONOUT= data sets and the STATUS column in tables produced when PRINT statements are processed indicate whether basic arcs are key or nonkey. Basic nonarc variables are always nonkey.

MAXL=*m*

If the working basis matrix is **LU** factored, **U** is an upper triangular matrix and **L** is a lower triangular matrix corresponding to a sequence of elementary matrix row operations required to change the working basis matrix into **U**. **L** and **U** enable substitution techniques to be used to solve the linear systems of the simplex algorithm. Among other things, the **LU** processing strives to keep the number of **L** elementary matrix row operation matrices small. A buildup in the number of these could indicate that fill-in is becoming excessive and the computations involving **L** and **U** will be hampered. Refactorization should be performed to restore **U** sparsity and reduce **L** information. When the number of **L** matrix row operations exceeds the value of the MAXL= option, a refactorization is done rather than one or more updates. The default value for MAXL= is 10 times the number of side constraints. The MAXL= option should not be used if INVD_2D is specified in the PROC NETFLOW statement.

MAXLUUPDATES=*m*
MLUU=*m*

In some iterations, PROC NETFLOW must either perform a series of single column updates or a complete refactorization of the working basis matrix. More than one column of the working basis matrix must change before the next simplex iteration can begin. The single column updates can often be done faster than a complete refactorization, especially if few updates are necessary, the working basis matrix is sparse, or a refactorization has been performed recently. If the number of columns that must change is less than the value specified in the MAXLUUPDATES= option, the updates are attempted; otherwise, a refactorization is done. Refactorization also occurs if the sum of the number of columns that must be changed and the number of **LU** updates done since the last refactorization exceeds the value of the REFACTFREQ= option. The MAXLUUPDATES= option should not be used if the INVD_2D option is specified in the PROC NETFLOW statement.

In some iterations, a series of single column updates are not able to complete the changes required for a working basis matrix because, ideally, all columns should change at once. If the update cannot be completed, PROC NETFLOW performs a refactorization. The default value is 5.

NOINTFIRST

indicates that of the arcs eligible to be swapped with the leaving arc, the one chosen to do so has the best cost. See the INTFIRST option.

REFACTFREQ=*r*
RFF=*r*

specifies the maximum number of **L** and **U** updates between refactorization of the working basis matrix to reinitialize **LU** factors. In most iterations, one or several Bartels-Golub updates can be performed. An update is performed more quickly than a complete refactorization. However, after a series of updates, the sparsity of the **U** factor is degraded. A refactorization is necessary to regain sparsity and to make subsequent computations and updates more efficient. The default value is 50. The REFACTFREQ= option should not be used if INVD_2D is specified in the PROC NETFLOW statement.

U=*u*

controls the choice of pivot during **LU** decomposition or Bartels-Golub update. When searching for a pivot, any element less than the value of the U= option times the largest element in its matrix row is excluded, or matrix rows are interchanged to improve numerical stability. The U= option should have values on or between ZERO2 and 1.0. Decreasing the value of the U= option biases the algorithm toward maintaining sparsity at the expense of numerical stability and vice-versa. Reid (1975) suggests that the value of 0.01 is acceptable and this is the default for the U= option. This option should not be used if INVD_2D is specified in the PROC NETFLOW statement.

Pricing Strategy Options

There are three main types of pricing strategies:

- PRICETYPEx=NOQ
- PRICETYPEx=BLAND
- PRICETYPEx=Q

The one that usually performs better than the others is PRICETYPEx=Q, so this is the default.

Because the pricing strategy takes a lot of computational time, you should experiment with the following options to find the optimum specification. These options influence how the pricing step of the simplex iteration is performed. See the section "Pricing Strategies" on page 528 for further information.

PRICETYPEx=BLAND or PTYPEx=BLAND

PRICETYPEx=NOQ or PTYPEx=NOQ

- PxSCAN=BEST
- PxSCAN=FIRST
- PxSCAN=PARTIAL and PxNPARTIAL=*p*

PRICETYPEx=Q or PTYPEx=Q
QSIZEx=*q* or Qx=*q*

REFRESHQx=*r*
REDUCEQSIZEx=*r*
REDUCEQx=*r*

- PxSCAN=BEST
- PxSCAN=FIRST
- PxSCAN=PARTIAL and PxNPARTIAL=*p*
- QxFILLSCAN=BEST
- QxFILLSCAN=FIRST
- QxFILLSCAN=PARTIAL and QxFILLNPARTIAL=*q*

For stage 2 optimization, you can specify P2SCAN=ANY, which is used in conjunction with the DUALFREQ= option.

Miscellaneous Options

FUTURE1
signals that PROC NETFLOW must output extra observations to the NODEOUT= and ARCOUT= data sets. These observations contain information about the solution found by doing optimization neglecting any side constraints. These two data sets then can be used as the NODEDATA= and ARCDATA= data sets, respectively, in subsequent PROC NETFLOW runs with the WARM option specified. See the section "Warm Starts" on page 542.

FUTURE2
signals that PROC NETFLOW must output extra observations to the DUALOUT= and CONOUT= data sets. These observations contain information about the solution found by optimization that considers side constraints. These two data sets can then be used as the NODEDATA= data set (also called the DUALIN= data set) and the ARCDATA= data sets, respectively, in subsequent PROC NETFLOW runs with the WARM option specified. See the section "Warm Starts" on page 542.

MOREOPT
The MOREOPT option turns off all optimality and infeasibility flags that may have been raised. Unless this is done, PROC NETFLOW will not do any optimization when a RUN statement is specified.

If PROC NETFLOW determines that the problem is infeasible, it will not do any more optimization unless you specify MOREOPT in a RESET statement. At the same time, you can try resetting options (particularly zero tolerances) in the hope that the infeasibility was raised incorrectly.

Consider the following example:

```
proc netflow
   nodedata=noded          /* supply and demand data   */
   arcdata=arcd1           /* the arc descriptions     */
   condata=cond1           /* the side constraints     */
   conout=solution;        /* output the solution      */
   run;
```

```
/* Netflow states that the problem is infeasible.     */
/* You suspect that the zero tolerance is too large   */
   reset zero2=1.0e-10 moreopt;
   run;
/* Netflow will attempt more optimization.           */
/* After this, if it reports that the problem is      */
/* infeasible, the problem really might be infeasible */
```

If PROC NETFLOW finds an optimal solution, you might want to do additional optimization to confirm that an optimum has really been reached. Specify the MOREOPT option in a RESET statement. Reset options, but in this case tighten zero tolerances.

NOFUTURE1
negates the FUTURE1 option.

NOFUTURE2
negates the FUTURE2 option.

NOSCRATCH
negates the SCRATCH option.

NOZTOL1
indicates that the majority of tests for roundoff error should not be done. Specifying the NOZTOL1 option and obtaining the same optimal solution as when the NOZTOL1 option is not specified in the PROC NETFLOW statement (or the ZTOL1 option is specified), verifies that the zero tolerances were not too high. Roundoff error checks that are critical to the successful functioning of PROC NETFLOW and any related readjustments are always done.

NOZTOL2
indicates that the majority of tests for roundoff error are not to be done during an optimization that considers side constraints. The reasons for specifying the NOZTOL2 option are the same as those for specifying the NOZTOL1 option for stage 1 optimization (see the NOZTOL1 option).

OPTIM_TIMER
indicates that the procedure is to issue a message to the SAS log giving the CPU time spent doing optimization. This includes the time spent preprocessing, performing optimization, and postprocessing. Not counted in that time is the rest of the procedure execution, which includes reading the data and creating output SAS data sets.

The time spent optimizing can be small compared to the total CPU time used by the procedure. This is especially true when the problem is quite small (e.g., fewer than 10,000 variables).

SCRATCH
specifies that you do not want PROC NETFLOW to enter or continue stage 1 of the algorithm. Rather than specify RESET SCRATCH, you can use the CONOPT statement.

VERBOSE=*v*
limits the number of similar messages that are displayed on the SAS log.

For example, when reading the ARCDATA= data set, PROC NETFLOW might have cause to issue the following message many times:

```
ERROR: The HEAD list variable value in obs i in ARCDATA is
       missing and the TAIL list variable value of this obs
       is nonmissing. This is an incomplete arc specification.
```

If there are many observations that have this fault, messages that are similar are issued for only the first VERBOSE= such observations. After the ARCDATA= data set has been read, PROC NETFLOW will issue the message

```
NOTE: More messages similar to the ones immediately above
      could have been issued but were suppressed as
      VERBOSE=v.
```

If observations in the ARCDATA= data set have this error, PROC NETFLOW stops and you have to fix the data. Imagine that this error is only a warning and PROC NETFLOW proceeded to other operations such as reading the CONDATA= data set. If PROC NETFLOW finds there are numerous errors when reading that data set, the number of messages issued to the SAS log are also limited by the VERBOSE= option.

If you have a problem with a large number of side constraints and for some reason you stop stage 2 optimization early, PROC NETFLOW indicates that constraints are violated by the current solution. Specifying VERBOSE=*v* allows at most *v* violated constraints to be written to the log. If there are more, these are not displayed.

When PROC NETFLOW finishes and messages have been suppressed, the message

```
NOTE: To see all messages, specify VERBOSE=vmin.
```

is issued. The value of *vmin* is the smallest value that should be specified for the VERBOSE= option so that *all* messages are displayed if PROC NETFLOW is run again with the same data and everything else (except the VERBOSE= option) unchanged. No messages are suppressed.

The default value for the VERBOSE= option is 12.

ZERO1=*z*
Z1=*z*

specifies the zero tolerance level in stage 1. If the NOZTOL1 option is not specified, values within *z* of zero are set to 0.0, where *z* is the value of the ZERO1= option. Flows close to the lower flow bound or capacity of arcs are reassigned those exact values. Two values are deemed to be close if one is within *z* of the other. The default value for the ZERO1= option is 0.000001. Any value specified for the ZERO1= option that is < 0.0 or > 0.0001 is invalid.

ZERO2=*z*
Z2=*z*

specifies the zero tolerance level in stage 2. If the NOZTOL2 option is not specified, values within *z* of zero are set to 0.0, where *z* is the value of the ZERO2= option.

Flows close to the lower flow bound or capacity of arcs are reassigned those exact values. If there are nonarc variables, values close to the lower or upper value bound of nonarc variables are reassigned those exact values. Two values are deemed to be close if one is within *z* of the other. The default value for the ZERO2= option is 0.000001. Any value specified for the ZERO2= option that is < 0.0 or > 0.0001 is invalid.

ZEROTOL=*z*

specifies the zero tolerance used when PROC NETFLOW must compare any real number with another real number, or zero. For example, if x and y are real numbers, then for x to be considered greater than y, x must be at least $y + z$. The ZEROTOL= option is used throughout any PROC NETFLOW run.

ZEROTOL=*z* controls the way PROC NETFLOW performs all double precision comparisons; that is, whether a double precision number is equal to, not equal to, greater than (or equal to), or less than (or equal to) zero or some other double precision number. A double precision number is deemed to be the same as another such value if the absolute difference between them is less than or equal to the value of the ZEROTOL= option.

The default value for the ZEROTOL= option is $1.0\text{E}{-14}$. You can specify the ZEROTOL= option in the NETFLOW or RESET statement. Valid values for the ZEROTOL= option must be > 0.0 and < 0.0001. Do not specify a value too close to zero as this defeats the purpose of the ZEROTOL= option. Neither should the value be too large, as comparisons might be incorrectly performed.

The ZEROTOL= option is different from the ZERO1= and ZERO2= options in that ZERO1= and ZERO2= options work when determining whether optimality has been reached, whether an entry in the updated column in the ratio test of the simplex method is zero, whether a flow is the same as the arc's capacity or lower bound, or whether the value of a nonarc variable is at a bound. The ZEROTOL= option is used in all other general double precision number comparisons.

ZTOL1

indicates that all tests for roundoff error are performed during stage 1 optimization. Any alterations are carried out. The opposite of the ZTOL1 option is the NOZTOL1 option.

ZTOL2

indicates that all tests for roundoff error are performed during stage 2 optimization. Any alterations are carried out. The opposite of the ZTOL2 option is the NOZTOL2 option.

Interior Point Algorithm Options

FACT_METHOD=*f*

enables you to choose the type of algorithm used to factorize and solve the main linear systems at each iteration of the interior point algorithm.

FACT_METHOD=LEFT_LOOKING is new for SAS 9.1.2. It uses algorithms described in George, Liu, and Ng (2001). Left looking is one of the main methods used

to perform Cholesky optimization and, along with some recently developed implementation approaches, can be faster and require less memory than other algorithms.

Specify FACT_METHOD=USE_OLD if you want the procedure to use the only factorization available prior to SAS 9.1.2.

TOLDINF=*t*
RTOLDINF=*t*
specifies the allowed amount of dual infeasibility. In the section "Interior Point Algorithmic Details" on page 567, the vector $infeas_d$ is defined. If all elements of this vector are $\leq t$, the solution is deemed feasible. $infeas_d$ is replaced by a zero vector, making computations faster. This option is the dual equivalent to the TOLPINF= option. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLPINF=*t*
RTOLPINF=*t*
specifies the allowed amount of primal infeasibility. This option is the primal equivalent to the TOLDINF= option. In the section "Interior Point: Upper Bounds" on page 575, the vector $infeas_b$ is defined. In the section "Interior Point Algorithmic Details" on page 567, the vector $infeas_c$ is defined. If all elements in these vectors are $\leq t$, the solution is deemed feasible. $infeas_b$ and $infeas_c$ are replaced by zero vectors, making computations faster. Increasing the value of the TOLPINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLTOTDINF=*t*
RTOLTOTDINF=*t*
specifies the allowed total amount of dual infeasibility. In the section "Interior Point Algorithmic Details" on page 567, the vector $infeas_d$ is defined. If $\sum_{i=1}^{n} infeas_{di} \leq t$, the solution is deemed feasible. $infeas_d$ is replaced by a zero vector, making computations faster. This option is the dual equivalent to the TOLTOTPINF= option. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

TOLTOTPINF=*t*
RTOLTOTPINF=*t*
specifies the allowed total amount of primal infeasibility. This option is the primal equivalent to the TOLTOTDINF= option. In the section "Interior Point: Upper Bounds" on page 575, the vector $infeas_b$ is defined. In the section "Interior Point Algorithmic Details" on page 567, the vector $infeas_c$ is defined. If $\sum_{i=1}^{n} infeas_{bi} \leq t$ and $\sum_{i=1}^{m} infeas_{ci} \leq t$, the solution is deemed feasible. $infeas_b$ and $infeas_c$ are replaced by zero vectors, making computations faster. Increasing the value of the TOLTOTPINF= option too much can lead to instability, but a modest increase can give the algorithm added flexibility and decrease the iteration count. Valid values for t are greater than $1.0\text{E}{-12}$. The default is $1.0\text{E}{-7}$.

CHOLTINYTOL=*c*
RCHOLTINYTOL=*c*
specifies the cut-off tolerance for Cholesky factorization of the $A\Theta A^{-1}$. If a diagonal value drops below *c*, the row is essentially treated as dependent and is ignored in the

factorization. Valid values for *c* are between $1.0\text{E}{-}30$ and $1.0\text{E}{-}6$. The default value is $1.0\text{E}{-}8$.

DENSETHR=*d*
RDENSETHR=*d*

specifies the density threshold for Cholesky processing. When the symbolic factorization encounters a column of L that has DENSETHR= proportion of nonzeros and the remaining part of L is at least 12×12, the remainder of L is treated as dense. In practice, the lower right part of the Cholesky triangular factor L is quite dense and it can be computationally more efficient to treat it as 100% dense. The default value for *d* is 0.7. A specification of $d \leq 0.0$ causes all dense processing; $d \geq 1.0$ causes all sparse processing.

PDSTEPMULT=*p*
RPDSTEPMULT=*p*

specifies the step-length multiplier. The maximum feasible step-length chosen by the Primal-Dual with Predictor-Corrector algorithm is multiplied by the value of the PDSTEPMULT= option. This number must be less than 1 to avoid moving beyond the barrier. An actual step length greater than 1 indicates numerical difficulties. Valid values for *p* are between 0.01 and 0.999999. The default value is 0.99995.

In the section "Interior Point Algorithmic Details" on page 567, the solution of the next iteration is obtained by moving along a direction from the current iteration's solution:

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where α is the maximum feasible step-length chosen by the interior point algorithm. If $\alpha \leq 1$, then α is reduced slightly by multiplying it by *p*. α is a value as large as possible but ≤ 1.0 and not so large that an x_i^{k+1} or s_i^{k+1} of some variable i is "too close" to zero.

PRSLTYPE=*p*
IPRSLTYPE=*p*

Preprocessing the linear programming problem often succeeds in allowing some variables and constraints to be temporarily eliminated from the LP that must be solved. This reduces the solution time and possibly also the chance that the optimizer will run into numerical difficulties. The task of preprocessing is inexpensive to do.

You control how much preprocessing to do by specifying PRSLTYPE=*p*, where *p* can be -1, 0, 1, 2, or 3.

-1 Do not perform preprocessing. For most problems, specifying PRSLTYPE=-1 is *not* recommended.

0 Given upper and lower bounds on each variable, the greatest and least contribution to the row activity of each variable is computed. If these are within the limits set by the upper and lower bounds on the row activity, then the row is redundant and can be discarded. Try to tighten the bounds on any of the variables it can. For example, if all coefficients in a constraint are positive and all variables have zero lower bounds, then the row's smallest contribution is zero. If the rhs value of this constraint is zero, then if the constraint type is $=$ or $\leq$, all the variables in that constraint can be fixed to zero. These variables and the constraint can be removed. If the constraint type is $\geq$, the constraint is redundant. If the rhs is negative and the constraint is $\leq$, the problem is infeasible. If just one variable in a row is not fixed, use the row to impose an implicit upper or lower bound on the variable and then eliminate the row. The preprocessor also tries to tighten the bounds on constraint right-hand sides.

1 When there are exactly two unfixed variables with coefficients in an equality constraint, solve for one in terms of the other. The problem will have one less variable. The new matrix will have at least two fewer coefficients and one less constraint. In other constraints where both variables appear, two coefs are combined into one. PRSLTYPE=0 reductions are also done.

2 It may be possible to determine that an equality constraint is not constraining a variable. That is, if all variables are nonnegative, then $x - \sum_i y_i = 0$ does not constrain x, since it must be nonnegative if all the y_i's are nonnegative. In this case, eliminate x by subtracting this equation from all others containing x. This is useful when the only other entry for x is in the objective function. Perform this reduction if there is at most one other nonobjective coefficient. PRSLTYPE=0 reductions are also done.

3 All possible reductions are performed. PRSLTYPE=3 is the default.

Preprocessing is iterative. As variables are fixed and eliminated, and constraints are found to be redundant and they too are eliminated, and as variable bounds and constraint right-hand sides are tightened, the LP to be optimized is modified to reflect these changes. Another iteration of preprocessing of the modified LP may reveal more variables and constraints that can be eliminated.

PRINTLEVEL2=*p*

is used when you want to see PROC NETFLOW's progress to the optimum. PROC NETFLOW will produce a table on the SAS log. A row of the table is generated during each iteration and may consist of values of

- the affine step complementarity
- the complementarity of the solution for the next iteration
- the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575)
- the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567)
- the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567)

As optimization progresses, the values in all columns should converge to zero. If you specify PRINTLEVEL2=2, all columns will appear in the table. If PRINTLEVEL2=1 is specified, only the affine step complementarity and the complementarity of the solution for the next iteration will appear. Some time is saved by not calculating the infeasibility values.

Interior Point Algorithm Options: Stopping Criteria

MAXITERB=*m*
IMAXITERB=*m*

specifies the maximum number of iterations of the interior point algorithm that can be performed. The default value for *m* is 100. One of the most remarkable aspects of the interior point algorithm is that for most problems, it usually needs to do a small number of iterations, *no matter the size of the problem*.

PDGAPTOL=*p*
RPDGAPTOL=*p*

specifies the primal-dual gap or duality gap tolerance. Duality gap is defined in the section "Interior Point Algorithmic Details" on page 567. If the relative gap $(duality\ gap/(c^T x))$ between the primal and dual objectives is smaller than the value of the PDGAPTOL= option and both the primal and dual problems are feasible, then PROC NETFLOW stops optimization with a solution that is deemed optimal. Valid values for *p* are between $1.0\text{E}{-12}$ and $1.0\text{E}{-1}$. The default is $1.0\text{E}{-7}$.

STOP_C=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq s$, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

STOP_DG=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq s$, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

STOP_IB=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq s$, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

STOP_IC=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $\leq s$, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

STOP_ID=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $\leq$ *s*, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

AND_STOP_C=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq$ *s*, *and* the conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

AND_STOP_DG=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq$ *s*, *and* the conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

AND_STOP_IB=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $\leq$ *s*, *and* the conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

AND_STOP_IC=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $\leq$ *s*, *and* the conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

AND_STOP_ID=*s*

is used to determine whether optimization should stop. At the beginning of each iteration, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $\leq$ *s*, *and* the conditions related to other AND_STOP parameters are also satisfied, optimization will stop. This option is discussed in the section "Stopping Criteria" on page 571.

KEEPGOING_C=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

KEEPGOING_DG=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

KEEPGOING_IB=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

KEEPGOING_IC=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

KEEPGOING_ID=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $> s$, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

AND_KEEPGOING_C=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if complementarity (the value of the Complem-ity column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, *and* the conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

AND_KEEPGOING_DG=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if the duality gap (the value of the Duality_gap column in the table produced when you specify PRINTLEVEL2=1 or PRINTLEVEL2=2) is $> s$, *and* the conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

AND_KEEPGOING_IB=*s*

is used to determine whether optimization should stop. If a stopping condition is

met, if total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575; this value appears in the Tot_infeasb column in the table produced when you specify PRINTLEVEL2=2) is $> s$, *and* the conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

AND_KEEPGOING_IC=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasc column in the table produced when you specify PRINTLEVEL2=2) is $> s$, *and* the conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

AND_KEEPGOING_ID=*s*

is used to determine whether optimization should stop. If a stopping condition is met, if total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567; this value appears in the Tot_infeasd column in the table produced when you specify PRINTLEVEL2=2) is $> s$, *and* the conditions related to other AND_KEEPGOING parameters are also satisfied, optimization will continue. This option is discussed in the section "Stopping Criteria" on page 571.

RHS Statement

RHS *variable* **;**

The RHS variable list is used when the dense format of the CONDATA= data set is used. The values of the SAS variable specified in the RHS list are constraint right-hand-side values. If the RHS list is not specified, the CONDATA= data set is searched and a SAS variable with the name _RHS_ is used. If there is no RHS list and no SAS variable named _RHS_, all constraints are assumed to have zero right-hand-side values. The RHS list variable must have numeric values.

ROW Statement

ROW *variables* **;**

The ROW list is used when either the sparse or the dense format of side constraints is being used. SAS variables in the ROW list have values that are constraint or special row names. The SAS variables in the ROW list must have character values.

If the dense data format is used, there must be only one SAS variable in this list. In this case, if a ROW list is not specified, the CONDATA= data set is searched and the SAS variable with the name _ROW_ or _CON_ is used.

If the sparse data format is used and the ROW statement is not specified, the CONDATA= data set is searched and SAS variables with names beginning with _ROW or _CON are used. The number of SAS variables in the ROW list must not be less than the number of SAS variables in the COEF list. The ith ROW list

variable is paired with the ith COEF list variable. If the number of ROW list variables is greater than the number of COEF list variables, the last ROW list variables have no COEF partner. These ROW list variables that have no corresponding COEF list variable are used in observations that have a TYPE list variable value. All ROW list variable values are tagged as having the type indicated. If there is no TYPE list variable, all ROW list variable values are constraint names.

RUN Statement

RUN ;

The RUN statement causes optimization to be started or resumed. The RUN statement has no options. If PROC NETFLOW is called and is not terminated because of an error or a QUIT statement, and you have not used a RUN statement, a RUN statement is assumed implicitly as the last statement of PROC NETFLOW. Therefore, PROC NETFLOW always performs optimization and saves the obtained (optimal) solution in the current output data sets.

SAVE Statement

SAVE *options* **;**

The options available with the SAVE statement of PROC NETFLOW are summarized by purpose in the following table.

Table 5.4. Functional Summary, SAVE Statement

Description	Statement	Option
Output Data Set Options:		
unconstrained primal solution data set	SAVE	ARCOUT=
unconstrained dual solution data set	SAVE	NODEOUT=
constrained primal solution data set	SAVE	CONOUT=
constrained dual solution data set	SAVE	DUALOUT=

The SAVE statement can be used to specify output data sets and create observations in these data sets. Use the SAVE statement if no optimization is to be performed before these output data sets are created.

The SAVE statement must be used to save solutions in data sets if there is no more optimization to do. If more optimization is to be performed, after which you want to save the solution, then do one of the following:

- Submit a RUN statement followed by a SAVE statement.
- Use the PROC NETFLOW or RESET statement to specify current output data sets. After optimization, output data sets are created and observations are automatically sent to the current output data sets.

Consider the following example:

```
proc netflow options; lists;
   reset maxit1=10 maxit2=25
         arcout=arcout0   nodeout=nodeout0
         conout=conout0   dualout=dualout0;
   run;
   /* Stage 1 optimization stops after iteration 10. */
   /* No output data sets are created yet.           */
   save arcout=arcout1 nodeout=nodeout1;
   /* arcout1 and nodeout1 are created.              */
   reset arcout=arcout2 maxit1=999999;
   run;
   /* The stage 1 optimum is reached.                */
   /* Arcout2 and nodeout0 are created.              */
   /* Arcout0 is not created as arcout=arcout2 over- */
   /* rides the arcout=arcout0 specified earlier.    */
   /* Stage 2 optimization stops after 25 iterations */
   /* as MAXIT2=25 was specified.                    */
   save conout=conout1;
   /* Conout1 is created.                            */
   reset maxit2=999999 dualout=null;
   run;
   /* The stage 2 optimum is reached.                */
   /* Conout0 is created.                            */
   /* No dualout is created as the last NETFLOW or   */
   /* reset statements dualout=data set specification*/
   /* was dualout=null.                              */
```

The data sets specified in the PROC NETFLOW and RESET statements are created when an optimal solution is found. The data sets specified in SAVE statements are created immediately.

The data sets in the preceding example are all distinct, but this need not be the case. The only exception to this is that the ARCOUT= data set and the NODEOUT= data set (or the CONOUT= data set and the DUALOUT= data set) that are being created at the same time must be distinct. Use the SHOW DATASETS statement to examine what data sets are current and when they were created.

The following options can appear in the SAVE statement:

ARCOUT= *SAS-data-set* (or **AOUT=** *SAS-data-set*)

NODEOUT= *SAS-data-set* (or **NOUT=** *SAS-data-set*)

CONOUT= *SAS-data-set* (or **COUT=** *SAS-data-set*)

DUALOUT= *SAS-data-set* (or **DOUT=** *SAS-data-set*)

SHOW Statement

SHOW *options / qualifiers* **;**

The options available with the SHOW statement of PROC NETFLOW are summarized by purpose in the following table.

Table 5.5. Functional Summary, SHOW Statement

Description	Statement	Option
SHOW Statement Options:		
show problem, optimization status	SHOW	STATUS
show network model parameters	SHOW	NETSTMT
show data sets that have been or will be created	SHOW	DATASETS
show options that pause optimization	SHOW	PAUSE
show simplex algorithm options	SHOW	SIMPLEX
show pricing strategy options	SHOW	PRICING
show miscellaneous options	SHOW	MISC
SHOW Statement Qualifiers:		
display information only on relevant options	SHOW	/ RELEVANT
display options for current stage only	SHOW	/ STAGE

The SHOW statement enables you to examine the status of the problem and values of the RESET statement options. All output of the SHOW statement appears on the SAS log. The amount of information displayed when a SHOW statement is processed can be limited if some of the options of the SHOW statement are specified. These options indicate whether the problem status or a specific category of the RESET options is of interest. If no options are specified, the problem status and information on all RESET statement options in every category is displayed. The amount of displayed information can be limited further by following any SHOW statement options with a slash (/) and one or both qualifiers, RELEVANT and STAGE.

STATUS

produces one of the following optimization status reports, whichever is applicable. The warning messages are issued only if the network or entire problem is infeasible.

```
NOTE: Optimization Status.
      Optimization has not started yet.

NOTE: Optimization Status.
      Optimizing network (ignoring any side constraints).
      Number of iterations=17
      Of these, 3 were degenerate
WARNING: This optimization has detected that the network is
         infeasible.
```

```
NOTE: Optimization Status.
      Found network optimum (ignoring side constraints)
      Number of iterations=23
      Of these, 8 were degenerate

NOTE: Optimization Status.
      Optimizing side constrained network.
      Number of iterations=27
      Of these, 9 were degenerate
WARNING: This optimization has detected that the problem is
         infeasible.

NOTE: Optimization Status.
      Found side constrained network optimum
      Number of iterations=6
      Of these, 0 were degenerate
```

DATASETS

produces a report on output data sets.

```
NOTE: Current output SAS data sets
      No output data sets have been specified

NOTE: Current output SAS data sets
      ARCOUT=libname.memname
      NODEOUT=libname.memname
      CONOUT=libname.memname
      DUALOUT=libname.memname


NOTE: Other SAS data sets specified in previous ARCOUT=, NODEOUT=,
      CONOUT=, or DUALOUT=.
      libname.memname
              .
              .
              .

NOTE: Current output SAS data sets (SHOW DATASETS)
      libname.memname
              .
              .
              .
NOTE: SAS data sets specified as ARCOUT= NODEOUT= CONOUT= or
      DUALOUT= data sets in previous PROC NETFLOW, SET, RESET
      and SAVE statements.
      The number following the data set specification was the
      iteration number when observations were placed into the
      data set.
      libname.memname      iteration_number
              .                    .
              .                    .
              .                    .
```

PAUSE

produces a report on the current settings of options used to make optimization pause.

```
NOTE: Options and parameters that stop optimization for reasons
      other than infeasibility or optimality (SHOW PAUSE)
      FEASIBLEPAUSE1=FALSE
      ENDPAUSE1=FALSE
      PAUSE1=999999
      MAXIT1=1000
      FEASIBLEPAUSE2=FALSE
      PAUSE2=999999
      MAXIT2=999999
```

SIMPLEX

produces the following:

```
NOTE: Options and parameters that control the primal simplex
      network algorithm (excluding those that affect the
      pricing strategies) (SHOW SIMPLEX)
      LRATIO1=FALSE
      BIGM1=NOTWOPHASE1=TRUE, TWOPHASE1=NOBIGM1=FALSE
      CYCLEMULT1=0.15
      PERTURB1=FALSE
      MINBLOCK1=2
      INTFIRST=TRUE
      LRATIO2=FALSE
      BIGM2=NOTWOPHASE2=TRUE, TWOPHASE2=NOBIGM2=FALSE
      REFACTFREQ=50
      U=0.1
      MAXLUUPDATES=6
      MAXL=40
```

PRICING

produces the following:

```
NOTE: Options and parameters that control the primal simplex
      network algorithm pricing strategies (SHOW PRICING)
      PRICETYPE1=Q
      P1SCAN=FIRST
      P1NPARTIAL=10
      Q1FILLSCAN=FIRST
      QSIZE1=24
      REFRESHQ1=0.75
      REDUCEQSIZE1=1
      Q1FILLNPARTIAL=10
      PRICETYPE2=Q
      P2SCAN=FIRST
      P2NPARTIAL=10
      DUALFREQ=4
      Q2FILLSCAN=FIRST
      QSIZE2=24
      REFRESHQ2=0.75
```

```
REDUCEQSIZE2=1
Q2FILLNPARTIAL=10
```

MISC
produces the following:

```
NOTE: Miscellaneous options and parameters (SHOW MISC)
      VERBOSE=12
      ZTOL1=TRUE
      ZERO1=1E-6
      FUTURE1=FALSE
      ZTOL2=TRUE
      ZERO2=1E-6
      FUTURE2=FALSE
```

Following a slash (/), the qualifiers below can appear in any SHOW statement.

RELEVANT
indicates that you want information only on relevant options of the RESET statement. The following will *not* be displayed if / RELEVANT is specified:

- information on noncurrent data sets
- the options that control the reasons why stage 1 optimization should be halted and the options that control the simplex algorithm during stage 1 optimization, if the unconstrained optimum has been reached or constrained optimization has been performed
- if P1SCAN=BEST or P1SCAN=FIRST, the P1NPARTIAL= option is irrelevant
- if PRICETYPE1=BLAND or PRICETYPE1=NOQ, the options QSIZE1=, Q1FILLSCAN=, REFRESHQ1=, and REDUCEQSIZE1= are irrelevant
- if Q1FILLSCAN=BEST or Q1FILLSCAN=FIRST, the Q1FILLNPARTIAL= option is irrelevant
- the options that control the reasons stage 2 optimization should be halted, the options that control the simplex algorithm during stage 2 optimization, if the constrained optimum has been reached
- if P2SCAN=BEST or P2SCAN=FIRST, the P2NPARTIAL= option is irrelevant
- if PRICETYPE2=BLAND or PRICETYPE2=NOQ, the options QSIZE2=, Q2FILLSCAN=, REFRESHQ2=, and REDUCEQSIZE2= are irrelevant
- if Q2FILLSCAN=BEST or Q2FILLSCAN=FIRST, the Q2FILLNPARTIAL= option is irrelevant

STAGE
indicates that you want to examine only the options that affect the optimization that is performed if a RUN statement is executed next. Before any optimization has been done, only stage 2 options are displayed if the problem has side constraints and the

SCRATCH option is used, or if the CONOPT statement is specified. Otherwise, stage 1 options are displayed. If still optimizing neglecting constraints, only stage 1 options will be displayed. If the unconstrained optimum has been reached and optimization that considers constraints has not been performed, stage 1 options are displayed. If the problem has constraints, stage 2 options are displayed. If any optimization that considers constraints has been performed, only stage 2 options are displayed.

SUPDEM Statement

SUPDEM *variable* **;**

The SAS variable in this list, which must be present in the NODEDATA= data set, contains supply and demand information for the nodes in the NODE list. A positive SUPDEM list variable value $s\,(s > 0)$ denotes that the node named in the NODE list variable can supply s units of flow. A negative SUPDEM list variable value $-d\,(d > 0)$ means that this node demands d units of flow. If a SAS variable is not explicitly specified, a SAS variable with the name _SUPDEM_ or _SD_ in the NODEDATA= data set is used as the SUPDEM variable. If a node is a transshipment node (neither a supply nor a demand node), an observation associated with this node need not be present in the NODEDATA= data set. If present, the SUPDEM list variable value must be zero or a missing value.

SUPPLY Statement

SUPPLY *variable* **;**

The SUPPLY statement identifies the SAS variable in the ARCDATA= data set that contains the supply at the node named in that observation's TAILNODE list variable. If a tail node does not supply flow, use zero or a missing value for the observation's SUPPLY list variable value. If a tail node has supply capability, a missing value indicates that the supply quantity is given in another observation. It is not necessary to have a SUPPLY statement if the name of this SAS variable is _SUPPLY_.

TAILNODE Statement

TAILNODE *variable* **;**

TAIL *variable* **;**

FROMNODE *variable* **;**

FROM *variable* **;**

The TAILNODE statement specifies the SAS variable that must be present in the ARCDATA= data set that has as values the names of tail nodes of arcs. The TAILNODE variable must have character values. It is not necessary to have a TAILNODE statement if the name of the SAS variable is _TAIL_ or _FROM_. If the TAILNODE list variable value is missing, it is assumed that the observation of ARCDATA= data set contains information concerning a nonarc variable.

TYPE Statement

TYPE *variable* **;**

CONTYPE *variable* **;**

The TYPE list, which is optional, names the variable that has as values keywords that indicate either the constraint type for each constraint or the type of special rows in the CONDATA= data set. The values of the TYPE list variable also indicate, in each observation of the CONDATA= data set, how values of the VAR or COEF list variables are to be interpreted and how the type of each constraint or special row name is determined. If the TYPE list is not specified, the CONDATA= data set is searched and a SAS variable with the name _TYPE_ is used. Valid keywords for the TYPE variable are given below. If there is no TYPE statement and no other method is used to furnish type information (see the DEFCONTYPE= option), all constraints are assumed to be of the type "less than or equal to" and no special rows are used. The TYPE list variable must have character values and can be used when the data in the CONDATA= data set is in either the sparse or the dense format. If the TYPE list variable value has a * as its first character, the observation is ignored because it is a comment observation.

TYPE List Variable Values

The following are valid TYPE list variable values. The letters in boldface denote the characters that PROC NETFLOW uses to determine what type the value suggests. You need to have at least these characters. In the following list, the minimal TYPE list variable values have additional characters to aid you in remembering these values.

<	less than or equal to ($\leq$)
=	equal to (=)
>	greater than or equal to ($\geq$)
CAPAC	capacity
COST	cost
EQ	equal to
FREE	free row (used only for linear programs solved by interior point)
GAIN	gain in arc flow (generalized networks)
GE	greater than or equal to
LE	less than or equal to
LOSS	loss in arc flow (generalized networks)
LOWERBD	lower flow or value bound
LOW*blank*	lower flow or value bound
MAXIMIZE	maximize (opposite of cost)
MINIMIZE	minimize (same as cost)
MULT	value of arc multiplier (generalized networks)
OBJECTIVE	objective function (same as cost)
RHS	rhs of constraint
TYPE	type of constraint
UPPCOST	reserved for future use
UNREST	unrestricted variable (used only for linear programs solved by interior point)

UPPER	upper value bound or capacity; second letter must not be N

The valid TYPE list variable values in function order are

- **LE** less than or equal to ($\leq$)
- **EQ** equal to (=)
- **GE** greater than or equal to ($\geq$)
- **CO**ST
  MINIMIZE
  MAXIMIZE
  OBJECTIVE
  cost or objective function coefficient
- **CA**PAC
  UPPER
  capacity or upper value bound
- **LOW**ERBD
  LOW*blank*
  lower flow or value bound
- **R**HS rhs of constraint
- **T**YPE type of constraint
- **MU**LT
  GAIN
  LOSS
  value of arc multiplier in a generalized network

A TYPE list variable value that has the first character $*$ causes the observation to be treated as a comment. If the first character is a negative sign, then $\leq$ is the type. If the first character is a zero, then $=$ is the type. If the first character is a positive number, then $\geq$ is the type.

VAR Statement

VAR *variables* **;**

The VAR variable list is used when the dense data format is used. The names of these SAS variables are also names of the arc and nonarc variables that have data in the CONDATA= data set. If no explicit VAR list is specified, all numeric variables not on other lists are put onto the VAR list. The VAR list variables must have numeric values. The values of the VAR list variables in some observations can be interpreted differently than in other observations. The values can be coefficients in the side constraints, costs and objective function coefficients, or bound data. How these numeric values are interpreted depends on the value of each observation's TYPE or ROW list variable value. If there are no TYPE list variables, the VAR list variable values are all assumed to be side constraint coefficients.

Details: NETFLOW Procedure

Input Data Sets

PROC NETFLOW is designed so that there are as few rules as possible that you must obey when inputting a problem's data. Raw data are acceptable. This should cut the amount of processing required to groom the data before it is input to PROC NETFLOW. Data formats are so flexible that, due to space restrictions, all possible forms for a problem's data are not shown here. Try any reasonable form for your problem's data; it should be acceptable. PROC NETFLOW will outline its objections.

There are several ways to supply the same piece of data. You do not have to restrict yourself to using any particular one. If you use several ways, PROC NETFLOW checks that the data are consistent each time the data are encountered. After all input data sets have been read, data are merged so that the problem is described completely. The order of the observations is not important in any of the input data sets.

ARCDATA= Data Set

See the section "Getting Started: NETFLOW Procedure" on page 446 and the section "Introductory Example" on page 447 for a description of this input data set.

Note: Information for an arc or nonarc variable can be specified in more than one observation. For example, consider an arc directed from node A toward node B that has a cost of 50, capacity of 100, and lower flow bound of 10 flow units. Some possible observations in the ARCDATA= data set may be

TAIL	_HEAD_	_COST_	_CAPAC_	_LO_
A	B	50	.	.
A	B	.	100	.
A	B	.	.	10
A	B	50	100	.
A	B	.	100	10
A	B	50	.	10
A	B	50	100	10

Similarly, for a nonarc variable with upperbd=100, lowerbd=10, and objective function coefficient=50, the _TAIL_ and _HEAD_ values are missing.

CONDATA= Data Set

Regardless of whether the data in the CONDATA= data set is in the sparse or dense format, you will receive a warning if PROC NETFLOW finds a constraint row that has no coefficients. You will also be warned if any nonarc variable has no constraint coefficients.

Dense Input Format

If the dense format is used, most SAS variables in the CONDATA= data set belong to the VAR list and have names of arc and nonarc variables. These names can be values of the NAME list SAS variables in the ARCDATA= data set, or names of

nonarc variables, or names in the form *tail_head*, or any combination of these three forms. Names in the form *tail_head* are default arc names, and if you use them, you must specify node names in the ARCDATA= data set (values of the TAILNODE and HEADNODE list SAS variables) using no lowercase letters.

There can be three other variables in the CONDATA= data set, belonging, respectively, to the ROW, TYPE, and RHS lists. The CONDATA= data set of the oil industry example in the section "Introductory Example" on page 447 uses the dense data format.

Consider the SAS code that creates a dense format CONDATA= data set that has data for three constraints. This data set was used in the section "Introductory Example" on page 447.

```
data cond1;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
      _type_ $ _rhs_;
   datalines;
-2  .  1 .  . . >= -15
 . -2  . .  1 . GE -15
 .  . -3 4  . . EQ   0
 .  .  . . -3 4  =   0
;
```

You can use nonconstraint type values to furnish data on costs, capacities, lower flow bounds (and, if there are nonarc variables, objective function coefficients and upper and lower bounds). You need not have such (or as much) data in the ARCDATA= data set. The first three observations in the following data set are examples of observations that provide cost, capacity and lower bound data.

```
data cond1b;
   input m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
         _type_ $ _rhs_;
   datalines;
63 81 200   . 220   . cost   .
95 80 175 140 100 100 capac  .
20 10  50   .  35   . lo     .
-2  .   1   .   .   . >=   -15
 . -2   .   .   1   . GE   -15
 .  .  -3   4   .   . EQ     0
 .  .   .   .  -3   4  =     0
;
```

If a ROW list variable is used, the data for a constraint can be spread over more than one observation. To illustrate, the data for the first constraint, (which is called con1), and the cost and capacity data (in special rows called costrow and caprow, respectively) are spread over more than one observation in the following data set.

```
data cond1c;
   input _row_ $
```

```
          m_e_ref1 m_e_ref2 thruput1 r1_gas thruput2 r2_gas
          _type_ $ _rhs_;
   datalines;
costrow 63   .    .    .    .    . .       .
costrow  . 81 200    .    .    . cost    .
.        .  .   .    . 220    . cost    .
caprow   .  .   .    .   .    . capac   .
caprow  95  . 175    . 100 100 .       .
caprow   . 80 175 140   .    . .       .
lorow   20 10  50    .  35    . lo      .
con1    -2  .   1    .   .    . .       .
con1     .  .   .    .   .    . >=    -15
con2     . -2   .    .   1    . GE    -15
con3     .  .  -3    4   .    . EQ      0
con4     .  .   .    .  -3    4  =      0
;
```

Using both ROW and TYPE lists, you can use special row names. Examples of these are "costrow" and "caprow" in the last data set. It should be restated that in any of the input data sets of PROC NETFLOW, the order of the observations does not matter. However, the CONDATA= data set can be read more quickly if PROC NETFLOW knows what type of constraint or special row a ROW list variable value is. For example, when the first observation is read, PROC NETFLOW does not know whether costrow is a constraint or special row and how to interpret the value 63 for the arc with the name m_e_ref1. When PROC NETFLOW reads the second observation, it learns that costrow has type cost and that the values 81 and 200 are costs. When the entire CONDATA= data set has been read, PROC NETFLOW knows the type of all special rows and constraints. Data that PROC NETFLOW had to set aside (such as the first observation 63 value and the costrow ROW list variable value, which at the time had unknown type, but is then known to be a cost special row) is reprocessed. During this second pass, if a ROW list variable value has unassigned constraint or special row type, it is treated as a constraint with DEFCONTYPE= (or DEFCONTYPE= default) type. Associated VAR list variable values as coefficients of that constraint.

Sparse Input Format

The side constraints usually become sparse as the problem size increases. When the sparse data format of the CONDATA= data set is used, only nonzero constraint coefficients must be specified. Remember to specify the SPARSECONDATA option in the PROC NETFLOW statement. With the sparse method of specifying constraint information, the names of arc and nonarc variables do not have to be valid SAS variable names.

A sparse format CONDATA= data set for the oil industry example in the section "Introductory Example" on page 447 is displayed in the following code.

```
title 'Setting Up Condata = Cond2 for PROC NETFLOW';
data cond2;
   input _column_ $ _row1 $ _coef1 _row2 $ _coef2 ;
   datalines;
m_e_ref1  con1  -2     .    .
m_e_ref2  con2  -2     .    .
thruput1  con1   1  con3  -3
r1_gas       .   .  con3   4
thruput2  con2   1  con4  -3
r2_gas       .   .  con4   4
_type_    con1   1  con2   1
_type_    con3   0  con4   0
_rhs_     con1 -15  con2 -15
;
```

Recall that the COLUMN list variable values "_type_" and "_rhs_" are the default values of the TYPEOBS= and RHSOBS= options. Also, the default rhs value of constraints (con3 and con4) is zero. The third to last observation has the value "_type_" for the COLUMN list variable. The _ROW1 variable value is con1, and the _COEF1_ variable has the value 1. This indicates that the constraint con1 is *greater than* or equal to type (because the value 1 is *greater than* zero). Similarly, the data in the second to last observation's _ROW2 and _COEF2 variables indicate that con2 is an *equality* constraint (0 *equals* zero).

An alternative, using a TYPE list variable is as follows:

```
title 'Setting Up Condata = Cond3 for PROC NETFLOW';
data cond3;
   input _column_ $ _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
m_e_ref1  con1  -2     .    . >=
m_e_ref2  con2  -2     .    . .
thruput1  con1   1  con3  -3 .
r1_gas       .   .  con3   4 .
thruput2  con2   1  con4  -3 .
r2_gas       .   .  con4   4 .
.         con3   .  con4   . eq
.         con1 -15  con2 -15 ge
;
```

If the COLUMN list variable is missing in a particular observation (the last two observations in the data set cond3, for instance), the constraints named in the ROW list variables all have the constraint type indicated by the value in the TYPE list variable. It is for this type of observation that you are allowed more ROW list variables than COEF list variables. If corresponding COEF list variables are not missing (for example, the last observation in the data set cond3), these values are the rhs values of those constraints. Therefore, you can specify both constraint type and rhs in the same observation.

As in the previous CONDATA= data set, if the COLUMN list variable is an arc or nonarc variable, the COEF list variable values are coefficient values for that arc or

nonarc variable in the constraints indicated in the corresponding ROW list variables. If in this same observation, the TYPE list variable contains a constraint type, all constraints named in the ROW list variables in that observation have this constraint type (for example, the first observation in the data set cond3). Therefore, you can specify both constraint type and coefficient information in the same observation.

Also note that DEFCONTYPE=EQ could have been specified, saving you from having to include in the data that CON3 and CON4 are of this type.

In the oil industry example, arc costs, capacities, and lower flow bounds are presented in the ARCDATA= data set. Alternatively, you could have used the following input data sets.

```
title3 'Setting Up Arcdata = Arcd2 for PROC NETFLOW';
data arcd2;
   input  _from_&$11. _to_&$15. ;
   datalines;
middle east  refinery 1
middle east  refinery 2
u.s.a.       refinery 1
u.s.a.       refinery 2
refinery 1   r1
refinery 2   r2
r1           ref1 gas
r1           ref1 diesel
r2           ref2 gas
r2           ref2 diesel
ref1 gas     servstn1 gas
ref1 gas     servstn2 gas
ref1 diesel  servstn1 diesel
ref1 diesel  servstn2 diesel
ref2 gas     servstn1 gas
ref2 gas     servstn2 gas
ref2 diesel  servstn1 diesel
ref2 diesel  servstn2 diesel
;

title 'Setting Up Condata = Cond4 for PROC NETFLOW';
data cond4;
   input _column_&$27. _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
.                            con1 -15   con2 -15    ge
.                          costrow   .      .   .  cost
.                                .   . caprow   . capac
middle east_refinery 1        con1  -2      .   .     .
middle east_refinery 2        con2  -2      .   .     .
refinery 1_r1                 con1   1   con3  -3     .
r1_ref1 gas                      .   .   con3   4     =
refinery 2_r2                 con2   1   con4  -3     .
r2_ref2 gas                      .   .   con4   4    eq
middle east_refinery 1     costrow  63 caprow  95     .
middle east_refinery 2     costrow  81 caprow  80     .
u.s.a._refinery 1          costrow  55      .   .     .
```

```
u.s.a._refinery 2            costrow  49      .   .     .
refinery 1_r1                costrow 200 caprow 175     .
refinery 2_r2                costrow 220 caprow 100     .
r1_ref1 gas                        .   . caprow 140     .
r1_ref1 diesel                     .   . caprow  75     .
r2_ref2 gas                        .   . caprow 100     .
r2_ref2 diesel                     .   . caprow  75     .
ref1 gas_servstn1 gas        costrow  15 caprow  70     .
ref1 gas_servstn2 gas        costrow  22 caprow  60     .
ref1 diesel_servstn1 diesel  costrow  18      .   .     .
ref1 diesel_servstn2 diesel  costrow  17      .   .     .
ref2 gas_servstn1 gas        costrow  17 caprow  35     .
ref2 gas_servstn2 gas        costrow  31      .   .     .
ref2 diesel_servstn1 diesel  costrow  36      .   .     .
ref2 diesel_servstn2 diesel  costrow  23      .   .     .
middle east_refinery 1             .  20      .   .    lo
middle east_refinery 2             .  10      .   .    lo
refinery 1_r1                      .  50      .   .    lo
refinery 2_r2                      .  35      .   .    lo
ref2 gas_servstn1 gas              .   5      .   .    lo
;
```

The first observation in the cond4 data set defines con1 and con2 as *greater than or equal to* ($\geq$) constraints that both (by coincidence) have rhs values of -15. The second observation defines the special row costrow as a cost row. When costrow is a ROW list variable value, the associated COEF list variable value is interpreted as a cost or objective function coefficient. PROC NETFLOW has to do less work if constraint names and special rows are defined in observations near the top of a data set, but this is not a strict requirement. The fourth to ninth observations contain constraint coefficient data. Observations 7 and 9 have TYPE list variable values that indicate that constraints con3 and con4 are equality constraints. The last five observations contain lower flow bound data. Observations that have an arc or nonarc variable name in the COLUMN list variable, a nonconstraint type TYPE list variable value, and a value in (one of) the COEF list variables are valid.

The following data set is equivalent to the cond4 data set.

```
title 'Setting Up Condata = Cond5 for PROC NETFLOW';
data cond5;
   input _column_&$27. _row1 $ _coef1 _row2 $ _coef2 _type_ $ ;
   datalines;
middle east_refinery 1          con1  -2 costrow  63     .
middle east_refinery 2          con2  -2   lorow  10     .
refinery 1_r1                      .   .    con3  -3     =
r1_ref1 gas                   caprow 140    con3   4     .
refinery 2_r2                   con2   1    con4  -3     .
r2_ref2 gas                        .   .    con4   4    eq
.                               CON1 -15    CON2 -15    GE
ref2 diesel_servstn1 diesel        .  36 costrow   .  cost
.                                  .   .  caprow   . capac
.                              lorow   .       .   .    lo
middle east_refinery 1        caprow  95   lorow  20     .
```

```
middle east_refinery 2        caprow  80 costrow  81     .
u.s.a._refinery 1                  .   .       .  55  cost
u.s.a._refinery 2             costrow  49       .   .     .
refinery 1_r1                    con1   1  caprow 175     .
refinery 1_r1                   lorow  50 costrow 200     .
refinery 2_r2                 costrow 220  caprow 100     .
refinery 2_r2                      .  35       .   .    lo
r1_ref1 diesel                caprow2  75       .   . capac
r2_ref2 gas                        .   .  caprow 100     .
r2_ref2 diesel                caprow2  75       .   .     .
ref1 gas_servstn1 gas         costrow  15  caprow  70     .
ref1 gas_servstn2 gas         caprow2  60 costrow  22     .
ref1 diesel_servstn1 diesel        .   . costrow  18     .
ref1 diesel_servstn2 diesel   costrow  17       .   .     .
ref2 gas_servstn1 gas         costrow  17   lorow   5     .
ref2 gas_servstn1 gas              .   . caprow2  35     .
ref2 gas_servstn2 gas              .  31       .   .  cost
ref2 diesel_servstn2 diesel        .   . costrow  23     .
;
```

If you have data for a linear programming program that has an embedded network, the steps required to change that data into a form that is acceptable by PROC NETFLOW are

1. Identify the nodal flow conservation constraints. The coefficient matrix of these constraints (a submatrix of the LP's constraint coefficient matrix) has only two nonzero elements in each column, -1 and 1.
2. Assign a node to each nodal flow conservation constraint.
3. The rhs values of conservation constraints are the corresponding node's supplies and demands. Use this information to create a NODEDATA= data set.
4. Assign an arc to each column of the flow conservation constraint coefficient matrix. The arc is directed from the node associated with the row that has the 1 element in it and directed toward to the node associated with the row that has the -1 element in it. Set up an ARCDATA= data set that has two SAS variables. This data set could resemble ARCDATA=arcd2. These will eventually be the TAILNODE and HEADNODE list variables when PROC NETFLOW is used. Each observation consists of the tail and head node of each arc.
5. Remove from the data of the linear program all data concerning the nodal flow conservation constraints.
6. Put the remaining data into a CONDATA= data set. This data set will probably resemble CONDATA=cond4 or CONDATA=cond5.

The Sparse Format Summary

The following list illustrates possible CONDATA= data set observation sparse formats. a1, b1, b2, b3 and c1 have as a _COLUMN_ variable value either the name of an arc (possibly in the form *tail_head*) or the name of a nonarc variable.

- If there is no TYPE list variable in the CONDATA= data set, the problem must be constrained and there is no nonconstraint data in the CONDATA= data set.

	COLUMN	_ROWx_	_COEFx_	_ROWy_ (no _COEFy_) (may not be in CONDATA)
a1	variable	constraint	lhs coef	
a2	_TYPE_ or TYPEOBS=	constraint	-1 0 1	
a3	_RHS_ or RHSOBS= or missing	constraint	rhs value	constraint or missing
a4	_TYPE_ or TYPEOBS=	constraint	missing	
a5	_RHS_ or RHSOBS= or missing	constraint	missing	

  Observations of the form a4 and a5 serve no useful purpose but are still allowed to make problem generation easier.

- If there are no ROW list variables in the data set, the problem has no constraints and the information is nonconstraint data. There must be a TYPE list variable and only one COEF list variable in this case. The COLUMN list variable has as values the names of arcs or nonarc variables and must not have missing values or special row names as values.

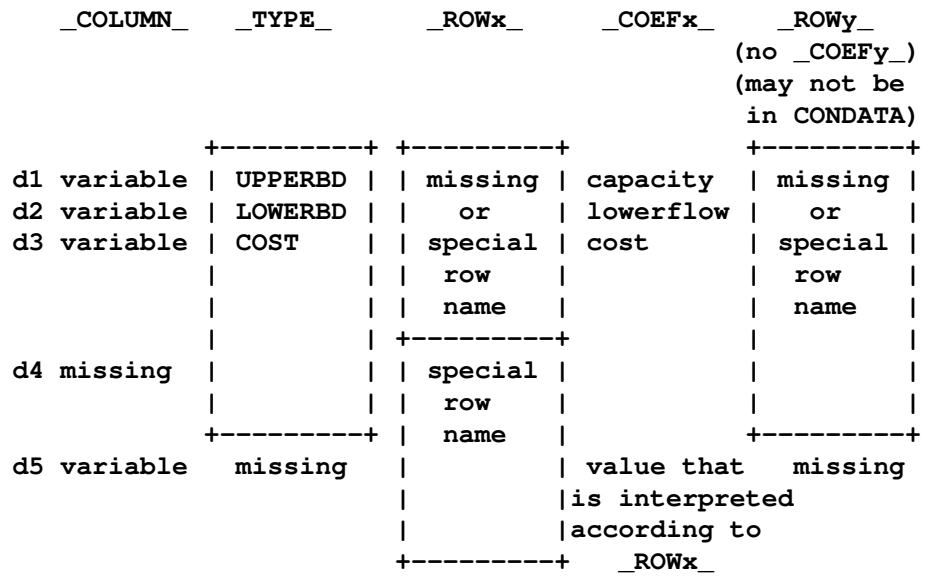
	COLUMN	_TYPE_	_COEFx_
b1	variable	UPPERBD	capacity
b2	variable	LOWERBD	lower flow
b3	variable	COST	cost

- Using a TYPE list variable for constraint data implies the following:

	COLUMN	_TYPE_	_ROWx_	_COEFx_	_ROWy_ (no _COEFy_) (may not be in CONDATA)
c1	variable	missing	c o n s t r a i n t	lhs coef	
c2	_TYPE_ or TYPEOBS=	missing		-1 0 1	
c3	_RHS_ or missing or RHSOBS=	missing		rhs value	constraint or missing
c4	variable	con type		lhs coef	
c5	_RHS_ or missing or RHSOBS=	con type		rhs value	
c6	missing	TYPE		-1 0 1	
c7	missing	RHS		rhs value	

  If the observation is of the form c4 or c5, and the _COEFx_ values are missing, the constraint is assigned the type data specified in the _TYPE_ variable.

- Using a TYPE list variable for arc and nonarc variable data implies the following:



Observations with form d1 to d5 can have ROW list variable values. Observation d4 must have ROW list variable values. The ROW value is put into the ROW name tree so that when dealing with observation d4 or d5, the COEF list variable value is interpreted according to the type of ROW list variable value. For example, the following three observations define the _ROWx_ variable values up_row, lo_row and co_row as being an upper value bound row, lower value bound row, and cost row, respectively.

```
_COLUMN_          _TYPE_          _ROWx_          _COEFx_

   .              UPPERBD         up_row             .
variable_a        LOWERBD         lo_row          lower flow
variable_b        COST            co_row          cost
```

PROC NETFLOW is now able to correctly interpret the following observation:

```
_COLUMN_ _TYPE_ _ROW1_ _COEF1_ _ROW2_ _COEF2_ _ROW3_ _COEF3_

 var_c      .    up_row  upval  lo_row  loval  co_row   cost
```

If the TYPE list variable value is a constraint type and the value of the COLUMN list variable equals the value of the TYPEOBS= option or the default value _TYPE_, the TYPE list variable value is ignored.

NODEDATA= Data Set

See the section "Getting Started: NETFLOW Procedure" on page 446 and the section "Introductory Example" on page 447 for a description of this input data set.

Output Data Sets

The procedure determines the flow that should pass through each arc as well as the value assigned to each nonarc variable. The goal is that the minimum flow bounds, capacities, lower and upper value bounds, and side constraints are not violated. This goal is reached when total cost incurred by such a flow pattern and value assignment is feasible and optimal. The solution found must also conserve flow at each node.

The ARCOUT= data set contains a solution obtained when performing optimization that does not consider any constraints. The NODEOUT= data set contains nodal dual variable information for this type of solution. You can choose to have PROC NETFLOW create the ARCOUT= data set and the NODEOUT= data set and save the optimum of the network or the nodal dual variable values before any optimization that considers the side constraints is performed.

If there are side constraints, the CONOUT= data set can be produced and contains a solution obtained after performing optimization that considers constraints. The DUALOUT= data set contains dual variable information for nodes and side constraints from the solution obtained after optimization that considers the constraints. The CONOUT= data set and DUALOUT= data set can be used to save the constrained optimal solution.

ARCOUT= and CONOUT= Data Sets

The ARCOUT= and CONOUT= data sets contain the same variables. Furthermore, the variables in the output data sets depend on whether or not the problem has a network component.

If the problem has a network component, the variables and their possible values in an observation are as follows:

FROM	a tail node of an arc. This is a missing value if an observation has information about a nonarc variable.
_TO_	a head node of an arc. This is a missing value if an observation has information about a nonarc variable.
COST	the cost of an arc or the objective function coefficient of a nonarc variable
CAPAC	the capacity of an arc or upper value bound of a nonarc variable
_LO_	the lower flow bound of an arc or lower value bound of a nonarc variable
NAME	a name of an arc or nonarc variable
SUPPLY	the supply of the tail node of the arc in the observation. This is a missing value if an observation has information about a nonarc variable.
DEMAND	the demand of the head node of the arc in the observation. This is a missing value if an observation has information about a nonarc variable.
FLOW	the flow through the arc or value of the nonarc variable
FCOST	flow cost, the product of _COST_ and _FLOW_
RCOST	the reduced cost of the arc or nonarc variable

ANUMB	the number of the arc (positive) or nonarc variable (nonpositive); used for warm starting PROC NETFLOW
TNUMB	the number of the tail node in the network basis spanning tree; used for warm starting PROC NETFLOW
STATUS	the status of the arc or nonarc variable

If the problem does not have a network component, the variables and their possible values in an observation are as follows:

OBJFN	the objective function coefficient of a variable
_UPPERBD	the upper value bound of a variable
_LOWERBD	the lower value bound of a variable
NAME	the name of a variable
VALUE	the value of the variable
FCOST	objective function value for that variable; the product of _OBJFN_ and _VALUE_

The variables present in the ARCDATA= data set are present in an ARCOUT= data set or a CONOUT= data set. For example, if there is a variable called tail in the ARCDATA= data set and you specified the SAS variable list

```
from tail;
```

then tail is a variable in the ARCOUT= and CONOUT= data sets instead of _FROM_. Any ID list variables also appear in the ARCOUT= and CONOUT= data sets.

NODEOUT= and DUALOUT= Data Sets

There are two types of observations in the NODEOUT= and DUALOUT= data sets. One type of observation contains information about a node. These are called *type N* observations. There is one such observation of this type for each node. The _NODE_ variable has a name of a node, and the _CON_ variable values in these observations are missing values.

The other type of observation contains information about constraints. These are called the *type C* observations. There is one such observation for each constraint. The _CON_ variable has a name of a constraint, and the _NODE_ variable values in these observations are missing values.

Many of the variables in the NODEOUT= and DUALOUT= data sets contain information used to warm start PROC NETFLOW. The variables _NODE_, _SD_, _DUAL_, _VALUE_, _RHS_, _TYPE_, and _CON_ contain information that might be of interest to you.

The NODEOUT= and DUALOUT= data sets look similar, as the same variables are in both. These variables and their values in an observation of each type are

NODE	Type N: the node name

	Type C: a missing value
_SD_	Type N: the supply (positive) or demand (negative) of the node
	Type C: a missing value
DUAL	Type N: the dual variable value of the node in _NODE_
	Type C: the dual variable value of the constraint named in _CON_
NNUMB	Type N: the number of the node named in _NODE_
	Type C: the number of the constraint named in _CON_
PRED	Type N: the predecessor in the network basis spanning tree of the node named in _NODE_
	Type C: the number of the node toward which the arc with number in _ARCID_ is directed, or the constraint number associated with the slack, surplus, or artificial variable basic in this row
TRAV	Type N: the traversal thread label of the node named in _NODE_
	Type C: a missing value
SCESS	Type N: the number of successors (including itself) in the network basis spanning tree of the node named in _NODE_
	Type C: a missing value
ARCID	Type N: if _ARCID_ is nonnegative, _ARCID_ is the number of the network basis spanning tree arc directed from the node with number _PRED_ to the node named in _NODE_. If _ARCID_ is negative, minus _ARCID_ is the number of the network basis spanning tree arc directed from the node named in _NODE_ to the node with number _PRED_.
	Type C: if _ARCID_ is positive, _ARCID_ is the number of the arc basic in a constraint row. If nonpositive, minus _ARCID_ is the number of the nonarc variable basic in a constraint row.
FLOW	Type N: the flow minus the lower flow bound of the arc _ARCID_
	Type C: the flow minus lower flow bound of the arc _ARCID_ or value lower bound of the nonarc variable value minus _ARCID_
FBQ	Type N: If _FBQ_ is positive, then _FBQ_ is the subscript in arc length arrays of the first arc directed toward the node named in _NODE_. PROC NETFLOW's arc length arrays are sorted so that data of arcs directed toward the same head node are together. If _FBQ_ is negative, no arcs are directed toward the node named in _NODE_. Arcs directed toward node i have subscripts in the arc length arrays between observations FBQ(i) and (FBQ($i+1$))-1, inclusive.
	Type C: a missing value
VALUE	Type N: a missing value
	Type C: the lhs value (the sum of the products of coefficient and flows or values) of the constraint named in _CON_
RHS	Type N: a missing value
	Type C: the rhs value of the constraint named in _CON_
TYPE	Type N: a missing value
	Type C: the type of the constraint named in _CON_

CON	Type N: a missing value Type C: the name of the constraint

If specified in variable lists, the variables in the input data sets are used instead of some of the previous variables. These variables are specified in the NODE, SUPDEM, RHS, TYPE, and ROW (if there is only one variable in the ROW list) lists and are used instead of _NODE_, _SD_, _RHS_, _TYPE_, and _CON_, respectively.

Case Sensitivity

Whenever the NETFLOW procedure has to compare character strings, whether they are node names, arc names, nonarc names, or constraint names, if the two strings have different lengths, or on a character by character basis the character is different *or has different cases*, PROC NETFLOW judges the character strings to be different.

Not only is this rule enforced when one or both character strings are obtained as values of SAS variables in PROC NETFLOW's input data sets, it also should be obeyed if one or both character strings were originally SAS variable names, or were obtained as the values of options or statements parsed to PROC NETFLOW. For example, if the network has only one node that has supply capability, or if you are solving a MAXFLOW or SHORTPATH problem, you can indicate that node using the SOURCE= option. If you specify

```
proc netflow source=NotableNode
```

then PROC NETFLOW looks for a value of the TAILNODE list variable that is `NotableNode`.

Version 6 of the SAS System converts text that makes up statements into uppercase. The name of the node searched for would be NOTABLENODE, even if this was your SAS code:

```
proc netflow source=NotableNode
```

If you want PROC NETFLOW to behave as it did in Version 6, specify

```
options validvarname=v6;
```

If the SPARSECONDATA option is not specified, and you are running SAS software Version 6 or have specified options validvarname=v6; using a later version, all NAME list variable values in the ARCDATA= data set are uppercased. This is because the SAS System has uppercased all SAS variable names, particularly those in the VAR list of the CONDATA= data set.

Entities that contain blanks must be enclosed in single or double quotes.

See the section "Cautions" on page 482 for additional discussion of case sensitivity.

Loop Arcs

When using the primal simplex network algorithm, loop arcs (arcs directed toward nodes from which they originate) are prohibited. Rather, introduce a dummy intermediate node in loop arcs. For example, replace arc (A,A) with (A,B) and (B,A). B is the name of a new node, and it must be distinct for each loop arc.

Multiple Arcs

Multiple arcs with the same tail and head nodes are prohibited. PROC NETFLOW checks to ensure there are no such arcs before proceeding with the optimization. Introduce a new dummy intermediate node in multiple arcs. This node must be distinct for each multiple arc. For example, if some network has three arcs directed from node A toward node B, then replace one of these three with arcs (A,C) and (C,B) and replace another one with (A,D) and (D,B). C and D are new nodes added to the network.

Pricing Strategies

The pricing strategy is the part of the simplex iteration that selects the nonbasic arc, constraint slack, surplus, or nonarc variable that should have a flow or value change, and perhaps enter the basis so that the total cost incurred is improved.

The pricing mechanism takes a large amount of computational effort, so it is important to use the appropriate pricing strategy for the problem under study. As in other large scale mathematical programming software, network codes can spend more than half of their execution time performing simplex iterations in the pricing step. Some compromise must be made between using a fast strategy and improving the quality of the flow or value change candidate selection, although more simplex iterations may need to be executed.

The configuration of the problem to be optimized has a great effect on the choice of strategy. If a problem is to be run repeatedly, experimentation on that problem to determine which scheme is best may prove worthwhile. The best pricing strategy to use when there is a large amount of work to do (for example, when a cold start is used) may not be appropriate when there is little work required to reach the optimum (such as when a warm start is used). If paging is necessary, then a pricing strategy that reduces the number of simplex iterations performed might have the advantage. The proportion of time spent doing the pricing step during stage 1 optimization is usually less than the same proportion when doing stage 2 optimization. Therefore, it is more important to choose a stage 2 pricing strategy that causes fewer, but not necessarily the fewest, iterations to be executed.

There are many similarities between the pricing strategies for optimizing an unconstrained problem (or when constraints are temporarily ignored) and the pricing mechanisms for optimizing considering constraints. To prevent repetition, options have a suffix or embedded x . Replace x with 1 for optimization without constraint consideration and 2 for optimization with constraint consideration.

There are three main types of pricing strategies:

- PRICETYPEx=NOQ
- PRICETYPEx=BLAND
- PRICETYPEx=Q

The pricing strategy that usually performs better than the others is PRICETYPEx=Q. For this reason, PRICETYPEx=Q is the default.

PRICETYPEx=NOQ

PRICETYPEx=NOQ is the least complex pricing strategy, but it is nevertheless quite efficient. In contrast to the specification of PRICETYPEx=Q, a candidate queue is not set up.

The PxSCAN= option controls the amount of additional candidate selection work done to find a better candidate after an eligible candidate has been found.

If PxSCAN=FIRST is specified, the search for candidates finishes when the first eligible candidate is found, with this exception: if a node has more than one eligible arc directed toward it, the best such arc is chosen.

If PxSCAN=BEST is specified, everything that is nonbasic is examined, and the best candidate of all is chosen.

If PxSCAN=PARTIAL is specified, once an eligible candidate is found, the scan continues for another PxNPARTIAL= cycles in the hope that during the additional scan, a better candidate is found. Examining all nonbasic arcs directed toward a single node is counted as only one cycle.

If PxSCAN=FIRST or PxSCAN=PARTIAL is specified, the scan for entering candidates starts where the last iteration's search left off. For example, if the last iteration's scan terminated after examining arcs that are directed toward the node with internal number i, the next iteration's scan starts by examining arcs directed toward the node with internal number $i+1$. If i is the largest node number, next iterations scan begins by scanning arcs directed toward node 1 (during stage 1) or scanning constraint slack or surplus variables, if any, or nonarc variables, if any, (during stage 2). During stage 2, if the scan terminated after examining the slack or surplus of constraint i, next iterations scan starts by examining the slack or surplus of the constraint with the internal number greater than i that has such a logical variable. If the scan terminated after examining the nonarc variable i, the next iterations scan starts by examining the nonarc variable with internal number $i+1$, (or arcs directed to the node with the smallest internal number if the nonarc variable with the greatest number has been examined). This is termed a *wraparound search*.

PRICETYPEx=Q

If PRICETYPEx=Q, a queue is set up. Candidates currently on the queue are tested at each iteration and either enter the basis or are removed from the queue. The size of the queue can be specified by using the QSIZEx= option. The default value for QSIZE1= is

```
QSIZE1=number of arcs/200
if (QSIZE1<24) QSIZE1=24
else if (QSIZE1>100) QSIZE1=100
```

The default value for QSIZE2= is

```
QSIZE2=(number of arcs+number of nonarc variables)/200
if (QSIZE2<24) QSIZE2=24
else if (QSIZE2>100) QSIZE2=100
```

controls the amount of additional candidate selection work done to find a better candidate after an eligible candidate has been found *in the queue*.

If you specify PxSCAN=BEST, the best eligible candidate found is removed from the queue. It can sustain a flow or value change and possibly enter the basis.

If you specify PxSCAN=FIRST, the first eligible candidate found is removed from the queue, and possibly sustains a flow or value change and enters the basis.

If you specify PxSCAN=PARTIAL, PxNPARTIAL= can then be specified as well. After an eligible candidate has been found, PxNPARTIAL= more queue members are examined and the best of the eligible candidates found is chosen.

When PxSCAN=FIRST or PxSCAN=PARTIAL, the scan of the queue is wraparound. When the member last added to the queue has been examined, the scan continues from the member that was first added to the queue.

When the queue is empty, or after QSIZEx= times REFRESHQx= iterations have been executed since the queue was last refreshed, new candidates are found and put onto the queue. Valid values for the REFRESHQx= options are greater than 0.0 and less than or equal to 1.0. The default for REFRESHQx is 0.75. If the scan cannot find enough candidates to fill the queue, the procedure reduces the value of QSIZEx=. If $qfound$ is the number of candidates found, the new QSIZEx= value is $qfound + ((old\ QSIZEx - qfound) \times REDUCEQSIZEx)$. Valid values of the REDUCEQSIZEx= option are between 0.0 and 1.0, inclusive. The default for REDUCEQSIZEx= is 1.0.

The QxFILLSCAN= option controls the amount of additional candidate selection work performed to find better candidates to put into the queue after the queue has been filled.

If you specify QxFILLSCAN=FIRST, the nonbasic arcs, and during stage 2 optimization, nonbasic constraint slack and surplus variables, and nonbasic nonarc variables are scanned; the scan stops when the queue is filled. If a node has more than one eligible arc directed toward it, the best such arc is put onto the queue. QxFILLSCAN=FIRST is the default.

If QxFILLSCAN=BEST is specified, everything that is nonbasic is scanned and the best eligible candidates are used to fill the queue.

If QxFILLSCAN=PARTIAL is specified, after the queue is full, the scan continues for another QxFILLNPARTIAL= cycles in the hope that during the additional

scan, better candidates are found to replace other candidates previously put onto the queue. QxFILLNPARTIAL=10 is the default. If QxFILLSCAN=FIRST or QxFILLSCAN=PARTIAL, the scan starts where the previous iteration ended; that is, it is wraparound.

In the following section, dual variables and reduced costs are explained. These help PROC NETFLOW determine whether an arc, constraint slack, surplus, or nonarc variable should have a flow or value change. P2SCAN=ANY and the DUALFREQ= option can be specified to control stage 2 pricing, and how often dual variables and reduced costs are calculated.

What usually happens when PRICETYPE2=Q is specified is that before the first iteration, the queue is filled with nonbasic variables that are eligible to enter the basis. At the start of each iteration, a candidate on the queue is examined and its reduced cost is calculated to ensure that it is still eligible to enter the basis. If it is ineligible to enter the basis, it is removed from the queue and another candidate on the queue is examined, until a candidate on the queue is found that can enter the basis. When this happens, a *minor* iteration occurs. If there are no candidates left on the queue, or several iterations have been performed since the queue was refreshed, new nonbasic variables that are eligible to enter the basis are found and are placed on the queue. When this occurs, the iteration is termed a *major* iteration. Dual variables are calculated or maintained every iteration.

During most optimizations, if a variable is put onto the queue during a major iteration, it usually remains eligible to enter the basis in later minor iterations. Specifying P2SCAN=ANY indicates that PROC NETFLOW should choose *any* candidate on the queue and use that as the entering variable. Reduced costs are not calculated. It is simply hoped that the chosen candidate is eligible. Sometimes, a candidate on the queue is chosen that has become ineligible and the optimization takes "a step backward" rather than "a step forward" toward the optimum. However, the disadvantages of incurring an occasional step backwards and the possible danger of never converging to the optimum are offset by not having to calculate reduced costs and, more importantly, not having to maintain dual variable values. The calculation of dual variables is one of two large linear equation systems that must be solved each iteration in the simplex iteration.

If P2SCAN=ANY is specified, dual variables are calculated after DUALFREQ= iterations have been performed since they were last calculated. These are used to calculate the reduced costs of all the candidates currently on the queue. Any candidate found to be ineligible to enter the basis is removed from the queue. DUALFREQ=4 is the default.

Once again, the practice of not maintaining correct dual variable values is dangerous because backward steps are allowed, so the optimization is not guaranteed to converge to the optimum. However, if PROC NETFLOW does not run forever, it can find the optimum much more quickly than when the P2SCAN= option is not ANY. Before concluding that any solution is optimal, PROC NETFLOW calculates true dual variable values and reduced costs and uses these to verify that the optimum is really at hand.

Whether P2SCAN=ANY is specified or not, dual variables are always calculated at the start of major iterations.

PRICETYPEx=BLAND

PRICETYPEx=BLAND is equivalent to specifying in the PROC NETFLOW or RESET statement all three options PRICETYPEx=NOQ, PxSCAN=FIRST, and LRATIOx, and the scans are not wraparound. Bland (1977) proved that this pivot rule prevents the simplex algorithm from cycling. However, because the pivots concentrate on the lower indexed arcs, constraint slack, surplus, and nonarc variables, optimization with PRICETYPEx=BLAND can make the optimization execute slowly.

Dual Variables, Reduced Costs, and Status

During optimization, dual variables and reduced costs are used to determine whether an arc, constraint slack, surplus, or nonarc variable should have a flow or value change. The ARCOUT= and CONOUT= data sets each have a variable called _RCOST_ that contains reduced cost values. In the CONOUT= data set, this variable also has the reduced costs of nonarc variables. For an arc, the reduced cost is the amount that would be added to the total cost if that arc were made to convey one more unit of flow. For a nonarc variable, the reduced cost is the amount that would be added to the total cost if the value currently assigned to that nonarc variable were increased by one.

During the optimization of a minimization problem, if an arc has a positive reduced cost, PROC NETFLOW takes steps to decrease the flow through it. If an arc has a negative reduced cost, PROC NETFLOW takes steps to increase the flow through it. At optimality, the reduced costs of arcs with flow at their respective lower bounds are nonnegative; otherwise, the optimizer would have tried to increase the flow, thereby decreasing the total cost. The _STATUS_ of each such nonbasic arc is LOWERBD NONBASIC. The reduced costs of arcs with flow at capacity are nonpositive. The _STATUS_ of each such nonbasic arc is UPPERBD NONBASIC. Even though it would decrease total cost, the optimizer cannot increase the flows through such arcs because of the capacity bound. Similar arguments apply for nonarc variables.

The reduced cost is also the amount that would be subtracted from the total cost if that arc was made to convey one less unit of flow. Similarly, a reduced cost is the amount subtracted from the total cost if the value currently assigned to that nonarc variable is decreased by one.

The dual variables and reduced costs can be used to detect whether multiple optimal solutions exist. A zero reduced cost of a nonbasic arc indicates the existence of multiple optimal solutions. A zero reduced cost indicates, by definition, that the flow through such arcs can be changed with zero change to the total cost. (Basic arcs and basic nonarc variables technically have zero reduced costs. A missing value is used for these so that reduced costs of nonbasic arcs and nonbasic nonarc variables that are zero are highlighted.)

The range over which costs can vary before the present solution becomes nonoptimal can be determined through examination of the reduced costs. For any nonbasic arc

with assigned flow equal to its lower bound, the amount by which the cost must be decreased before it becomes profitable for this arc to convey additional flow is the value of its reduced cost. The cost reduction necessary for a nonbasic arc currently assigned capacity flow to undergo a worthwhile flow decrease is the absolute value of its reduced cost. In both cases, this minimum cost reduction changes the reduced cost to zero. Any further reduction promotes a possible basis change.

The reduced cost of an arc (t, h) is $rc_{t,h} = c_{t,h} - \pi_t + \pi_h$ where π_i is the dual value for node i and $c_{t,h}$ is the cost of the arc with tail node t and head node h.

If the problem has side constraints and arc (t, h) has nonzero lhs coefficients, then the following term must be subtracted from $rc_{t,h}$:

$$\sum_i condual_i H_{i,(t,h)}$$

where $condual_i$ is the dual variable of constraint i, and $H_{i,(t,h)}$ is the coefficient of arc (t, h) in constraint i.

If d_n is the objective function coefficient of nonarc variable n, the reduced cost is $rc_n = d_n - \sum_i condual_i Q_{i,n}$, where $Q_{i,n}$ is the coefficient of nonarc variable n in constraint i.

The Working Basis Matrix

Let $\mathbf{T}$ be the basis matrix of NPSC. The following partitioning is done:

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

where

- n is the number of nodes.
- k is the number of side constraints.
- $\mathbf{A}$ $(n \times n)$ is the network component of the basis. Most of the columns of this matrix are columns of the problem's node-arc incidence matrix. The arcs associated with columns of $\mathbf{A}$, called key basic variables or key arcs, form a spanning tree. The data structures of the spanning tree of this submatrix of the basis $\mathbf{T}$ enable the computations involving $\mathbf{T}$ and the manner in which $\mathbf{T}$ is updated to be very efficient, especially those dealing with $\mathbf{A}$ (or $\mathbf{A}^{-1}$).
- $\mathbf{C}$ $(k \times n)$ are the key arcs' side constraint coefficient columns.
- $\mathbf{B}$ $(n \times k)$ are the node-arc incidence matrix columns of the nontree arcs. The columns of $\mathbf{B}$ having nonzero elements are associated with basic nonspanning tree arcs.
- $\mathbf{D}$ $(k \times k)$ are the constraint coefficient columns of nonkey basic variables. Nonkey basic variables not only include nontree basic arcs but also basic slack, surplus, artificial, or nonarc variables.

It is more convenient to factor $\mathbf{T}$ by block triangular matrices $\mathbf{P}$ and $\mathbf{M}$, such that $\mathbf{P} = \mathbf{TM}$. The matrices $\mathbf{P}$ and $\mathbf{M}$ are used instead of $\mathbf{T}$ because they are less burdensome to work with. You can perform block substitution when solving the simplex iteration linear systems of equations

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D_w} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where $\mathbf{D_w} = \mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}$ and is called the working basis matrix.

To perform block substitution, you need the tree data structure of the $\mathbf{A}$ matrix, and also the $\mathbf{C}$, $\mathbf{B}$, and $\mathbf{D_w}$ matrices. Because the $\mathbf{C}$ matrix consists of columns of the constraint coefficient matrix, the maintenance of $\mathbf{C}$ from iteration to iteration simply entails changing information specifying which columns of the constraint coefficient matrix compose $\mathbf{C}$.

The $\mathbf{A}^{-1}\mathbf{B}$ matrix is usually very sparse. Fortunately, the information in $\mathbf{A}^{-1}\mathbf{B}$ can be initialized easily using the tree structures. In most iterations, only one column is replaced by a new one. The values of the elements of the new column may already be known from preceding steps of the simplex iteration.

The working basis matrix is the submatrix that presents the most computational complexity. However, PROC NETFLOW usually can use classical simplex pivot techniques. In many iterations, only one column of $\mathbf{D_w}$ changes. Sometimes it is not necessary to update $\mathbf{D_w}$ or its inverse at all.

If INVD_2D is specified in the PROC NETFLOW statement, only one row and one column may need to be changed in the $\mathbf{D_w^{-1}}$ before the next simplex iteration can begin. The new contents of the changed column are already known. The new elements of the row that changes are influenced by the contents of a row of $\mathbf{A}^{-1}\mathbf{B}$ that is very sparse.

If INVD_2D is not specified in the PROC NETFLOW statement, the Bartels-Golub update can be used to update the $\mathbf{LU}$ factors of $\mathbf{D_w}$. The choice must be made whether to perform a series of updates (how many depends on the number of nonzeros in a row of $\mathbf{A}^{-1}\mathbf{B}$), or refactorization.

Flow and Value Bounds

The capacity and lower flow bound of an arc can be equal. Negative arc capacities and lower flow bounds are permitted. If both arc capacities and lower flow bounds are negative, the lower flow bound must be at least as negative as the capacity. An arc (A,B) that has a negative flow of $-f$ units can be interpreted as an arc that conveys f units of flow from node B to node A.

The upper and lower value bounds of a nonarc variable can be equal. Negative upper and lower bounds are permitted. If both are negative, the lower bound must be at least as negative as the upper bound.

Tightening Bounds and Side Constraints

If any piece of data is furnished to PROC NETFLOW more than once, PROC NETFLOW checks for consistency so that no conflict exists concerning the data values. For example, if the cost of some arc is seen to be one value and as more data are read, the cost of the same arc is seen to be another value, PROC NETFLOW issues an error message on the SAS log and stops. There are two exceptions:

- The bounds of arcs and nonarc variables are made as tight as possible. If several different values are given for the lower flow bound of an arc, the greatest value is used. If several different values are given for the lower bound of a nonarc variable, the greatest value is used. If several different values are given for the capacity of an arc, the smallest value is used. If several different values are given for the upper bound of a nonarc variable, the smallest value is used.
- Several values can be given for inequality constraint right-hand sides. For a particular constraint, the lowest rhs value is used for the rhs if the constraint is of *less than or equal to* type. For a particular constraint, the greatest rhs value is used for the rhs if the constraint is of *greater than or equal to* type.

Reasons for Infeasibility

Before optimization commences, PROC NETFLOW tests to ensure that the problem is not infeasible by ensuring that, with respect to supplies, demands, and arc flow bounds, flow conservation can be obeyed at each node.

- Let *IN* be the sum of lower flow bounds of arcs directed toward a node plus the node's supply. Let *OUT* be the sum of capacities of arcs directed from that node plus the node's demand. If *IN* exceeds *OUT*, not enough flow can leave the node.
- Let *OUT* be the sum of lower flow bounds of arcs directed from a node plus the node's demand. Let *IN* be the total capacity of arcs directed toward the node plus the node's supply. If *OUT* exceeds *IN*, not enough flow can arrive at the node.

Reasons why a network problem can be infeasible are similar to those previously mentioned but apply to a set of nodes rather than for an individual node.

Consider the network illustrated in Figure 5.13.

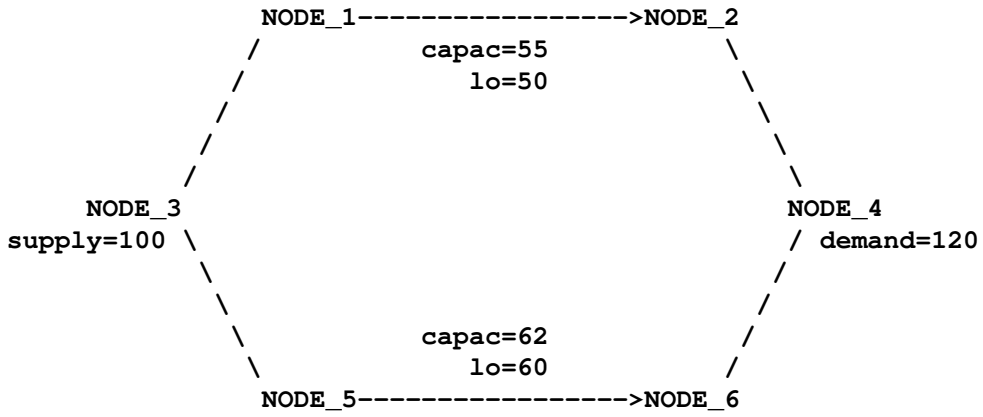


Figure 5.13. An Infeasible Network

The demand of NODE_4 is 120. That can never be satisfied because the maximal flow through arcs (NODE_1, NODE_2) and (NODE_5, NODE_6) is 117. More specifically, the implicit supply of NODE_2 and NODE_6 is only 117, which is insufficient to satisfy the demand of other nodes (real or implicit) in the network.

Furthermore, the lower flow bounds of arcs (NODE_1, NODE_2) and (NODE_5, NODE_6) are greater than the flow that can reach the tail nodes of these arcs, that, by coincidence, is the total supply of the network. The implicit demand of nodes NODE_1 and NODE_5 is 110, which is greater than the amount of flow that can reach these nodes.

When PROC NETFLOW detects that the problem is infeasible, it indicates why the solution, obtained after optimization stopped, is infeasible. It can report that the solution cannot obey flow conservation constraints and which nodes these conservation constraints are associated with. If applicable, the side constraints that the solution violates are also output.

If stage 1 optimization obtains a feasible solution to the network, stage 2 optimization can determine that the problem is infeasible and note that some flow conservation constraint is broken while all side constraints are satisfied. The infeasibility messages issued by PROC NETFLOW pertain to why the *current* solution is infeasible, not quite the same as the reasons why the *problem* is infeasible. However, the messages highlight *areas* in the problem where the infeasibility can be tracked down. If the problem is infeasible, make PROC NETFLOW do a stage 1 unconstrained optimization by removing the CONDATA= data set specification in the PROC NETFLOW statement. If a feasible network solution is found, then the side constraints are the source of the infeasibility in the problem.

Missing S Supply and Missing D Demand Values

In some models, you may want a node to be either a supply or demand node but you want the node to supply or demand the optimal number of flow units. To indicate that a node is such a supply node, use a missing S value in the SUPPLY list variable in the ARCDATA= data set or the SUPDEM list variable in the NODEDATA= data set. To indicate that a node is such a demand node, use a missing D value in the DEMAND list variable in the ARCDATA= data set or the SUPDEM list variable in the NODEDATA= data set.

Suppose the oil example in the section "Introductory Example" on page 447 is changed so that crude oil can be obtained from either the Middle East or U.S.A. in any amounts. You should specify that the node "middle east" is a supply node, but you do not want to stipulate that it supplies 100 units, as before. The node "u.s.a." should also remain a supply node, but you do not want to stipulate that it supplies 80 units. You must specify that these nodes have missing S supply capabilities.

```
title  'Oil Industry Example';
title3 'Crude Oil can come from anywhere';
data miss_s;
   missing S;
   input   _node_&$15. _sd_;
   datalines;
middle east         S
u.s.a.              S
servstn1 gas      -95
servstn1 diesel   -30
servstn2 gas      -40
servstn2 diesel   -15
;
```

The following PROC NETFLOW run uses the same ARCDATA= and CONDATA= data sets used in the section "Introductory Example" on page 447.

```
proc netflow
   nodedata=miss_s       /* the supply (missing S) and */
                         /* demand data                */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
print some_arcs('middle east' 'u.s.a.',_all_)/short;

proc print;
   sum _fcost_;
   run;
```

The following messages appear on the SAS log:

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
```

```
NOTE: Of these, 2 have unspecified (.S) supply capability.
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 0 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 9 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 50040 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of arc and nonarc variable side
      constraint coefficients= 8 .
NOTE: Number of iterations, optimizing with
      constraints= 3 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= 50075 .
```

The PRINT statement reports the arcs directed away from the supply nodes, shown in Figure 5.14. The amount of crude obtained from the Middle East and U.S.A. is 30 and 150 units, respectively.

Oil Industry Example

Crude Oil can come from anywhere

The NETFLOW Procedure

_N_	_from_	_to_	_cost_	_capac_	_lo_	_name_
1	middle east	refinery 1	63	95	20	m_e_ref1
2	u.s.a.	refinery 1	55	99999999	0	
3	middle east	refinery 2	81	80	10	m_e_ref2
4	u.s.a.	refinery 2	49	99999999	0	

_N_	_FLOW_
1	20
2	125
3	10
4	25

Figure 5.14. Print Statement, Oil Example, Missing S Supplies

The CONOUT= data set is shown in Figure 5.15.

Obs	_from_	_to_	_cost_	_capac_	_lo_	_name_	_SUPPLY_
1	refinery 1	r1	200	175	50	thruput1	.
2	refinery 2	r2	220	100	35	thruput2	.
3	r1	ref1 diesel	0	75	0		.
4	r1	ref1 gas	0	140	0	r1_gas	.
5	r2	ref2 diesel	0	75	0		.
6	r2	ref2 gas	0	100	0	r2_gas	.
7	middle east	refinery 1	63	95	20	m_e_ref1	S
8	u.s.a.	refinery 1	55	99999999	0		S
9	middle east	refinery 2	81	80	10	m_e_ref2	S
10	u.s.a.	refinery 2	49	99999999	0		S
11	ref1 diesel	servstn1 diesel	18	99999999	0		.
12	ref2 diesel	servstn1 diesel	36	99999999	0		.
13	ref1 gas	servstn1 gas	15	70	0		.
14	ref2 gas	servstn1 gas	17	35	5		.
15	ref1 diesel	servstn2 diesel	17	99999999	0		.
16	ref2 diesel	servstn2 diesel	23	99999999	0		.
17	ref1 gas	servstn2 gas	22	60	0		.
18	ref2 gas	servstn2 gas	31	99999999	0		.

Obs	_DEMAND_	_FLOW_	_FCOST_	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_
1	.	145.00	29000.00	.	7	2	KEY_ARC BASIC
2	.	35.00	7700.00	17	8	3	LOWERBD NONBASIC
3	.	36.25	0.00	.	10	5	KEY_ARC BASIC
4	.	108.75	0.00	.	9	5	KEY_ARC BASIC
5	.	8.75	0.00	.	12	6	KEY_ARC BASIC
6	.	26.25	0.00	.	11	6	KEY_ARC BASIC
7	.	20.00	1260.00	8	2	1	LOWERBD NONBASIC
8	.	125.00	6875.00	.	3	4	KEY_ARC BASIC
9	.	10.00	810.00	32	4	1	LOWERBD NONBASIC
10	.	25.00	1225.00	.	5	4	KEY_ARC BASIC
11	30	30.00	540.00	.	17	8	KEY_ARC BASIC
12	30	0.00	0.00	12	18	10	LOWERBD NONBASIC
13	95	68.75	1031.25	.	13	7	KEY_ARC BASIC
14	95	26.25	446.25	.	14	9	NONKEY ARC BASIC
15	15	6.25	106.25	.	19	8	KEY_ARC BASIC
16	15	8.75	201.25	.	20	10	KEY_ARC BASIC
17	40	40.00	880.00	.	15	7	KEY_ARC BASIC
18	40	0.00	0.00	7	16	9	LOWERBD NONBASIC
			========				
			50075.00				

Figure 5.15. Missing S SUPDEM Values in NODEDATA

The optimal supplies of nodes “middle east” and “u.s.a.” are 30 and 150 units, respectively. For this example, the same optimal solution is obtained if these nodes had supplies less than these values (each supplies 1 unit, for example) and the THRUNET option was specified in the PROC NETFLOW statement. With the THRUNET option active, when total supply exceeds total demand, the specified nonmissing demand values are the lowest number of flow units that must be absorbed by the corresponding node. This is demonstrated in the following PROC NETFLOW run. The missing S is most useful when nodes are to supply optimal numbers of flow units and it turns out that for some nodes, the optimal supply is 0.

```
data miss_s_x;
   missing S;
   input   _node_&$15. _sd_;
   datalines;
middle east         1
u.s.a.              1
servstn1 gas      -95
servstn1 diesel   -30
servstn2 gas      -40
servstn2 diesel   -15
;

proc netflow
   thrunet
   nodedata=miss_s_x     /* No supply (missing S)      */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
print some_arcs('middle east' 'u.s.a.',_all_)/short;

proc print;
   sum _fcost_;
   run;
```

The following messages appear on the SAS log. Note that the Total supply= 2, not 0 as in the last run.

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 2 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 13 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 50040 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of arc and nonarc variable side
      constraint coefficients= 8 .
NOTE: Number of iterations, optimizing with
      constraints= 3 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= 50075 .
```

The PRINT statement and the CONDATA= data set are very similar; the supplies of the supply nodes are 1, not missing S. Otherwise, the solutions are identical.

If total supply exceeds total demand, any missing S values are ignored. If total demand exceeds total supply, any missing D values are ignored.

Balancing Total Supply and Total Demand

When Total Supply Exceeds Total Demand

When total supply of a network problem exceeds total demand, PROC NETFLOW can add an extra node (called the *excess node*) to the problem and set the demand at that node equal to the difference between total supply and total demand. There are three ways that this excess node can be joined to the network. All three ways entail PROC NETFLOW generating a set of arcs (henceforth referred to as the *generated* arcs) that are directed toward the excess node. The total amount of flow in generated arcs equals the demand of the excess node. The generated arcs originate from one of three sets of nodes.

When you specify the THRUNET option, the set of nodes that generated arcs originate from are all demand nodes, even those demand nodes with unspecified demand capability. You indicate that a node has unspecified demand capability by using a missing D value instead of an actual value for demand data (discussed in the section "Missing S Supply and Missing D Demand Values" on page 537). The value specified as the demand of a demand node is in effect a lower bound of the number of flow units that node can actually demand. For missing D demand nodes, this lower bound is zero.

If you do not specify the THRUNET option, the way in which the excess node is joined to the network depends on whether there are demand nodes with unspecified demand capability (nodes with missing D demand).

If there are missing D demand nodes, these nodes are the set of nodes that generated arcs originate from. The value specified as the demand of a demand node, if not missing D, is the number of flow units that node actually demands. For a missing D demand node, the actual demand of that node may be zero or greater.

If there are no missing D demand nodes, the set of nodes that generated arcs originate from are the set of supply nodes. The value specified as the supply of a supply node is in effect an upper bound of the number of flow units that node can actually supply. For missing S supply nodes (discussed in the section "Missing S Supply and Missing D Demand Values" on page 537), this upper bound is zero, so missing S nodes when total supply exceeds total demand are transshipment nodes, nodes that neither supply nor demand flow.

When Total Supply Is Less Than Total Demand

When total supply of a network problem is less than total demand, PROC NETFLOW can add an extra node (called the *excess node*) to the problem and set the supply at that node equal to the difference between total demand and total supply. There are three ways that this excess node can be joined to the network. All three ways entail PROC NETFLOW generating a set of arcs (henceforth referred to as the *generated* arcs) that originate from the excess node. The total amount of flow in generated arcs

equals the supply of the excess node. The generated arcs are directed toward one of three sets of nodes.

When you specify the THRUNET option, the set of nodes that generated arcs are directed toward are all supply nodes, even those supply nodes with unspecified supply capability. You indicate that a node has unspecified supply capability by using a missing S value instead of an actual value for supply data (discussed in the section "Missing S Supply and Missing D Demand Values" on page 537). The value specified as the supply of a supply node is in effect a lower bound of the number of flow units that node can actually supply. For missing S supply nodes, this lower bound is zero.

If you do not specify the THRUNET option, the way in which the excess node is joined to the network depends on whether there are supply nodes with unspecified supply capability (nodes with missing S supply).

If there are missing S supply nodes, these nodes are the set of nodes that generated arcs are directed toward. The value specified as the supply of a supply node, if not missing S, is the number of flow units that node actually supplys. For a missing S supply node, the actual supply of that node may be zero or greater.

If there are no missing S supply nodes, the set of nodes that generated arcs are directed toward are the set of demand nodes. The value specified as the demand of a demand node is in effect an upper bound of the number of flow units that node can actually demand. For missing D demand nodes, (discussed in the section "Missing S Supply and Missing D Demand Values" on page 537), this upper bound is zero, so missing D nodes when total supply is less than total demand are transshipment nodes, nodes that neither supply nor demand flow.

Warm Starts

Using a warm start can increase the overall speed of PROC NETFLOW when it is used repetitively on problems with similar structure. It is most beneficial when a solution of a previous optimization is close to the optimum of the same network with some of its parameters, for example, arc costs, changed. Whether a problem is changed or not, a nonoptimal solution resulting from a previous optimization can be used to restart optimization, thereby saving PROC NETFLOW from having to repeat work to reach the warm start already available.

Time also is saved in the data structure initialization part of the NETFLOW procedure's execution. Information about the previous optimal solution, particularly concerning the size of the problem, a description of the basis spanning tree structure, and what is basic in constraint rows, is known. Information about which nonbasic arcs have capacity flow and which nonbasic nonarc variables are at their respective upper bounds also makes up part of the warm start. The procedure can place arc data into the internal arc length arrays in precisely defined locations, in order of ascending head node internal number. It is not necessary to have multiple passes through the data because literals such as node, nonarc variable, arc, constraint, and special row names are defined and meaning is attached to each. This saves a considerable amount of memory as well. None of the pre-optimization feasibility checks need be repeated.

Warm starts also are useful if you want to determine the effect of arcs being closed to carrying flow. The costs of these arcs are set high enough to ensure that the next optimal solution never has flow through them. Similarly, the effect of opening arcs can be determined by changing the cost of such arcs from an extreme to a reasonable value.

Specify the FUTURE1 or FUTURE2 option to ensure that additional data about a solution to be used as a warm start are output to output data sets. If the FUTURE1 option is specified, extra observations with information on what is to be the warm start are set up for the NODEOUT= and ARCOUT= data sets. The warm start solution in these data sets is a solution obtained after optimization neglecting side constraints. Any cost list variable value in the ARCOUT= data set (and, if there are side constraints, any constraint data in the CONDATA= data set) can be changed before the solution is used as a warm start in a subsequent PROC NETFLOW run. Any nonarc variable data in the CONDATA= data set can be changed at this time as well. New nonarc variables not present in the original problem when the warm start was generated can also be added to the CONDATA= data set before the problem is warm started.

If the FUTURE2 option is specified, extra variables containing information on what will be the warm start solution are set up for the DUALOUT= and CONOUT= data sets. The warm start solution in these data sets is obtained after optimization that considers side constraints has been performed. Part of the warm start is concerned with the constraint part of the basis. Only cost list variable values in the CONOUT= data set can be changed before the solution is used as a warm start in a subsequent PROC NETFLOW run.

If a primal simplex optimization is to use a warm start, the WARM option must be specified in the PROC NETFLOW statement. Otherwise, the primal simplex network algorithm processes the data for a cold start and the extra information is not used.

The ARCDATA= data set is either the ARCOUT= data set from a previous run of PROC NETFLOW with the FUTURE1 option specified (if an unconstrained warm start is used) or the CONOUT= data set from a previous run of PROC NETFLOW with the FUTURE2 option specified (if the warm start was obtained after optimization that considers side constraints was used).

The NODEDATA= data set is the NODEOUT= data set from a previous run of PROC NETFLOW with FUTURE1 specified if an unconstrained warm start is being used. Otherwise, the DUALIN= is the DUALOUT= data sets from a previous run of PROC NETFLOW with FUTURE2 specified, if the warm start was obtained after optimization that considers side constraints was used.

You never need to alter the NODEOUT= data set or the DUALOUT= data set between the time they are generated and when they are used as a warm start. The results would be unpredictable if incorrect changes were made to these data sets, or if a NODEDATA= or a DUALIN= data set were used with an ARCDATA= data set of a different solution.

It is possible, and often useful, to specify WARM and either FUTURE1 or FUTURE2, or both, in the same PROC NETFLOW statement if a new warm start

is to be generated from the present warm start.

The extent of the changes allowed to a primal simplex warm start between the time it is generated and when it is used depends on whether the warm start describes an unconstrained or constrained solution. The following list describes parts of a constrained or an unconstrained warm start that can be altered:

- COST list variable values
- the value of an arc's capacity, as long as the new capacity value is not less than the lower flow bound or the flow through the arc
- any nonarc variable information, in an unconstrained warm start
- for an unconstrained warm start, any side constraint data

The changes that can be made in constraint data for a constrained warm start are more restrictive than those for an unconstrained warm start. The lhs coefficients, type, and rhs value of a constraint can be changed as long as that constraint's slack, surplus, or artificial variable is basic. The constraint name cannot be changed.

Example of a Warm Start

The following sample SAS session demonstrates how the warm start facilities are used to obtain optimal solutions to an unconstrained network where some arc cost changes occur or optimization is halted before the optimum is found.

```
                /* data already in data sets node0 and arc0 */
proc netflow
   nodedata=node0  /* if supply_demand information  */
                   /* is in this SAS data set       */
   arcdata=arc0;
               /* variable list specifications go here    */
               /* assume that they are not necessary here */
               /* if they are, they must be included in   */
               /* all the PROC NETFLOW calls that follow  */
   reset
      future1
      nodeout=node2  /* nodeout and arcout are necessary  */
                     /* when FUTURE1 is used              */
      arcout=arc1;
proc print
      data=arc1;     /* display the optimal solution      */
proc fsedit
      data=arc1;     /* change some arc costs             */
data arc2;
   reset arc1;
      oldflow=_flow_;
      oldfc=_fcost_;
               /* make duplicates of the flow and flowcost*/
               /* variables. If a id list was explicitly  */
               /* specified, add oldflow and oldfc to this*/
               /* list so that they appear in subsequently*/
               /* created arcout= data sets               */
```

The following PROC NETFLOW uses the warm start created previously, performs 250 stage 2 iterations and saves that solution, which (as FUTURE1, ARCOUT=, and NODEOUT= are specified) can be used as a warm start in another PROC NETFLOW run.

```
proc netflow
     warm
     nodedata=node2
     arcdata=arc2;
   reset
     maxit1=250
     future1;
   run;
   save
     nodeout=savelib.node3
     arcout=savelib.arc3;
            /* optimization halted because 250 iterations */
            /* were performed to resume optimization,     */
            /* possibly in another session (the output    */
            /* data sets were saved in a SAS library      */
            /* called savelib)                            */
```

Using the latest warm start, PROC NETFLOW is re-invoked to find the optimal solution.

```
proc netflow
     warm
     nodedata=savelib.node3
     arcdata=savelib.arc3;
   reset
     future1
     nodeout=node4
     arcout=arc4;
   run;
```

If this problem has constraints with data in a data set called CON0, then in each of the previous PROC NETFLOW statements, specify CONDATA=CON0. Between PROC NETFLOW runs, you can change constraint data. In each of the RESET statements, you could specify the CONOUT= data set to save the last (possibly optimal) solution reached by the optimizer if it reaches stage 2. You could specify FUTURE2 and the DUALOUT= data set to generate a constrained warm start.

```
proc netflow
     warm
     nodedata=node4
     arcdata=arc4
     condata=con0;
   reset
     maxit2=125 /* optional, here as a reason why         */
                /* optimum will not be obtained          */
     scratch    /* optional, but warm start might be good */
                /* enough to start stage 2 optimization   */
     future2
   run;
     /* optimization halted after 125 stage 2 iterations  */
   save dualout=dual1 conout=conout1;
```

Stage 2 optimization halted before optimum was reached. Now you can make cost and nonarc variable objective function coefficient changes. Then to restart optimization, use

```
proc netflow
     warm
     condata=con0
           /* NB. NETFLOW reads constraint data only     */
     dualin=dual1
     arcdata=con1;
   reset
     future2
     dualout=dual2
     conout=con2;
   run;
```

How to Make the Data Read of PROC NETFLOW More Efficient

This section contains information useful when you want to solve large constrained network problems. However, much of this information is also useful if you have a large linear programming problem. All of the options described in this section that are not directly applicable to networks (options such as ARCS_ONLY_ARCDATA, ARC_SINGLE_OBS, NNODES=, and NARCS=) can be specified to improve the speed at which LP data is read.

Large Constrained Network Problems

Many of the models presented to PROC NETFLOW are enormous. They can be considered large by linear programming standards; problems with thousands of variables and constraints. When dealing with side constrained network programming problems, models can have not only a linear programming component of that magnitude, but also a larger, possibly *much* larger, network component.

The majority of a network problem's decision variables are arcs. Like an LP decision variable, an arc has an objective function coefficient, upper and lower value bounds,

and a name. Arcs can have coefficients in constraints. Therefore, an arc is quite similar to an LP variable and places the same memory demands on optimization software as an LP variable. But a typical network model has many more arcs and nonarc variables than the typical LP model has variables. And arcs have tail and head nodes. Storing and processing node names require huge amounts of memory. To make matters worse, node names occupy memory at times when a large amount of other data should reside in memory as well.

While memory requirements are lower for a model with embedded network component compared with the equivalent LP *once optimization starts*, the same is usually not true *during the data read*. Even though nodal flow conservation constraints in the LP should not be specified in the constrained network formulation, the memory requirements to read the latter are greater because each arc (unlike an LP variable) originates at one node, and is directed toward another.

Paging

PROC NETFLOW has facilities to read data when the available memory is insufficient to store all the data at once. PROC NETFLOW does this by allocating memory for different purposes, for example, to store an array or receive data read from an input SAS data set. After that memory has filled, the information is sent to disk and PROC NETFLOW can resume filling that memory with new information. Often, information must be retrieved from disk so that data previously read can be examined or checked for consistency. Sometimes, to prevent any data from being lost, or to retain any changes made to the information in memory, the contents of the memory must be sent to disk before other information can take its place. This process of swapping information to and from disk is called paging. Paging can be very time-consuming, so it is crucial to minimize the amount of paging performed.

There are several steps you can take to make PROC NETFLOW read the data of network and linear programming models more efficiently, particularly when memory is scarce and the amount of paging must be reduced. PROC NETFLOW will then be able to tackle large problems in what can be considered reasonable amounts of time.

The Order of Observations

PROC NETFLOW is quite flexible in the ways data can be supplied to it. Data can be given by any reasonable means. PROC NETFLOW has convenient defaults that can save you work when generating the data. There can be several ways to supply the same piece of data, and some pieces of data can be given more than once. PROC NETFLOW reads everything, then merges it all together. However, this flexibility and convenience come at a price; PROC NETFLOW may not assume the data has a characteristic that, if possessed by the data, could save time and memory during the data read. There are several options that indicate the data has some exploitable characteristic.

For example, an arc cost can be specified once or several times in the ARCDATA= or CONDATA= data set, or both. Every time it is given in ARCDATA, a check is made to ensure that the new value is the same as any corresponding value read in a previous observation of ARCDATA. Every time it is given in CONDATA, a check is made to ensure that the new value is the same as the value read in a previous observation of

CONDATA, or previously in ARCDATA. It would save PROC NETFLOW time if it knew that arc cost data would be encountered only once while reading ARCDATA, so performing the time-consuming check for consistency would not be necessary. Also, if you indicate that CONDATA contains data for constraints only, PROC NETFLOW will not expect any arc information, so memory will not be allocated to receive such data while reading CONDATA. This memory is used for other purposes and this might lead to a reduction in paging. If applicable, use the ARC_SINGLE_OBS or the CON_SINGLE_OBS option, or both, and the NON_REPLIC=COEFS specification to improve how ARCDATA and CONDATA are read.

PROC NETFLOW allows the observations in input data sets to be in any order. However, major time savings can result if you are prepared to order observations in particular ways. Time spent by the SORT procedure to sort the input data sets, particularly the CONDATA= data set, may be more than made up for when PROC NETFLOW reads them, because PROC NETFLOW has in memory information possibly used when the previous observation was read. PROC NETFLOW can assume a piece of data is either similar to that of the last observation read or is new. In the first case, valuable information such as an arc or a nonarc variable number or a constraint number is retained from the previous observation. In the last case, checking the data with what has been read previously is not necessary.

Even if you do not sort the CONDATA= data set, grouping observations that contain data for the same arc or nonarc variable or the same row pays off. PROC NETFLOW establishes whether an observation being read is similar to the observation just read.

Practically, several input data sets for PROC NETFLOW might have this characteristic, because it is natural for data for each constraint to be grouped together (dense format of CONDATA) or data for each column to be grouped together (sparse format of CONDATA). If data for each arc or nonarc is spread over more than one observation of the ARCDATA= data set, it is natural to group these observations together.

Use the GROUPED= option to indicate whether observations of the ARCDATA= data set, CONDATA= data set, or both are grouped in a way that can be exploited during data read.

Time is saved if the type data for each row appears near the top of the CONDATA= data set, especially if it has the sparse format. Otherwise, when reading an observation, if PROC NETFLOW does not know if a row is a constraint or special row, the data is set aside. Once the data set has been completely read, PROC NETFLOW must reprocess the data it set aside. By then, it knows the type of each constraint or row or, if its type was not provided, it is assumed to have a default type.

Better Memory Utilization

In order for PROC NETFLOW to make better utilization of available memory, you can now specify options that indicate the approximate size of the model. PROC NETFLOW then knows what to expect. For example, if you indicate that the problem has no nonarc variables, PROC NETFLOW will not allocate memory to store nonarc data. That memory is utilized better for other purposes. Memory is often allocated to receive or store data of some type. If you indicate that the model does not have

much data of a particular type, the memory that would otherwise have been allocated to receive or store that data can be used to receive or store data of another type.

- NNODES= approximate number of nodes
- NARCS= approximate number of arcs
- NNAS= approximate number of nonarc variables or LP variables
- NCONS= approximate number of constraints
- NCOEFS= approximate number of constraint coefficients

These options will sometimes be referred to as Nxxxx= options.

You do not need to specify all these options for the model, but the more you do, the better. If you do not specify some or all of these options, PROC NETFLOW guesses the size of the problem by using what it already knows about the model. Sometimes PROC NETFLOW guesses the size of the model by looking at the number of observations in the ARCDATA= and CONDATA= data sets. However, PROC NETFLOW uses rough rules of thumb; that typical models are proportioned in certain ways (for example, if there are constraints, then arcs and nonarcs usually have 5 constraint coefficients). If your model has an unusual shape or structure, you are encouraged to use these options.

If you do use the options and you do not know the exact values to specify, *overestimate* the values. For example, if you specify NARCS=10000 but the model has 10100 arcs, when dealing with the last 100 arcs, PROC NETFLOW might have to page out data for 10000 arcs each time one of the last arcs must be dealt with. Memory could have been allocated for all 10100 arcs without affecting (much) the rest of the data read, so NARCS=10000 could be more of a hindrance than a help.

The point of these Nxxxx= options is to indicate the model size when PROC NETFLOW does not know it. When PROC NETFLOW knows the "real" value, that value is used instead of Nxxxx= .

When PROC NETFLOW is given a constrained solution warm start, PROC NETFLOW knows from the warm start information all model size parameters, so Nxxxx= options are not used. When an unconstrained warm start is used and the SAME_NONARC_DATA is specified, PROC NETFLOW knows the number of nonarc variables, so that is used instead of the value of the NNAS= option.

ARCS_ONLY_ARCDATA indicates that data for only arcs are in the ARCDATA= data set. Memory would not be wasted to receive data for nonarc and LP variables.

Use the memory usage parameters:

- The BYTES= option specifies the size of PROC NETFLOW main working memory in number of bytes.
- The MAXARRAYBYTES= option specifies the maximum number of bytes that an array can occupy.

- The MEMREP option indicates that memory usage report is to be displayed on the SAS log.

Specifying the BYTES= parameter is particularly important. Specify as large a number as possible, but not such a large number of bytes that will cause PROC NETFLOW (rather, the SAS System running underneath PROC NETFLOW) to run out of memory. Use the MAXARRAYBYTES= option if the model is very large or "disproportionate." Try increasing or decreasing the MAXARRAYBYTES= option. Limiting the amount of memory for use by big arrays is good if they would take up too much memory to the detriment of smaller arrays, buffers, and other things that require memory. However, too small a value of the MAXARRAYBYTES= option might cause PROC NETFLOW to page a big array excessively. Never specify a value for the MAXARRAYBYTES= option that is smaller than the main node length array. PROC NETFLOW reports the size of this array on the SAS log if you specify the MEMREP option. The MAXARRAYBYTES= option influences paging not only in the data read, but also during optimization. It is often better if optimization is performed as fast as possible, even if the read is made slower as a consequence.

Use Defaults to Reduce the Amount of Data

Use as much as possible the parameters that specify default values. For example, if there are several arcs with the same cost value *c*, use DEFCOST=*c* for arcs that have that cost. Use missing values in the COST variable in ARCDATA instead of *c*. PROC NETFLOW ignores missing values, but must read, store, and process nonmissing values, even if they are equal to a default option or could have been equal to a default parameter had it been specified. Sometimes, using default parameters makes the need for some SAS variables in the ARCDATA= and CONDATA= data sets no longer necessary, or reduces the quantity of data that must be read. The default options are

- DEFCOST= default cost of arcs, objective function of nonarc variables or LP variables
- DEFMINFLOW= default lower flow bound of arcs, lower bound of nonarc variables or LP variables
- DEFCAPACITY= default capacity of arcs, upper bound of nonarc variables or LP variables
- DEFCONTYPE=LE DEFCONTYPE= <=
  DEFCONTYPE=EQ DEFCONTYPE= =
  DEFCONTYPE=GE DEFCONTYPE= >= (default constraint type)

The default options themselves have defaults. For example, you do not need to specify DEFCOST=0 in the PROC NETFLOW statement. You should still have missing values in the COST variable in ARCDATA for arcs that have zero costs.

If the network has only one supply node, one demand node, or both, use

- SOURCE= name of single node that has supply capability
- SUPPLY= the amount of supply at SOURCE

- SINK= name of single node that demands flow
- DEMAND= the amount of flow SINK demands

Do not specify that a constraint has zero right-hand-side values. That is the default. The only time it might be practical to specify a zero rhs is in observations of CONDATA read early so that PROC NETFLOW can infer that a row is a constraint. This could prevent coefficient data from being put aside because PROC NETFLOW did not know the row was a constraint.

Names of Things

To cut data read time and memory requirements, reduce the number of bytes in the longest node name, longest arc name, and longest constraint name to 8 bytes or less. The longer a name, the more bytes must be stored and compared with other names.

If an arc has no constraint coefficients, do not give it a name in the NAME list variable in the ARCDATA= data set. Names for such arcs serve no purpose.

PROC NETFLOW can have a default name for each arc. If an arc is directed from node *tailname* toward node *headname*, the default name for that arc is *tailname_headname*. If you do not want PROC NETFLOW to use these default arc names, specify NAMECTRL=1. Otherwise, PROC NETFLOW must use memory for storing node names and these node names must be searched often.

If you want to use the default *tailname_headname name*, that is, NAMECTRL=2 or NAMECTRL=3, do not use underscores in node names. If a CONDATA has a dense format and has a variable in the VAR list A_B_C_D, or if the value A_B_C_D is encountered as a value of the COLUMN list variable when reading CONDATA that has the sparse format, PROC NETFLOW first looks for a node named A. If it finds it, it looks for a node called B_C_D. It then looks for a node with the name A_B and possibly a node with name C_D. A search for a node named A_B_C and possibly a node named D is done. Underscores could have caused PROC NETFLOW to look unnecessarily for nonexistant nodes. Searching for node names can be expensive, and the amount of memory to store node names large. It might be better to assign the arc name A_B_C_D directly to an arc by having that value as a NAME list variable value for that arc in ARCDATA and specify NAMECTRL=1.

Other Ways to Speed Up Data Reads

Use warm starts as much as possible.

- WARM indicates that the input SAS data sets contain a warm start.

The data read of a warm start is much faster than a cold start data read. The model size is known before the read starts. The observations of the NODEDATA= or DUALIN= data sets have observations ordered by node name and constraint name. Information is stored directly in the data structures used by PROC NETFLOW. For a cold start, much of preprocessing must be performed before the information can be stored in the same way. And using a warm start can greatly reduce the time PROC NETFLOW spends doing optimization.

- SAME_NONARC_DATA is an option that excludes data from processing.

This option indicates that the warm start nonarc variable data in ARCDATA is read and any nonarc variable data in CONDATA is to be ignored. Use this option if it is applicable, or when CONDATA has no nonarc variable data, or such data is duplicated in ARCDATA. ARCDATA is always read before CONDATA.

Arcs and nonarc variables can have associated with them values or quantities that have no bearing with the optimization. This information is given in ARCDATA in the ID list variables. For example, in a distribution problem, information such as truck number and driver's name can be associated with each arc. This is useful when a solution is saved in an output SAS data set. However, PROC NETFLOW needs to reserve memory to process this information when data is being read. For large problems when memory is scarce, it might be better to remove ancillary data from ARCDATA. After PROC NETFLOW runs, use SAS software to merge this information into the output data sets that contain the optimal solution.

Macro Variable _ORNETFL

The NETFLOW procedure always creates and initializes a SAS macro called _ORNETFL. After each PROC NETFLOW run, you can examine this macro by specifying **`%put &_ORNETFL;`** and see whether PROC NETFLOW ran correctly or what error or difficulty it encountered.

The value of _ORNETFL consists of five parts:

- ERROR_STATUS, indicating the existence or absence of any errors
- OPT_STATUS, the stage of the optimization, or what solution has been found
- OBJECTIVE=*objective*, the total cost or profit of the current solution. If PROC NETFLOW is solving a maximal flow problem, MAXFLOW=*maxflow*, the amount of the current solution's maximal flow, will follow. If PROC NETFLOW is solving a minimal flow problem (MAXFLOW and MAXIMIZE specified at the same time), MINFLOW=*minflow*, the amount of the current solution's minimal flow, will follow instead.
- SOLUTION, describing the nature of the current solution
- information about the interior point algorithm if this algorithm is used. ITERATIONS=*n*, the number of iterations required to solve the problem. ITERATING_TIME=*Ti*, the time in seconds taken by the interior point algorithm to perform iterations for solving the problem. SOLUTION_TIME=*Ts*, the time in seconds taken by the procedure to presolve the problem, perform interior point iterations, and postsolve the problem.

The value of _ORNETFL is of the following form:

ERROR_STATUS=*charstr* OPT_STATUS=*charstr* OBJECTIVE=*objective*
SOLUTION=*charstr* ITERATIONS=*n* ITERATING_TIME=*Ti*
SOLUTION_TIME=*Ts*

The last three terms appear only if the interior point algorithm is used. Nontrailing blank characters that are unnecessary are removed. Ideally, at the end a PROC NETFLOW run, _ORNETFL should have the value

```
ERROR_STATUS=OK OPT_STATUS=OPTIMAL OBJECTIVE=x
SOLUTION=OPTIMAL
```

if the interior point algorithm is not used, or the value

```
ERROR_STATUS=OK OPT_STATUS=OPTIMAL OBJECTIVE=x
SOLUTION=OPTIMAL ITERATIONS=x ITERATING_TIME=x SOLUTION_TIME=x
```

if the interior point algorithm is used.

If the preprocessor detects that a problem with a network component is infeasible, and you specify that the interior point algorithm should be used, _ORNETFL has the following value:

```
ERROR_STATUS=OK SOLUTION=INFEASIBLE
ITERATIONS=0 ITERATING_TIME=0 SOLUTION_TIME=0
```

The same value is assigned to the _ORNETFL macro variable if the preprocessor detects that an LP problem is infeasible.

Table 5.6 lists alternate values for the _ORNETFL value parts.

Table 5.6. PROC NETFLOW _ORNETFL Macro Values

Keyword	Value	Meaning
ERROR_STATUS	OK	no errors
	MEMORY	memory request failed
	IO	input/output error
	DATA	error in the data
	BUG	error with PROC NETFLOW
	SEMANTIC	semantic error
	SYNTAX	syntax error
	UNKNOWN	unknown error
OPT_STATUS	START	no optimization has been done
	STAGE_1	performing stage 1 optimization
	UNCON_OPT	reached unconstrained optimum, but there are side constraints
	STAGE_2	performing stage 2 optimization
	OPTIMAL	reached the optimum
OBJECTIVE	*objective*	total cost or profit
MINFLOW	*minflow*	if MAXFLOW MAXIMIZE is specified
MAXFLOW	*maxflow*	if MAXFLOW is specified
SOLUTION	NONOPTIMAL	more optimization is required

Keyword	Value	Meaning
	STAGE_2_REQUIRED	reached unconstrained optimum, stage 2 optimization is required
	OPTIMAL	have determined the optimum
	INFEASIBLE	infeasible; no solution exists
	UNRESOLVED_OPTIMALITY _OR_FEASIBILITY	the optimization process stops before optimality or infeasibility can be proven.
	MAXITERB_OPTION _STOPPED_OPTIMIZATION	the interior point algorithm stops after performing maximal number of iterations specified by the MAXITERB= option

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the NETFLOW procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

The Interior Point Algorithm: NETFLOW Procedure

Introduction

The simplex algorithm, developed shortly after World War II, was the main method used to solve linear programming problems. Over the last fifteen years, the interior point algorithm has been developed to also solve linear programming problems. From the start it showed great theoretical promise, and considerable research in the area resulted in practical implementations that performed competitively with the simplex algorithm. More recently, interior point algorithms have evolved to become superior to the simplex algorithm, in general, especially when the problems are large.

The interior point algorithm has been implemented in PROC NETFLOW. This algorithm can be used to solve linear programs as well as network problems. When PROC NETFLOW detects that the problem has no network component, it automatically invokes the interior point algorithm to solve the problem. The data required by PROC NETFLOW for a linear program resembles the data for nonarc variables and constraints for constrained network problems.

If PROC NETFLOW does detect a network component to the problem (the problem has arcs), you must specify the option INTPOINT in the PROC NETFLOW statement if you want to use the interior point algorithm. PROC NETFLOW first converts the constrained network model into an equivalent linear programming formulation, solves that, then converts the LP back to the network model. These models remain conceptually easy since they are based on network diagrams that represent the problem pictorially. This procedure accepts the network specification in a format that is particularly suited to networks. This not only simplifies problem description but also aids in the interpretation of the solution. The conversions to and from the equivalent LP are done "behind the scenes."

There are many variations of interior point algorithms. PROC NETFLOW uses the Primal-Dual with Predictor-Corrector algorithm. This algorithm and related theory can be found in the texts by Roos, Terlaky, and Vial (1997), Wright (1996), and Ye (1996).

The remainder of this section is split into two parts. In the first part, how you use PROC NETFLOW's interior point algorithm to solve network problems is described. In the second part, using PROC NETFLOW to solve linear programming problems (its interior point algorithm must be used) is described. Both parts are organized similarly:

- The way data are supplied to PROC NETFLOW is outlined in a "Getting Started" subsection.
- An "Introductory Example" is solved to demonstrate how the data is set up, how PROC NETFLOW is used to compute the solution, and how the optimum is saved.
- More sophisticated ways to use PROC NETFLOW interactively are detailed in an "Interactivity" subsection.

- A "Functional Summary" lists the statements and options that can be used to control PROC NETFLOW. Of particular interest are the options used to control the optimizer, and the way the solution is saved into output data sets or is displayed.

The Linear Programs section has additional subsections:

- "Mathematical Description of LP"
- "Interior Point Algorithmic Details," a brief theory of the algorithm containing information about the options that can be specified to control the interior point algorithm.
- "Syntax" subsection, which is a subset of the syntax when the simplex algorithm is used. Gone are the statements and lists relevant only when the simplex algorithm is used.

Network Models: Interior Point Algorithm

The data required by PROC NETFLOW for a network problem is identical whether the simplex algorithm or the interior point algorithm is used as the optimizer. By default, the simplex algorithm is used for problems with a network component. To use the interior point algorithm, all you need to do is specify the INTPOINT option in the PROC NETFLOW statement. You can optionally specify some options that control the interior point algorithm, of which there are only a few. The interior point algorithm is remarkably robust when reasonable choices are made during the design and implementation, so it does not need to be tuned to the same extent as the simplex algorithm.

When to Use INTPOINT: Network Models: Interior Point Algorithm

PROC NETFLOW uses the primal simplex network algorithm and the primal partitioning algorithm to solve constrained network problems. These algorithms are fast, since they take advantage of algebraic properties of the network component of the problem.

If the network component of the model is large compared to the side constraint component, PROC NETFLOW's optimizer can store what would otherwise be a large matrix as a spanning tree computer data structure. Computations involving the spanning tree data structure can be performed much faster than those using matrices. Only the nonnetwork part of the problem, hopefully quite small, needs to be manipulated by PROC NETFLOW as matrices.

In contrast, LP optimizers must contend with matrices that can be large for large problems. Arithmetic operations on matrices often accumulate rounding errors that cause difficulties for the algorithm. So in addition to the performance improvements, network optimization is generally more numerically stable than LP optimization.

The nodal flow conservation constraints do not need to be specified in the network model. They are implied by the network structure. However, flow conservation constraints do make up the data for the equivalent LP model. If you have an LP that is

small after the flow conservation constraints are removed, that problem is a definite candidate for solution by PROC NETFLOW's specialized simplex method.

However, some constrained network problems are solved more quickly by the interior point algorithm than the network optimizer in PROC NETFLOW. Usually, they have a large number of side constraints or nonarc variables. These models are more like LPs than network problems. The network component of the problem is so small that PROC NETFLOW's network simplex method cannot recoup the effort to exploit that component rather than treat the whole problem as an LP. If this is the case, it is worthwhile to get PROC NETFLOW to convert a constrained network problem to the equivalent LP and use its interior point algorithm. This conversion must be done before any optimization has been performed (specify the INTPOINT option in the PROC NETFLOW statement).

Even though some network problems are better solved by converting them to an LP, the input data and the output solution are more conveniently maintained as networks. You retain the advantages of casting problems as networks: ease of problem generation and expansion when more detail is required. The model and optimal solutions are easy to understand, as a network can be drawn.

Getting Started: Network Models: Interior Point Algorithm

To solve network programming problems with side constraints using PROC NETFLOW, you save a representation of the network and the side constraints in three SAS data sets. These data sets are then passed to PROC NETFLOW for solution. There are various forms that a problem's data can take. You can use any one or a combination of several of these forms.

The NODEDATA= data set contains the names of the supply and demand nodes and the supply or demand associated with each. These are the elements in the column vector b in problem (NPSC).

The ARCDATA= data set contains information about the variables of the problem. Usually these are arcs, but there can be data related to nonarc variables in the ARCDATA= data set as well. If there are no arcs, this is a linear programming problem.

An arc is identified by the names of its tail node (where it originates) and head node (where it is directed). Each observation can be used to identify an arc in the network and, optionally, the cost per flow unit across the arc, the arc's lower flow bound, capacity, and name. These data are associated with the matrix F and the vectors c, l, and u in problem (NPSC).

Note: Although F is a node-arc incidence matrix, it is specified in the ARCDATA= data set by arc definitions. Do not explicitly specify these flow conservation constraints as constraints of the problem.

In addition, the ARCDATA= data set can be used to specify information about nonarc variables, including objective function coefficients, lower and upper value bounds, and names. These data are the elements of the vectors d, m, and v in problem (NPSC). Data for an arc or nonarc variable can be given in more than one observation.

Supply and demand data also can be specified in the ARCDATA= data set. In such a case, the NODEDATA= data set may not be needed.

The CONDATA= data set describes the side constraints and their right-hand sides. These data are elements of the matrices H and Q and the vector r. Constraint types are also specified in the CONDATA= data set. You can include in this data set upper bound values or capacities, lower flow or value bounds, and costs or objective function coefficients. It is possible to give all information about some or all nonarc variables in the CONDATA= data set.

An arc or nonarc variable is identified in this data set by its name. If you specify an arc's name in the ARCDATA= data set, then this name is used to associate data in the CONDATA= data set with that arc. Each arc also has a default name that is the name of the tail and head node of the arc concatenated together and separated by an underscore character; *tail_head*, for example.

If you use the dense side constraint input format and want to use the default arc names, these arc names are names of SAS variables in the VAR list of the CONDATA= data set.

If you use the sparse side constraint input format (described later as well) and want to use the default arc names, these arc names are values of the COLUMN list SAS variable of the CONDATA= data set.

When using the interior point algorithm, the execution of PROC NETFLOW has two stages. In the preliminary (zeroth) stage, the data are read from the NODEDATA= data set, the ARCDATA= data set, and the CONDATA= data set. Error checking is performed. The model is converted into an equivalent linear program.

In the next stage, the linear program is preprocessed. This is optional but highly recommended. Preprocessing analyzes the model and tries to determine before optimization whether variables can be "fixed" to their optimal values. Knowing that, the model can be modified and these variables dropped out. It can be determined that some constraints are redundant. Sometimes, preprocessing succeeds in reducing the size of the problem, thereby making the subsequent optimization easier and faster.

The optimal solution to the linear program is then found. The linear program is converted back to the original constrained network problem, and the optimum for this is derived from the optimum of the equivalent linear program. If the problem was preprocessed, the model is now post-processed, where fixed variables are reintroduced. The solution can be saved in the CONOUT= data set. This data set is also named in the PROC NETFLOW, RESET, and SAVE statements.

The interior point algorithm cannot efficiently be warm started, so options such as FUTURE1 and FUTURE2 options are irrelevant.

Introductory Example: Network Models: Interior Point Algorithm

Consider the following transshipment problem for an oil company in the section "Introductory Example" on page 447. Recall that crude oil is shipped to refineries where it is processed into gasoline and diesel fuel. The gasoline and diesel fuel are then distributed to service stations. At each stage there are shipping, processing, and

distribution costs. Also, there are lower flow bounds and capacities. In addition, there are side constraints to model crude mix stipulations, and model the limitations on the amount of Middle Eastern crude that can be processed by each refinery and the conversion proportions of crude to gasoline and diesel fuel. The network diagram is reproduced in Figure 5.16.

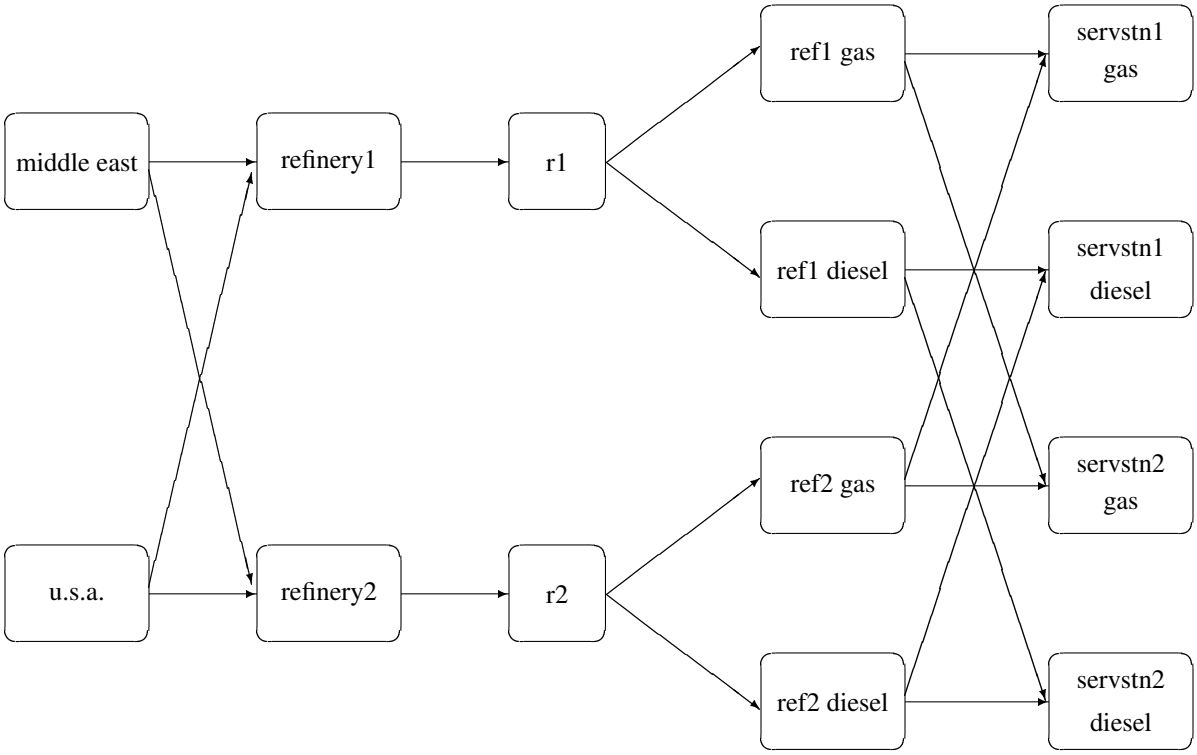


Figure 5.16. Oil Industry Example

To solve this problem with PROC NETFLOW, a representation of the model is saved in three SAS data sets that are identical to the data sets supplied to PROC NETFLOW when the simplex algorithm was used.

To find the minimum cost flow through the network that satisfies the supplies, demands, and side constraints, invoke PROC NETFLOW as follows:

```
proc netflow
   intpoint              /* <<<--- Interior Point used */
   nodedata=noded        /* the supply and demand data */
   arcdata=arcd1         /* the arc descriptions       */
   condata=cond1         /* the side constraints       */
   conout=solution;      /* the solution data set      */
   run;
```

The following messages, which appear on the SAS log, summarize the model as read by PROC NETFLOW and note the progress toward a solution:

```
NOTE: Number of nodes= 14 .
NOTE: Number of supply nodes= 2 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 180 , total demand= 180 .
NOTE: Number of arcs= 18 .
NOTE: Number of <= side constraints= 0 .
NOTE: Number of == side constraints= 2 .
NOTE: Number of >= side constraints= 2 .
NOTE: Number of side constraint coefficients= 8 .
NOTE: The following messages relate to the equivalent Linear
      Programming problem solved by the Interior Point
      algorithm.
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 16 .
NOTE: Number of >= constraints= 2 .
NOTE: Number of constraint coefficients= 44 .
NOTE: Number of variables= 18 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 5.
NOTE: After preprocessing, number of >= constraints= 2.
NOTE: The preprocessor eliminated 11 constraints from the
      problem.
NOTE: The preprocessor eliminated 25 constraint coefficients
      from the problem.
NOTE: After preprocessing, number of variables= 8.
NOTE: The preprocessor eliminated 10 variables from the
      problem.
NOTE: 2 columns, 0 rows and 2 coefficients were added to the
      problem to handle unrestricted variables, variables that
      are split, and constraint slack or surplus variables.
NOTE: There are 13 nonzero elements in A * A transpose.
NOTE: Of the 7 rows and columns, 2 are sparse.
NOTE: There are 6 nonzero superdiagonal elements in the
      sparse rows of the factored A * A transpose. This
      includes fill-in.
NOTE: There are 2 operations of the form
      u[i,j]=u[i,j]-u[q,j]*u[q,i]/u[q,q] to factorize the
      sparse rows of A * A transpose.
NOTE: Bound feasibility attained by iteration 1.
NOTE: Dual feasibility attained by iteration 1.
NOTE: Constraint feasibility attained by iteration 2.
NOTE: The Primal-Dual Predictor-Corrector Interior Point
      algorithm performed 7 iterations.
NOTE: Objective= 50875.01279.
NOTE: The data set WORK.SOLUTION has 18 observations and
      10 variables.
```

The first set of messages provide statistics on the size of the equivalent linear programming problem. The number of variables may not equal the number of arcs if the problem has nonarc variables. This example has none. To convert a network to an equivalent LP problem, a flow conservation constraint must be created for each node (including an excess or bypass node, if required). This explains why the number of equality side constraints and the number of constraint coefficients change when the

interior point algorithm is used.

If the preprocessor was successful in decreasing the problem size, some messages will report how well it did. In this example, the model size was cut in half!

The following set of messages describe aspects of the interior point algorithm. Of particular interest are those concerned with the Cholesky factorization of AA^T where A is the coefficient matrix of the final LP. It is crucial to preorder the rows and columns of this matrix to prevent *fill-in* and reduce the number of row operations to undertake the factorization. See the section "Interior Point Algorithmic Details" on page 567 for more explanation.

Unlike PROC LP, which displays the solution and other information as output, PROC NETFLOW saves the optimum in output SAS data sets you specify. For this example, the solution is saved in the SOLUTION data set. It can be displayed with PROC PRINT as

```
proc print data=solution;
   var _from_ _to_ _cost_ _capac_ _lo_ _name_
       _supply_ _demand_ _flow_ _fcost_ ;
   sum _fcost_;
   title3 'Constrained Optimum';
   run;
```

Constrained Optimum

Obs	_from_	_to_	_cost_	_capac_	_lo_	_name_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	refinery 1	r1	200	175	50	thruput1	.	.	145.000	28999.98
2	refinery 2	r2	220	100	35	thruput2	.	.	35.000	7700.02
3	r1	ref1 diesel	0	75	0		.	.	36.250	0.00
4	r1	ref1 gas	0	140	0	r1_gas	.	.	108.750	0.00
5	r2	ref2 diesel	0	75	0		.	.	8.750	0.00
6	r2	ref2 gas	0	100	0	r2_gas	.	.	26.250	0.00
7	middle east	refinery 1	63	95	20	m_e_ref1	100	.	80.000	5039.99
8	u.s.a.	refinery 1	55	99999999	0		80	.	65.000	3575.00
9	middle east	refinery 2	81	80	10	m_e_ref2	100	.	20.000	1620.02
10	u.s.a.	refinery 2	49	99999999	0		80	.	15.000	735.00
11	ref1 diesel	servstn1 diesel	18	99999999	0		.	30	30.000	540.00
12	ref2 diesel	servstn1 diesel	36	99999999	0		.	30	0.000	0.01
13	ref1 gas	servstn1 gas	15	70	0		.	95	68.750	1031.26
14	ref2 gas	servstn1 gas	17	35	5		.	95	26.250	446.24
15	ref1 diesel	servstn2 diesel	17	99999999	0		.	15	6.250	106.25
16	ref2 diesel	servstn2 diesel	23	99999999	0		.	15	8.750	201.24
17	ref1 gas	servstn2 gas	22	60	0		.	40	40.000	879.99
18	ref2 gas	servstn2 gas	31	99999999	0		.	40	0.000	0.01
										50875.01

Figure 5.17. CONOUT=SOLUTION

Notice that, in the solution data set (Figure 5.17), the optimal flow through each arc in the network is given in the variable named _FLOW_, and the cost of flow through each arc is given in the variable _FCOST_. As expected, the miminal total cost of

the solution found by the interior point algorithm is equal to miminal total cost of the solution found by the simplex algorithm. In this example, the solutions are the same (within several significant digits), but sometimes the solutions can be different.

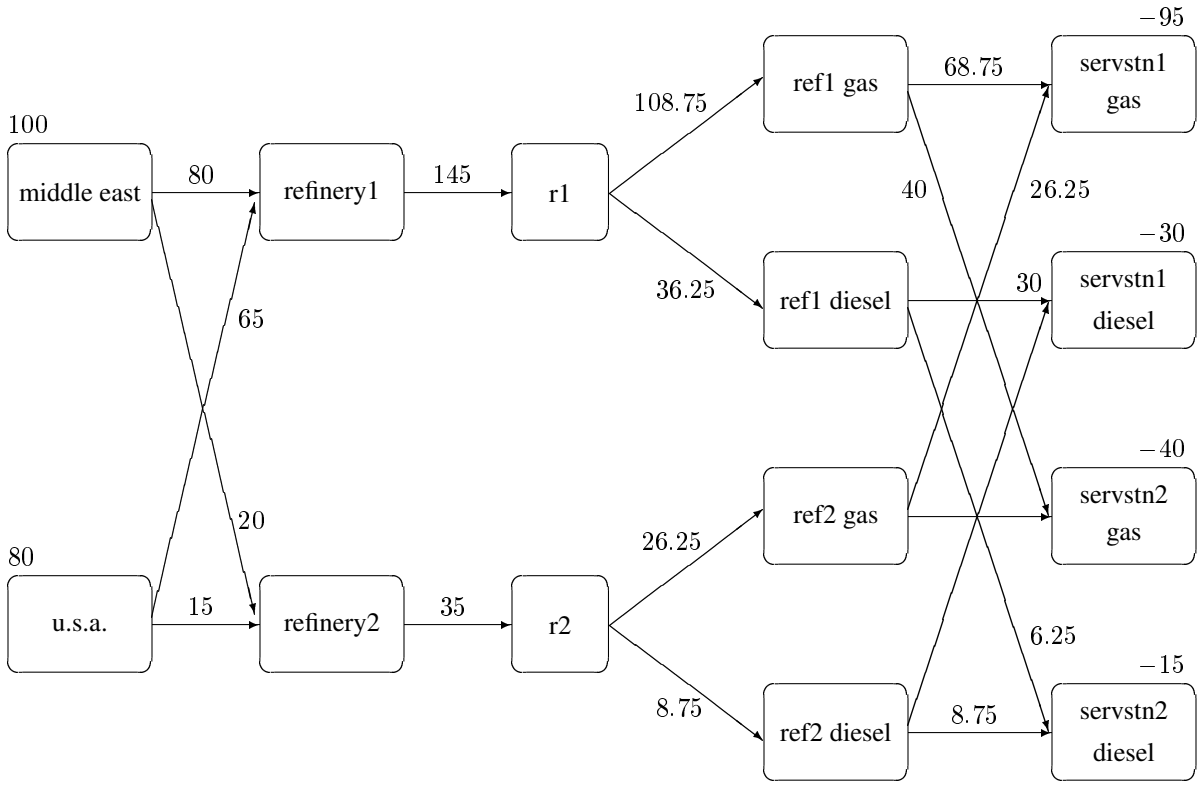


Figure 5.18. Oil Industry Solution

Interactivity: Network Models: Interior Point Algorithm

PROC NETFLOW can be used interactively. You begin by giving the PROC NETFLOW statement with INTPOINT specified, and you must specify the ARCDATA= data set. The CONDATA= data set must also be specified if the problem has side constraints. If necessary, specify the NODEDATA= data set.

The variable lists should be given next. If you have variables in the input data sets that have special names (for example, a variable in the ARCDATA= data set named _TAIL_ that has tail nodes of arcs as values), it may not be necessary to have many or any variable lists.

So far, this is the same as when the simplex algorithm is used, except the INTPOINT option is specified in the PROC NETFLOW statement. The PRINT, QUIT, SAVE, SHOW, RESET, and RUN statements follow and can be listed in any order. The QUIT statements can be used only once. The others can be used as many times as needed.

The CONOPT and PIVOT statements are not relevant to the interior point algorithm and should not be used.

Use the RESET or SAVE statement to change the name of the output data set. There is only one output data set, the CONOUT= data set. With the RESET statement, you can also indicate the reasons why optimization should stop (for example, you can indicate the maximum number of iterations that can be performed). PROC NETFLOW then has a chance to either execute the next statement, or, if the next statement is one that PROC NETFLOW does not recognize (the next PROC or DATA step in the SAS session), do any allowed optimization and finish. If no new statement has been submitted, you are prompted for one. Some options of the RESET statement enable you to control aspects of the interior point algorithm. Specifying certain values for these options can reduce the time it takes to solve a problem. Note that any of the RESET options can be specified in the PROC NETFLOW statement.

The RUN statement starts optimization. Once the optimization has started, it runs until the optimum is reached. The RUN statement should be specified at most once.

The QUIT statement immediately stops PROC NETFLOW. The SAVE statement has options that allow you to name the output data set; information about the current solution is put in this output data set. Use the SHOW statement if you want to examine the values of options of other statements. Information about the amount of optimization that has been done and the STATUS of the current solution can also be displayed using the SHOW statement.

The PRINT statement makes PROC NETFLOW display parts of the problem. The way the PRINT statements are specified are identical whether the interior point algorithm or the simplex algorithm is used, however there are minor differences in what is displayed for each arc, nonarc variable or constraint coefficient.

PRINT ARCS produces information on all arcs. PRINT SOME_ARCS limits this output to a subset of arcs. There are similar PRINT statements for nonarc variables and constraints:

```
PRINT NONARCS;
PRINT SOME_NONARCS;
PRINT CONSTRAINTS;
PRINT SOME_CONS;
```

PRINT CON_ARCS enables you to limit constraint information that is obtained to members of a set of arcs and that have nonzero constraint coefficients in a set of constraints. PRINT CON_NONARCS is the corresponding statement for nonarc variables.

For example, an interactive PROC NETFLOW run might look something like this:

```
proc netflow
       intpoint      /* use the Interior Point algorithm */
       arcdata=data set
       other options;
   variable list specifications;  /* if necessary        */
   reset options;
   print options;                 /* look at problem     */
run;                              /* do the optimization */
```

```
print options;     /* look at the optimal solution     */
save options;      /* keep optimal solution            */
```

If you are interested only in finding the optimal solution, have used SAS variables that have special names in the input data sets, and want to use default settings for everything, then the following statement is all you need.

proc netflow intpoint arcdata= *data set* **;**

Functional Summary: Network Models, Interior Point Algorithm

The following table outlines the options available for the NETFLOW procedure when the interior point algorithm is being used, classified by function.

Table 5.7. Functional Summary, Network Models

Description	Statement	Option
Input Data Set Options:		
arcs input data set	PROC NETFLOW	ARCDATA=
nodes input data set	PROC NETFLOW	NODEDATA=
constraint input data set	PROC NETFLOW	CONDATA=
Output Data Set Option:		
constrained solution data set	PROC NETFLOW	CONOUT=
Data Set Read Options:		
CONDATA has sparse data format	PROC NETFLOW	SPARSECONDATA
default constraint type	PROC NETFLOW	DEFCONTYPE=
special COLUMN variable value	PROC NETFLOW	TYPEOBS=
special COLUMN variable value	PROC NETFLOW	RHSOBS=
used to interpret arc and nonarc variable names	PROC NETFLOW	NAMECTRL=
no new nonarc variables	PROC NETFLOW	SAME_NONARC_DATA
no nonarc data in ARCDATA	PROC NETFLOW	ARCS_ONLY_ARCDATA
data for an arc found once in ARCDATA	PROC NETFLOW	ARC_SINGLE_OBS
data for a constraint found once in CONDATA	PROC NETFLOW	CON_SINGLE_OBS
data for a coefficient found once in CONDATA	PROC NETFLOW	NON_REPLIC=
data is grouped, exploited during data read	PROC NETFLOW	GROUPED=
Problem Size Specification Options:		
approximate number of nodes	PROC NETFLOW	NNODES=
approximate number of arcs	PROC NETFLOW	NARCS=
approximate number of nonarc variables	PROC NETFLOW	NNAS=
approximate number of coefficients	PROC NETFLOW	NCOEFS=
approximate number of constraints	PROC NETFLOW	NCONS=
Network Options:		
default arc cost	PROC NETFLOW	DEFCOST=
default arc capacity	PROC NETFLOW	DEFCAPACITY=

Description	Statement	Option
default arc lower flow bound	PROC NETFLOW	DEFMINFLOW=
network's only supply node	PROC NETFLOW	SOURCE=
SOURCE's supply capability	PROC NETFLOW	SUPPLY=
network's only demand node	PROC NETFLOW	SINK=
SINK's demand	PROC NETFLOW	DEMAND=
convey excess supply/demand through network	PROC NETFLOW	THRUNET
find maximal flow between SOURCE and SINK	PROC NETFLOW	MAXFLOW
cost of bypass arc for MAXFLOW problem	PROC NETFLOW	BYPASSDIVIDE=
find shortest path from SOURCE to SINK	PROC NETFLOW	SHORTPATH
Memory Control Options:		
issue memory usage messages to SAS log	PROC NETFLOW	MEMREP
number of bytes to use for main memory	PROC NETFLOW	BYTES=
proportion of memory for arrays	PROC NETFLOW	COREFACTOR=
maximum bytes for a single array	PROC NETFLOW	MAXARRAYBYTES=
Interior Point Algorithm Options:		
use interior point algorithm	PROC NETFLOW	INTPOINT
factorization method	RESET	FACT_METHOD=
allowed amount of dual infeasibility	RESET	TOLDINF=
allowed amount of primal infeasibility	RESET	TOLPINF=
allowed total amount of dual infeasibility	RESET	TOLTOTDINF=
allowed total amount of primal infeasibility	RESET	TOLTOTPINF=
cut-off tolerance for Cholesky factorization	RESET	CHOLTINYTOL=
density threshold for Cholesky processing	RESET	DENSETHR=
step-length multiplier	RESET	PDSTEPMULT=
preprocessing type	RESET	PRSLTYPE=
print optimization progress on SAS log	RESET	PRINTLEVEL2=
write optimization time to SAS log	RESET	OPTIM_TIMER
Interior Point Stopping Criteria Options:		
maximum number of interior point iterations	RESET	MAXITERB=
primal-dual (duality) gap tolerance	RESET	PDGAPTOL=
stop because of complementarity	RESET	STOP_C=
stop because of duality gap	RESET	STOP_DG=
stop because of $infeas_b$	RESET	STOP_IB=
stop because of $infeas_c$	RESET	STOP_IC=
stop because of $infeas_d$	RESET	STOP_ID=
stop because of complementarity	RESET	AND_STOP_C=
stop because of duality gap	RESET	AND_STOP_DG=
stop because of $infeas_b$	RESET	AND_STOP_IB=
stop because of $infeas_c$	RESET	AND_STOP_IC=
stop because of $infeas_d$	RESET	AND_STOP_ID=
stop because of complementarity	RESET	KEEPGOING_C=
stop because of duality gap	RESET	KEEPGOING_DG=

Description	Statement	Option
stop because of $infeas_b$	RESET	KEEPGOING_IB=
stop because of $infeas_c$	RESET	KEEPGOING_IC=
stop because of $infeas_d$	RESET	KEEPGOING_ID=
stop because of complementarity	RESET	AND_KEEPGOING_C=
stop because of duality gap	RESET	AND_KEEPGOING_DG=
stop because of $infeas_b$	RESET	AND_KEEPGOING_IB=
stop because of $infeas_c$	RESET	AND_KEEPGOING_IC=
stop because of $infeas_d$	RESET	AND_KEEPGOING_ID=
PRINT Statement Options:		
display everything	PRINT	PROBLEM
display arc information	PRINT	ARCS
display nonarc variable information	PRINT	NONARCS
display variable information	PRINT	VARIABLES
display constraint information	PRINT	CONSTRAINTS
display information for some arcs	PRINT	SOME_ARCS
display information for some nonarc variables	PRINT	SOME_NONARCS
display information for some variables	PRINT	SOME_VARIABLES
display information for some constraints	PRINT	SOME_CONS
display information for some constraints associated with some arcs	PRINT	CON_ARCS
display information for some constraints associated with some nonarc variables	PRINT	CON_NONARCS
display information for some constraints associated with some variables	PRINT	CON_VARIABLES
PRINT Statement Qualifiers:		
produce a short report	PRINT	/ SHORT
produce a long report	PRINT	/ LONG
display arcs/variables with zero flow/value	PRINT	/ ZERO
display arcs/variables with nonzero flow/value	PRINT	/ NONZERO
SHOW Statement Options:		
show problem, optimization status	SHOW	STATUS
show network model parameters	SHOW	NETSTMT
show data sets that have been or will be created	SHOW	DATASETS
Miscellaneous Options:		
infinity value	PROC NETFLOW	INFINITY=
scale constraint row, nonarc variable column coefficients, or both	PROC NETFLOW	SCALE=
maximization instead of minimization	PROC NETFLOW	MAXIMIZE

Linear Programming Models: Interior Point Algorithm

By default, the interior point algorithm is used for problems without a network component, that is, a linear programming problem. You do not need to specify the INTPOINT option in the PROC NETFLOW statement (although you will do no harm if you do).

Data for a linear programming problem resembles the data for side constraints and nonarc variables supplied to PROC NETFLOW when solving a constrained network problem. It is also very similar to the data required by the LP procedure.

Mathematical Description of LP

If the network component of NPSC is removed, the result is the mathematical description of the linear programming problem. If an LP has g variables, and k constraints, then the formal statement of the problem solved by PROC NETFLOW is

$$\begin{array}{ll} \text{minimize} & d^T z \\ \text{subject to} & Qz \{\geq, =, \leq\} r \\ & m \leq z \leq v \end{array}$$

where

- d is the $g \times 1$ objective function coefficient vector
- z is the $g \times 1$ variable value vector
- Q is the $k \times g$ constraint coefficient matrix for variables, where $Q_{i,j}$ is the coefficient of variable j in the ith constraint
- r is the $k \times 1$ side constraint right-hand-side vector
- m is the $g \times 1$ variable value lower bound vector
- v is the $g \times 1$ variable value upper bound vector

Interior Point Algorithmic Details

After preprocessing, the linear program to be solved is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

This is the *primal* problem. The matrices d, z, and Q of NPSC have been renamed c, x, and A respectively, as these symbols are by convention used more, the problem to be solved is different from the original because of preprocessing, and there has been a change of primal variable to transform the LP into one whose variables have zero lower bounds. To simplify the algebra here, assume that variables have infinite bounds, and constraints are equalities. (Interior point algorithms do efficiently handle finite bounds, and it is easy to introduce primal slack variables to change inequalities into equalities.) The problem has n variables; i is a variable number, k is an iteration number, and if used as a subscript or superscript it denotes "of iteration k".

There exists an equivalent problem, the *dual* problem, stated as

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c \\ & s \geq 0 \end{array}$$

where y are dual variables, and s are dual constraint slacks.

The interior point algorithm solves the system of equations to satisfy the Karush-Kuhn-Tucker (KKT) conditions for optimality:

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ x^T s &= 0 \\ x &\geq 0 \\ s &\geq 0 \end{aligned}$$

These are the conditions for feasibility, with the *complementarity* condition $x^T s = 0$ added. Complementarity forces the optimal objectives of the primal and dual to be equal, $c^T x_{opt} = b^T y_{opt}$, as

$$\begin{aligned} &0 = x_{opt}^T s_{opt} = s_{opt}^T x_{opt} = (c - A^T y_{opt})^T x_{opt} = \\ &c^T x_{opt} - y_{opt}^T (A x_{opt}) = c^T x_{opt} - b^T y_{opt} \end{aligned}$$

Before the optimum is reached, a solution (x, y, s) may not satisfy the KKT conditions:

- Primal constraints may be violated, $infeas_c = b - Ax \neq 0$.
- Dual constraints may be violated, $infeas_d = c - A^T y - s \neq 0$.
- Complementarity may not be satisfied, $x^T s = c^T x - b^T y \neq 0$. This is called the *duality gap*.

The interior point algorithm works by using Newton's method to find a direction to move $(\Delta x^k, \Delta y^k, \Delta s^k)$ from the current solution (x^k, y^k, s^k) toward a better solution:

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where α is the *step length* and is assigned a value as large as possible but ≤ 1.0 and not so large that an x_i^{k+1} or s_i^{k+1} is "too close" to zero. The direction in which to move is found using the following:

$$\begin{aligned} A\Delta x^k &= -infeas_c \\ A^T \Delta y^k + \Delta s^k &= -infeas_d \end{aligned}$$

$$S^k \Delta x^k + X^k \Delta s^k = -X^k S^k e$$

where $S = \text{diag}(s)$, $X = \text{diag}(x)$, and e is a vector with all elements equal to 1.

To greatly improve performance, the third equation is changed to

$$S^k \Delta x^k + X^k \Delta s^k = -X^k S^k e + \sigma_k \mu_k e$$

where $\mu_k = 1/n X^k S^k e$, the average complementarity, and $0 \leq \sigma_k \leq 1$.

The effect now is to find a direction in which to move to reduce infeasibilities and to reduce the complementarity toward zero, but if any $x_i^k s_i^k$ is too close to zero, it is "nudged out" to μ, and any $x_i^k s_i^k$ that is larger than μ is "nudged into" μ. A σ_k close to or equal to 0.0 biases a direction toward the optimum, and a value for σ_k close to or equal to 1.0 "centers" the direction toward a point where all pairwise products $x_i^k s_i^k = \mu$. Such points make up the *central path* in the interior. Although centering directions make little, if any, progress in reducing μ and moving the solution closer to the optimum, substantial progress toward the optimum can usually be made in the next iteration.

The central path is crucial to why the interior point algorithm is so efficient. This path "guides" the algorithm to the optimum through the interior of feasible space. Without centering, the algorithm would find a series of solutions near each other close to the boundary of feasible space. Step lengths along the direction would be small and many more iterations would probably be required to reach the optimum.

That in a nutshell is the primal-dual interior point algorithm. Varieties of the algorithm differ in the way α and σ_k are chosen and the direction adjusted during each iteration. A wealth of information can be found in the texts by Roos, Terlaky, and Vial (1997), Wright (1996), and Ye (1996).

The calculation of the direction is the most time-consuming step of the interior point algorithm. Assume the kth iteration is being performed, so the subscript and superscript k can be dropped from the algebra:

$$\begin{aligned} A\Delta x &= -infeas_c \\ A^T \Delta y + \Delta s &= -infeas_d \\ S\Delta x + X\Delta s &= -XSe + \sigma\mu e \end{aligned}$$

Rearranging the second equation,

$$\Delta s = -infeas_d - A^T \Delta y$$

Rearranging the third equation,

$$\Delta s = X^{-1}(-S\Delta x - XSe + \sigma\mu e)$$
$$\Delta s = -\Theta\Delta x - Se + X^{-1}\sigma\mu e$$

where $\Theta = SX^{-1}$.

Equating these two expressions for Δs and rearranging,

$$-\Theta\Delta x - Se + X^{-1}\sigma\mu e = -infeas_d - A^T\Delta y$$
$$-\Theta\Delta x = Se - X^{-1}\sigma\mu e - infeas_d - A^T\Delta y$$
$$\Delta x = \Theta^{-1}(-Se + X^{-1}\sigma\mu e + infeas_d + A^T\Delta y)$$
$$\Delta x = \rho + \Theta^{-1}A^T\Delta y$$

where $\rho = \Theta^{-1}(-Se + X^{-1}\sigma\mu e + infeas_d)$.

Substituting into the first direction equation,

$$A\Delta x = -infeas_c$$
$$A(\rho + \Theta^{-1}A^T\Delta y) = -infeas_c$$
$$A\Theta^{-1}A^T\Delta y = -infeas_c - A\rho$$
$$\Delta y = (A\Theta^{-1}A^T)^{-1}(-infeas_c - A\rho)$$

Θ, ρ, Δy, Δx and Δs are calculated in that order. The hardest term is the factorization of the $(A\Theta^{-1}A^T)$ matrix to determine Δy. Fortunately, although the *values* of $(A\Theta^{-1}A^T)$ are different for each iteration, the *locations* of the nonzeros in this matrix remain fixed; the nonzero locations are the same as those in the matrix (AA^T). This is due to $\Theta^{-1} = XS^{-1}$ being a diagonal matrix, which has the effect of merely scaling the columns of (AA^T).

The fact that the nonzeros in $A\Theta^{-1}A^T$ have a constant pattern is exploited by all interior point algorithms, and is a major reason for their excellent performance. Before iterations begin, AA^T is examined and its rows and columns are permuted so that during Cholesky Factorization, the number of *fill-ins* created is smaller. A list of arithmetic operations to perform the factorization is saved in concise computer data structures (working with memory locations rather than actual numerical values). This is called *symbolic factorization*. During iterations, when memory has been initialized with numerical values, the operations list is performed sequentially. Determining how the factorization should be performed again and again is unnecessary.

The Primal-Dual Predictor-Corrector Interior Point Algorithm

The variant of the interior point algorithm implemented in PROC NETFLOW is a Primal-Dual Predictor-Corrector interior point algorithm. At first, Newton's method is used to find a direction to move $(\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff})$, but calculated as if μ is zero, that is, a step with no centering, known as an *affine* step:

$$A\Delta x^k_{aff} = -infeas_c$$
$$A^T\Delta y^k_{aff} + \Delta s^k_{aff} = -infeas_d$$
$$S^k\Delta x^k_{aff} + X^k\Delta s^k_{aff} = -X^kS^ke$$

$$(x^k_{aff}, y^k_{aff}, s^k_{aff}) = (x^k, y^k, s^k) + \alpha(\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff})$$

where α is the *step length* as before.

Complementarity x^Ts is calculated at $(x^k_{aff}, y^k_{aff}, s^k_{aff})$ and compared with the complementarity at the starting point (x^k, y^k, s^k), and the success of the affine step is gauged. If the affine step was successful in reducing the complementarity by a substantial amount, the need for centering is not great, and the value of σ_k in the following linear system is assigned a value close to zero. If, however, the affine step was unsuccessful, centering would be beneficial, and the value of σ_k in the following linear system is assigned a value closer to 1.0. The value of σ_k is therefore adaptively altered depending on the progress made toward the optimum.

A second linear system is solved to determine a centering vector $(\Delta x^k_c, \Delta y^k_c, \Delta s^k_c)$ from $(x^k_{aff}, y^k_{aff}, s^k_{aff})$:

$$A\Delta x^k_c = 0$$
$$A^T\Delta y^k_c + \Delta s^k_c = 0$$
$$S^k\Delta x^k_c + X^k\Delta s^k_c = -X^kS^ke$$
$$S^k\Delta x^k + X^k\Delta s^k = -X^k_{aff}S^k_{aff}e + \sigma_k\mu_ke$$

Then

$$(\Delta x^k, \Delta y^k, \Delta s^k) = (\Delta x^k_{aff}, \Delta y^k_{aff}, \Delta s^k_{aff}) + (\Delta x^k_c, \Delta y^k_c, \Delta s^k_c)$$

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)$$

where, as before, α is the *step length* assigned a value as large as possible but not so large that an x^{k+1}_i or s^{k+1}_i is "too close" to zero.

Although the Predictor-Corrector variant entails solving two linear system instead of one, fewer iterations are usually required to reach the optimum. The additional overhead of calculating the second linear system is small, as the factorization of the $(A\Theta^{-1}A^T)$ matrix has already been performed to solve the first linear system.

Stopping Criteria

There are several reasons why PROC NETFLOW stops interior point optimization. Optimization stops when

- the number of iteration equals MAXITERB=m

- the relative gap $(duality\ gap/(c^T x))$ between the primal and dual objectives is smaller than the value of the PDGAPTOL= option, and both the primal and dual problems are feasible. Duality gap is defined in the section "Interior Point Algorithmic Details" on page 567.

PROC NETFLOW may stop optimization when it detects that the rate at which the complementarity or duality gap is being reduced is too slow, that is, there are consecutive iterations when the complementarity or duality gap has stopped getting smaller and the infeasibilities, if nonzero, have also stalled. Sometimes, this indicates the problem is infeasible.

The reasons to stop optimization outlined in the previous paragraph will be termed the *usual* stopping conditions in the following explanation.

However, when solving some problems, especially if the problems are large, the usual stopping criteria are inappropriate. PROC NETFLOW might stop prematurely. If it were allowed to perform additional optimization, a better solution would be found. On other occasions, PROC NETFLOW might do too much work. A sufficiently good solution might be reached several iterations before PROC NETFLOW eventually stops.

You can see PROC NETFLOW's progress to the optimum by specifying PRINTLEVEL2=2. PROC NETFLOW will produce a table on the SAS log. A row of the table is generated during each iteration and consists of values of the affine step complementarity, the complementarity of the solution for the next iteration, the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi}$ (see the $infeas_b$ array in the section "Interior Point: Upper Bounds" on page 575), the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci}$ (see the $infeas_c$ array in the section "Interior Point Algorithmic Details" on page 567), and the total dual infeasibility $\sum_{i=1}^{n} infeas_{di}$ (see the $infeas_d$ array in the section "Interior Point Algorithmic Details" on page 567). As optimization progresses, the values in all columns should converge to zero.

To tailor stopping criteria to your problem, you can use two sets of parameters: the STOP_x and the KEEPGOING_x parameters. The STOP_x parameters (STOP_C, STOP_DG, STOP_IB, STOP_IC, and STOP_ID) are used to test for some condition at the beginning of each iteration and if met, to stop immediately. The KEEPGOING_x parameters (KEEPGOING_C, KEEPGOING_DG, KEEPGOING_IB, KEEPGOING_IC, and KEEPGOING_ID) are used when PROC NETFLOW would ordinarily stop but does not if some conditions are not met.

For the sake of conciseness, a set of options will be referred to as the part of the option name they have in common followed by the suffix x. For example, STOP_C, STOP_DG, STOP_IB, STOP_IC, and STOP_ID will collectively be referred to as STOP_x.

At the beginning of each iteration, PROC NETFLOW will test whether complementarity is $\leq$ STOP_C (provided you have specified a STOP_C parameter) and if it is, PROC NETFLOW will stop. If the duality gap is $\leq$ STOP_DG (provided you have specified a STOP_DG parameter), PROC NETFLOW will stop immediately. This is true as well for the other STOP_x parameters that are related to infeasibilities, STOP_IB, STOP_IC, and STOP_ID.

For example, if you want PROC NETFLOW to stop optimizing for the usual stopping conditions, plus the additional condition, complementarity ≤ 100 *or* duality gap ≤ 0.001, then use

```
proc netflow stop_c=100 stop_dg=0.001
```

If you want PROC NETFLOW to stop optimizing for the usual stopping conditions, plus the additional condition, complementarity ≤ 1000 *and* duality gap ≤ 0.001 *and* constraint infeasibility ≤ 0.0001, then use

```
proc netflow
   and_stop_c=1000 and_stop_dg=0.01 and_stop_ic=0.0001
```

Unlike the STOP_x parameters that cause PROC NETFLOW to stop when any one of them is satisfied, the corresponding AND_STOP_x parameters (AND_STOP_C, AND_STOP_DG, AND_STOP_IB, AND_STOP_IC, and AND_STOP_ID) cause PROC NETFLOW to stop only if all (more precisely, all that are specified) options are satisfied. For example, if PROC NETFLOW should stop when

- complementarity ≤ 100 *or* duality gap ≤ 0.001 or
- complementarity ≤ 1000 *and* duality gap ≤ 0.001 *and* constraint infeasibility ≤ 0.000

then use

```
proc netflow
   stop_c=100 stop_dg=0.001
   and_stop_c=1000 and_stop_dg=0.01 and_stop_ic=0.0001
```

Just as the STOP_x parameters have AND_STOP_x partners, the KEEPGOING_x parameters have AND_KEEPGOING_x partners. The role of the KEEPGOING_x and AND_KEEPGOING_x parameters is to prevent optimization from stopping too early, even though a usual stopping criterion is met.

When PROC NETFLOW detects that it should stop for a usual stopping condition, it performs the following tests:

- It will test whether complementarity is $>$ KEEPGOING_C (provided you have specified a KEEPGOING_C parameter), and if it is, PROC NETFLOW will perform more optimization.
- Otherwise, PROC NETFLOW will then test whether the primal-dual gap is $>$ KEEPGOING_DG (provided you have specified a KEEPGOING_DG parameter), and if it is, PROC NETFLOW will perform more optimization.
- Otherwise, PROC NETFLOW will then test whether the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi} >$ KEEPGOING_IB (provided you have specified a KEEPGOING_IB parameter), and if it is, PROC NETFLOW will perform more optimization.

- Otherwise, PROC NETFLOW will then test whether the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci} >$ KEEPGOING_IC (provided you have specified a KEEPGOING_IC parameter), and if it is, PROC NETFLOW will perform more optimization.
- Otherwise, PROC NETFLOW will then test whether the total dual infeasibility $\sum_{i=1}^{n} infeas_{di} >$ KEEPGOING_ID (provided you have specified a KEEPGOING_ID parameter), and if it is, PROC NETFLOW will perform more optimization.
- Otherwise it will test whether complementarity is $>$ AND_KEEPGOING_C (provided you have specified an AND_KEEPGOING_C parameter), *and* the primal-dual gap is $>$ AND_KEEPGOING_DG (provided you have specified an AND_KEEPGOING_DG parameter), *and* the total bound infeasibility $\sum_{i=1}^{n} infeas_{bi} >$ AND_KEEPGOING_IB (provided you have specified an AND_KEEPGOING_IB parameter), *and* the total constraint infeasibility $\sum_{i=1}^{m} infeas_{ci} >$ AND_KEEPGOING_IC (provided you have specified an AND_KEEPGOING_IC parameter) *and* the total dual infeasibility $\sum_{i=1}^{n} infeas_{di} >$ AND_KEEPGOING_ID (provided you have specified an AND_KEEPGOING_ID parameter), and if it is, PROC NETFLOW will perform more optimization.

If all these tests to decide whether more optimization should be performed are false, optimization is stopped.

For example,

```
proc netflow
   stop_c=1000
   and_stop_c=2000 and_stop_dg=0.01
   and_stop_ib=1 and_stop_ic=1 and_stop_id=1
   keepgoing_c=1500
   and_keepgoing_c=2500 and_keepgoing_dg=0.05
   and_keepgoing_ib=1 and_keepgoing_ic=1 and_keepgoing_id=1
```

At the beginning of each iteration, PROC NETFLOW will stop if

- complementarity $\leq$ 1000 or
- complementarity $\leq$ 2000 and duality gap $\leq$ 0.01 and the total bound, constraint, and dual infeasibilities are each $\leq$ 1

When PROC NETFLOW determines it should stop because a usual stopping condition is met, it will stop only if

- complementarity $\leq$ 1500 or
- complementarity $\leq$ 2500 and duality gap $\leq$ 0.05 and the total bound, constraint, and dual infeasibilities are each $\leq$ 1

Interior Point: Upper Bounds

If the LP model had upper bounds ($0 \leq x \leq u$ where u is the upper bound vector), then the primal and dual problems, the duality gap, and the KKT conditions would have to be expanded.

The primal linear program to be solved is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

where $0 \leq x \leq u$ is split into $x \geq 0$ and $x \leq u$. Let z be primal slack so that $x + z = u$, and associate dual variables w with these constraints. The interior point algorithm solves the system of equations to satisfy the Karush-Kuhn-Tucker (KKT) conditions for optimality:

$$\begin{aligned} Ax &= b \\ x + z &= u \\ A^T y + s - w &= c \\ x^T s &= 0 \\ z^T w &= 0 \\ x, s, z, w &\geq 0 \end{aligned}$$

These are the conditions for feasibility, with the *complementarity* conditions $x^T s = 0$ and $z^T w = 0$ added. Complementarity forces the optimal objectives of the primal and dual to be equal, $c^T x_{opt} = b^T y_{opt} - u^T w_{opt}$, as

$$0 = z_{opt}^T w_{opt} = (u - x_{opt})^T w_{opt} = u^T w_{opt} - x_{opt}^T w_{opt}$$

$$0 = x_{opt}^T s_{opt} = s_{opt}^T x_{opt} = (c - A^T y_{opt} + w_{opt})^T x_{opt} =$$
$$c^T x_{opt} - y_{opt}^T (A x_{opt}) + w_{opt})^T x_{opt} = c^T x_{opt} - b^T y_{opt} + u^T w_{opt}$$

Before the optimum is reached, a solution (x, y, s, z, w) might not satisfy the KKT conditions:

- Primal bound constraints may be violated, $infeas_b = u - x - z \neq 0$.
- Primal constraints may be violated, $infeas_c = b - Ax \neq 0$.
- Dual constraints may be violated, $infeas_d = c - A^T y - s + w \neq 0$.
- Complementarity conditions may not be satisfied, $x^T s \neq 0$ and $z^T w \neq 0$.

The calculations of the interior point algorithm can easily be derived in a fashion similar to calculations for when an LP has no upper bounds. See the paper by Lustig, Marsten, and Shanno (1992). An important point is that upper bounds can be handled by specializing the algorithm and *not* by generating the constraints $x + z = u$ and adding these to the main primal constraints $Ax = b$.

Getting Started: Linear Programming Models: Interior Point Algorithm

To solve linear programming problem using PROC NETFLOW, you save a representation of the variables and the constraints in one or two SAS data sets. These data sets are then passed to PROC NETFLOW for solution. There are various forms that a problem's data can take. You can use any one or a combination of several of these forms.

The ARCDATA= data set contains information about the variables of the problem. Although this data set is called ARCDATA, it contains data for no arcs. Instead, all data in this data set are related to variables.

The ARCDATA= data set can be used to specify information about variables, including objective function coefficients, lower and upper value bounds, and names. These data are the elements of the vectors d, m, and v in problem (NPSC). Data for a variable can be given in more than one observation.

When the data for a constrained network problem is being provided, the ARCDATA= data set always contains information necessary for arcs, their tail and head nodes, and optionally the supply and demand information of these nodes. When the data for a linear programming problem is being provided, none of this information is present, as the model has no arcs. This is the way PROC NETFLOW decides which type of problem it is to solve.

PROC NETFLOW was originally designed to solve models with networks, so an ARCDATA= data set is always expected. If an ARCDATA= data set is not specified, by default the last data set created before PROC NETFLOW is invoked is assumed to be an ARCDATA= data set. However, these characteristics of PROC NETFLOW are not helpful when a linear programming problem is being solved and all data is provided in a single data set specified by the CONDATA= data set, and that data set is not the last data set created before PROC NETFLOW starts. In this case, you must specify that an ARCDATA= data set and a CONDATA= data set are both equal to the input data set. PROC NETFLOW then knows that a linear programming problem is to be solved, and the data reside in one data set.

The CONDATA= data set describes the constraints and their right-hand sides. These data are elements of the matrix Q and the vector r.

Constraint types are also specified in the CONDATA= data set. You can include in this data set variable data such as upper bound values, lower value bounds, and objective function coefficients. It is possible to give all information about some or all variables in the CONDATA= data set.

A variable is identified in this data set by its name. If you specify a variable's name in the ARCDATA= data set, then this name is used to associate data in the CONDATA= data set with that variable.

If you use the dense constraint input format, these variable names are names of SAS variables in the VAR list of the CONDATA= data set.

If you use the sparse constraint input format, these variable names are values of the COLUMN list SAS variable of CONDATA= data set.

When using the interior point algorithm, the execution of PROC NETFLOW has two stages. In the preliminary (zeroth) stage, the data are read from the ARCDATA= data set (if used) and the CONDATA= data set. Error checking is performed. In the next stage, the linear program is preprocessed, then the optimal solution to the linear program is found. The solution is saved in the CONOUT= data set. This data set is also named in the PROC NETFLOW, RESET, and SAVE statements.

See the section "Getting Started: Network Models: Interior Point Algorithm" on page 557 for a fuller description of the stages of the interior point algorithm.

Introductory Example: Linear Programming Models: Interior Point Algorithm

Consider the linear programming problem in the section "An Introductory Example" on page 164 in the chapter on the LP procedure.

```
data dcon1;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2
         _type_ $ _rhs_;
   datalines;
profit            -175 -165 -205  0  0  0 300 300 max      .
naphtha_l_conv     .035 .030 .045 -1  0  0   0   0  eq      0
naphtha_i_conv     .100 .075 .135  0 -1  0   0   0  eq      0
heating_o_conv     .390 .300 .430  0  0 -1   0   0  eq      0
recipe_1             0    0    0  0 .3 .7  -1   0  eq      0
recipe_2             0    0    0 .2  0 .8   0  -1  eq      0
available          110  165   80  .  .  .   .   . upperbd .
;
```

To find the minimum cost solution and to examine all or parts of the optimum, you use PRINT statements.

- **print problem/short;** outputs information for all variables and all constraint coefficients. See Figure 5.19 and Figure 5.20.
- **print some_variables(j:)/short;** is information about a set of variables, (in this case, those with names that start with the character string preceding the colon). See Figure 5.21.
- **print some_cons(recipe_1)/short;** is information about a set of constraints (here, that set only has one member, the constraint called recipe_1). See Figure 5.22.
- **print con_variables(_all_,brega)/short;** lists the constraint information for a set of variables (here, that set only has one member, the variable called brega). See Figure 5.23.
- **print con_variables(recipe:,n: jet_1)/short;** coefficient information for those in a set of constraints belonging to a set of variables. See Figure 5.24.

```
proc netflow
   condata=dcon1
   conout=solutn1;
run;
print problem/short;
print some_variables(j:)/short;
print some_cons(recipe_1)/short;
print con_variables(_all_,brega)/short;
print con_variables(recipe:,n: jet_1)/short;
```

The following messages, which appear on the SAS log, summarize the model as read by PROC NETFLOW and note the progress toward a solution:

```
NOTE: Number of variables= 8 .
NOTE: Number of <= constraints= 0 .
NOTE: Number of == constraints= 5 .
NOTE: Number of >= constraints= 0 .
NOTE: Number of constraint coefficients= 18 .
NOTE: After preprocessing, number of <= constraints= 0.
NOTE: After preprocessing, number of == constraints= 0.
NOTE: After preprocessing, number of >= constraints= 0.
NOTE: The preprocessor eliminated 5 constraints from the
      problem.
NOTE: The preprocessor eliminated 18 constraint
      coefficients from the problem.
NOTE: After preprocessing, number of variables= 0.
NOTE: The preprocessor eliminated 8 variables from the
      problem.
WARNING: Optimization is unnecessary as the problem no
      longer has any variables and rows.
NOTE: Preprocessing could have caused that.
NOTE: Objective= 1544.
NOTE: The data set WORK.SOLUTN1 has 8 observations and
      6 variables.
```

The NETFLOW Procedure

_N_	_NAME_	_OBJFN_	_UPPERBD	_LOWERBD	_VALUE_
1	a_heavy	-165	165	0	0
2	a_light	-175	110	0	110
3	brega	-205	80	0	80
4	heatingo	0	99999999	0	77.3
5	jet_1	300	99999999	0	60.65
6	jet_2	300	99999999	0	63.33
7	naphthai	0	99999999	0	21.8
8	naphthal	0	99999999	0	7.45

Figure 5.19. PRINT PROBLEM/SHORT;

The NETFLOW Procedure

_N_	_id_	_type_	_rhs_	_NAME_	_OBJFN_	_UPPERBD
1	heating_o_conv	EQ	0	a_light	-175	110
2	heating_o_conv	EQ	0	a_heavy	-165	165
3	heating_o_conv	EQ	0	brega	-205	80
4	heating_o_conv	EQ	0	heatingo	0	99999999
5	naphtha_i_conv	EQ	0	a_light	-175	110
6	naphtha_i_conv	EQ	0	a_heavy	-165	165
7	naphtha_i_conv	EQ	0	brega	-205	80
8	naphtha_i_conv	EQ	0	naphthai	0	99999999
9	naphtha_l_conv	EQ	0	a_light	-175	110
10	naphtha_l_conv	EQ	0	a_heavy	-165	165
11	naphtha_l_conv	EQ	0	brega	-205	80
12	naphtha_l_conv	EQ	0	naphthal	0	99999999
13	recipe_1	EQ	0	naphthai	0	99999999
14	recipe_1	EQ	0	heatingo	0	99999999
15	recipe_1	EQ	0	jet_1	300	99999999
16	recipe_2	EQ	0	naphthal	0	99999999
17	recipe_2	EQ	0	heatingo	0	99999999
18	recipe_2	EQ	0	jet_2	300	99999999

_N_	_LOWERBD	_VALUE_	_COEF_
1	0	110	0.39
2	0	0	0.3
3	0	80	0.43
4	0	77.3	-1
5	0	110	0.1
6	0	0	0.075
7	0	80	0.135
8	0	21.8	-1
9	0	110	0.035
10	0	0	0.03
11	0	80	0.045
12	0	7.45	-1
13	0	21.8	0.3
14	0	77.3	0.7
15	0	60.65	-1
16	0	7.45	0.2
17	0	77.3	0.8
18	0	63.33	-1

Figure 5.20. PRINT PROBLEM/SHORT; (continued)

The NETFLOW Procedure

_N_	_NAME_	_OBJFN_	_UPPERBD	_LOWERBD	_VALUE_
1	jet_1	300	99999999	0	60.65
2	jet_2	300	99999999	0	63.33

Figure 5.21. PRINT SOME_VARIABLES(J:)/SHORT;

The NETFLOW Procedure

_N_	_id_	_type_	_rhs_	_NAME_	_OBJFN_	_UPPERBD
1	recipe_1	EQ	0	naphthai	0	99999999
2	recipe_1	EQ	0	heatingo	0	99999999
3	recipe_1	EQ	0	jet_1	300	99999999

_N_	_LOWERBD	_VALUE_	_COEF_
1	0	21.8	0.3
2	0	77.3	0.7
3	0	60.65	-1

Figure 5.22. PRINT SOME_CONS(RECIPE_1)/SHORT;

The NETFLOW Procedure

_N_	_id_	_type_	_rhs_	_NAME_	_OBJFN_	_UPPERBD
1	heating_o_conv	EQ	0	brega	-205	80
2	naphtha_i_conv	EQ	0	brega	-205	80
3	naphtha_l_conv	EQ	0	brega	-205	80

_N_	_LOWERBD	_VALUE_	_COEF_
1	0	80	0.43
2	0	80	0.135
3	0	80	0.045

Figure 5.23. PRINT CON_VARIABLES(_ALL_,BREGA)/SHORT;

The NETFLOW Procedure

_N_	_id_	_type_	_rhs_	_NAME_	_OBJFN_	_UPPERBD
1	recipe_1	EQ	0	naphthai	0	99999999
2	recipe_1	EQ	0	jet_1	300	99999999
3	recipe_2	EQ	0	naphthal	0	99999999

_N_	_LOWERBD	_VALUE_	_COEF_
1	0	21.8	0.3
2	0	60.65	-1
3	0	7.45	0.2

Figure 5.24. PRINT CON_VARIABLES(RECIPE:,N: JET_1)/SHORT;

Unlike PROC LP, which displays the solution and other information as output, PROC NETFLOW saves the optimum in output SAS data sets you specify. For this example, the solution is saved in the SOLUTN1 data set. It can be displayed with PROC PRINT as

```
proc print data=solutn1;
   var _name_ _objfn_ _upperbd _lowerbd _value_ _fcost_;
   sum _fcost_;
   title3 'LP Optimum';
   run;
```

Notice, in the CONOUT=SOLUTN1 (Figure 5.25), the optimal value through each variable in the linear program is given in the variable named _VALUE_, and the cost of value for each variable is given in the variable _FCOST_.

LP Optimum

Obs	_NAME_	_OBJFN_	_UPPERBD	_LOWERBD	_VALUE_	_FCOST_
1	a_heavy	-165	165	0	0.00	0
2	a_light	-175	110	0	110.00	-19250
3	brega	-205	80	0	80.00	-16400
4	heatingo	0	99999999	0	77.30	0
5	jet_1	300	99999999	0	60.65	18195
6	jet_2	300	99999999	0	63.33	18999
7	naphthai	0	99999999	0	21.80	0
8	naphthal	0	99999999	0	7.45	0
						=======
						1544

Figure 5.25. CONOUT=SOLUTN1

The same model can be specified in the sparse format as in the following scon2 dataset. This format enables you to omit the zero coefficients.

```
data scon2;
   format _type_ $8. _col_ $8. _row_ $16. ;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
max       .          profit              .
eq        .          napha_l_conv        .
eq        .          napha_i_conv        .
eq        .          heating_oil_conv    .
eq        .          recipe_1            .
eq        .          recipe_2            .
upperbd   .          available           .
.         a_light    profit           -175
.         a_light    napha_l_conv     .035
.         a_light    napha_i_conv     .100
.         a_light    heating_oil_conv .390
.         a_light    available         110
.         a_heavy    profit           -165
.         a_heavy    napha_l_conv     .030
.         a_heavy    napha_i_conv     .075
.         a_heavy    heating_oil_conv .300
.         a_heavy    available         165
.         brega      profit           -205
.         brega      napha_l_conv     .045
.         brega      napha_i_conv     .135
.         brega      heating_oil_conv .430
.         brega      available          80
.         naphthal   napha_l_conv       -1
.         naphthal   recipe_2           .2
.         naphthai   napha_i_conv       -1
.         naphthai   recipe_1           .3
.         heatingo   heating_oil_conv   -1
.         heatingo   recipe_1           .7
```

```
.         heatingo      recipe_2                .8
.         jet_1         profit                300
.         jet_1         recipe_1               -1
.         jet_2         profit                300
.         jet_2         recipe_2               -1
;
```

To find the minimum cost solution, invoke PROC NETFLOW (note the SPARSECONDATA option which must be specified) as follows:

```
proc netflow
   sparsecondata
   condata=scon2
   conout=solutn2;
   run;
```

A data set that is used as an ARCDATA= data set can be initialized as follows:

```
data vars3;
   input _name_ $ profit available;
   datalines;
a_heavy  -165 165
a_light  -175 110
brega    -205  80
heatingo    0   .
jet_1     300   .
jet_2     300   .
naphthai    0   .
naphthal    0   .
;
```

The following CONDATA= data set is the original dense format CONDATA= dcon1 data set with the variable information removed. (You could have left some or all of that information in CONDATA as PROC NETFLOW "merges" data, but doing that and checking for consistency uses time.)

```
data dcon3;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2
         _type_ $ _rhs_;
   datalines;
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0  eq     0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0  eq     0
heating_o_conv    .390 .300 .430  0  0 -1   0   0  eq     0
recipe_1             0    0    0  0 .3 .7  -1   0  eq     0
recipe_2             0    0    0 .2  0 .8   0  -1  eq     0
;
```

It is important to note that it is now necessary to specify the MAXIMIZE option; otherwise, PROC NETFLOW will optimize to the minimum (which, incidently, has

a total objective = -3539.25). You must indicate that the SAS variable profit in the ARCDATA=vars3 data set has values that are objective function coefficients, by specifying the OBJFN statement. The UPPERBD must be specified as the SAS variable available that has as values upper bounds.

```
proc netflow
     maximize          /* ***** necessary ***** */
     arcdata=vars3
     condata=dcon3
     conout=solutn3;
   objfn profit;
   upperbd available;
   run;
```

The ARCDATA=vars3 data set can become more concise by noting that the model variables heatingo, naphthai, and naphthal have zero objective function coefficients (the default) and default upper bounds, so those observations need not be present.

```
data vars4;
   input _name_ $ profit available;
   datalines;
a_heavy  -165 165
a_light  -175 110
brega    -205  80
jet_1     300   .
jet_2     300   .
;
```

The CONDATA=dcon3 data set can become more concise by noting that all the constraints have the same type (eq) and zero (the default) rhs values. This model is a good candidate for using the DEFCONTYPE= option.

The DEFCONTYPE= option can be useful not only when *all* constraints have the same type as is the case here, but also when *most* constraints have the same type, or if you prefer to change the default type from $\leq$ to = or $\geq$. The essential constraint type data in CONDATA= data set is that which overrides the DEFCONTYPE= type you specified.

```
data dcon4;
   input _id_ $17.
         a_light a_heavy brega naphthal naphthai
         heatingo jet_1 jet_2;
   datalines;
naphtha_l_conv    .035 .030 .045 -1  0  0   0   0
naphtha_i_conv    .100 .075 .135  0 -1  0   0   0
heating_o_conv    .390 .300 .430  0  0 -1   0   0
recipe_1             0    0    0  0 .3 .7  -1   0
recipe_2             0    0    0 .2  0 .8   0  -1
;
```

```
proc netflow
     maximize defcontype=eq
     arcdata=vars3
     condata=dcon3
     conout=solutn3;
   objfn profit;
   upperbd available;
   run;
```

Several different ways of using an ARCDATA= data set and a sparse format CONDATA= data set for this linear program follow. The following CONDATA= data set is the result of removing the profit and available data from the original sparse format CONDATA=scon2 data set.

```
data scon5;
   format _type_ $8. _col_ $8. _row_ $16. ;
   input _type_ $ _col_ $ _row_ $ _coef_;
   datalines;
eq       .            napha_l_conv             .
eq       .            napha_i_conv             .
eq       .            heating_oil_conv         .
eq       .            recipe_1                 .
eq       .            recipe_2                 .
.        a_light      napha_l_conv           .035
.        a_light      napha_i_conv           .100
.        a_light      heating_oil_conv       .390
.        a_heavy      napha_l_conv           .030
.        a_heavy      napha_i_conv           .075
.        a_heavy      heating_oil_conv       .300
.        brega        napha_l_conv           .045
.        brega        napha_i_conv           .135
.        brega        heating_oil_conv       .430
.        naphthal     napha_l_conv             -1
.        naphthal     recipe_2                 .2
.        naphthai     napha_i_conv             -1
.        naphthai     recipe_1                 .3
.        heatingo     heating_oil_conv         -1
.        heatingo     recipe_1                 .7
.        heatingo     recipe_2                 .8
.        jet_1        recipe_1                 -1
.        jet_2        recipe_2                 -1
;

proc netflow
     maximize
     sparsecondata
     arcdata=vars3    /* or arcdata=vars4 */
     condata=scon5
     conout=solutn5;
   objfn profit;
   upperbd available;
   run;
```

The CONDATA=scon5 data set can become more concise by noting that all the constraints have the same type (eq) and zero (the default) rhs values. Use the DEFCONTYPE= option again. Once the first 5 observations of the CONDATA=scon5 data set are removed, the _type_ SAS variable has values that are missing in the remaining observations. Therefore, this SAS variable can be removed.

```
data scon6;
   input _col_ $ _row_&$16. _coef_;
   datalines;
a_light  napha_l_conv          .035
a_light  napha_i_conv          .100
a_light  heating_oil_conv      .390
a_heavy  napha_l_conv          .030
a_heavy  napha_i_conv          .075
a_heavy  heating_oil_conv      .300
brega    napha_l_conv          .045
brega    napha_i_conv          .135
brega    heating_oil_conv      .430
naphthal napha_l_conv           -1
naphthal recipe_2                .2
naphthai napha_i_conv           -1
naphthai recipe_1                .3
heatingo heating_oil_conv       -1
heatingo recipe_1                .7
heatingo recipe_2                .8
jet_1    recipe_1               -1
jet_2    recipe_2               -1
;

proc netflow
     maximize
     defcontype=eq
     sparsecondata
     arcdata=vars3     /* or arcdata=vars4 */
     condata=scon6
     conout=solutn6;
   objfn profit;
   upperbd available;
   run;
```

Interactivity: Linear Programming Models: Interior Point algorithm

PROC NETFLOW can be used interactively. You begin by giving the PROC NETFLOW statement, and you must specify the CONDATA= data set. If necessary, specify the ARCDATA= data set.

The variable lists should be given next. If you have variables in the input data sets that have special names (for example, a variable in the ARCDATA= data set named _COST_ that has objective function coefficients as values), it may not be necessary to have many or any variable lists.

The PRINT, QUIT, SAVE, SHOW, RESET, and RUN statements follow and can be listed in any order. The QUIT statements can be used only once. The others can be used as many times as needed.

The CONOPT and PIVOT are not relevant to the interior point algorithm and should not be used.

Use the RESET or SAVE statement to change the name of the output data set. There is only one output data set, the CONOUT= data set. With the RESET statement, you can also indicate the reasons why optimization should stop, (for example, you can indicate the maximum number of iterations that can be performed). PROC NETFLOW then has a chance to either execute the next statement or, if the next statement is one that PROC NETFLOW does not recognize (the next PROC or DATA step in the SAS session), do any allowed optimization and finish. If no new statement has been submitted, you are prompted for one. Some options of the RESET statement enable you to control aspects of the interior point algorithm. Specifying certain values for these options can reduce the time it takes to solve a problem. Note that any of the RESET options can be specified in the PROC NETFLOW statement.

The RUN statement starts optimization. Once the optimization has started, it runs until the optimum is reached. The RUN statement should be specified at most once.

The QUIT statement immediately stops PROC NETFLOW. The SAVE statement has options that enable you to name the output data set; information about the current solution is saved in this output data set. Use the SHOW statement if you want to examine the values of options of other statements. Information about the amount of optimization that has been done and the STATUS of the current solution can also be displayed using the SHOW statement.

The PRINT statement instructs PROC NETFLOW to display parts of the problem. The ways the PRINT statements are specified are identical whether the interior point algorithm or the simplex algorithm is used; however, there are minor differences in what is displayed for each variable or constraint coefficient.

PRINT VARIABLES produces information on all arcs. PRINT SOME_VARIABLES limits this output to a subset of variables. There are similar PRINT statements for constraints:

```
PRINT CONSTRAINTS;
PRINT SOME_CONS;
```

PRINT CON_VARIABLES enables you to limit constraint information that is obtained to members of a set of variables that have nonzero constraint coefficients in a set of constraints.

For example, an interactive PROC NETFLOW run might look something like this:

```
proc netflow
       condata=data set
       other options;
   variable list specifications;     /* if necessary */
   reset options;
```

```
   print options;     /* look at problem              */
run;                  /* do some optimization         */
   print options;     /* look at the optimal solution */
   save options;      /* keep optimal solution        */
```

If you are interested only in finding the optimal solution, have used SAS variables that have special names in the input data sets, and want to use default setting for everything, then the following statement is all you need:

proc netflow condata= *data set* **;**

Functional Summary: Linear Programming Models: Interior Point Algorithm

The following table outlines the options available for the NETFLOW procedure when the interior point algorithm is being used to solve a linear programming problem, classified by function.

Table 5.8. Functional Summary, Linear Programming Models

Description	Statement	Option
Input Data Set Options:		
arcs input data set	PROC NETFLOW	ARCDATA=
constraint input data set	PROC NETFLOW	CONDATA=
Output Data Set Option:		
solution data set	PROC NETFLOW	CONOUT=
Data Set Read Options:		
CONDATA has sparse data format	PROC NETFLOW	SPARSECONDATA
default constraint type	PROC NETFLOW	DEFCONTYPE=
special COLUMN variable value	PROC NETFLOW	TYPEOBS=
special COLUMN variable value	PROC NETFLOW	RHSOBS=
data for a constraint found once in CONDATA	PROC NETFLOW	CON_SINGLE_OBS
data for a coefficient found once in CONDATA	PROC NETFLOW	NON_REPLIC=
data is grouped, exploited during data read	PROC NETFLOW	GROUPED=
Problem Size Specification Options:		
approximate number of variables	PROC NETFLOW	NNAS=
approximate number of coefficients	PROC NETFLOW	NCOEFS=
approximate number of constraints	PROC NETFLOW	NCONS=
Network Options:		
default variable objective function coefficient	PROC NETFLOW	DEFCOST=
default variable upper bound	PROC NETFLOW	DEFCAPACITY=
default variable lower bound	PROC NETFLOW	DEFMINFLOW=
Memory Control Options:		
issue memory usage messages to SAS log	PROC NETFLOW	MEMREP

Description	Statement	Option
number of bytes to use for main memory	PROC NETFLOW	BYTES=
proportion of memory for arrays	PROC NETFLOW	COREFACTOR=
maximum bytes for a single array	PROC NETFLOW	MAXARRAYBYTES=
Interior Point Algorithm Options:		
use interior point algorithm	PROC NETFLOW	INTPOINT
factorization method	RESET	FACT_METHOD=
allowed amount of dual infeasibility	RESET	TOLDINF=
allowed amount of primal infeasibility	RESET	TOLPINF=
allowed total amount of dual infeasibility	RESET	TOLTOTDINF=
allowed total amount of primal infeasibility	RESET	TOLTOTPINF=
cut-off tolerance for Cholesky factorization	RESET	CHOLTINYTOL=
density threshold for Cholesky processing	RESET	DENSETHR=
step-length multiplier	RESET	PDSTEPMULT=
preprocessing type	RESET	PRSLTYPE=
print optimization progress on SAS log	RESET	PRINTLEVEL2=
write optimization time to SAS log	RESET	OPTIM_TIMER
Interior Point Stopping Criteria Options:		
maximum number of interior point iterations	RESET	MAXITERB=
primal-dual (duality) gap tolerance	RESET	PDGAPTOL=
stop because of complementarity	RESET	STOP_C=
stop because of duality gap	RESET	STOP_DG=
stop because of $infeas_b$	RESET	STOP_IB=
stop because of $infeas_c$	RESET	STOP_IC=
stop because of $infeas_d$	RESET	STOP_ID=
stop because of complementarity	RESET	AND_STOP_C=
stop because of duality gap	RESET	AND_STOP_DG=
stop because of $infeas_b$	RESET	AND_STOP_IB=
stop because of $infeas_c$	RESET	AND_STOP_IC=
stop because of $infeas_d$	RESET	AND_STOP_ID=
stop because of complementarity	RESET	KEEPGOING_C=
stop because of duality gap	RESET	KEEPGOING_DG=
stop because of $infeas_b$	RESET	KEEPGOING_IB=
stop because of $infeas_c$	RESET	KEEPGOING_IC=
stop because of $infeas_d$	RESET	KEEPGOING_ID=
stop because of complementarity	RESET	AND_KEEPGOING_C=
stop because of duality gap	RESET	AND_KEEPGOING_DG=
stop because of $infeas_b$	RESET	AND_KEEPGOING_IB=
stop because of $infeas_c$	RESET	AND_KEEPGOING_IC=
stop because of $infeas_d$	RESET	AND_KEEPGOING_ID=
PRINT Statement Options:		
display everything	PRINT	PROBLEM
display variable information	PRINT	VARIABLES

Description	Statement	Option
display constraint information	PRINT	CONSTRAINTS
display information for some variables	PRINT	SOME_VARIABLES
display information for some constraints	PRINT	SOME_CONS
display information for some constraints associated with some variables	PRINT	CON_VARIABLES
PRINT Statement Qualifiers:		
produce a short report	PRINT	/ SHORT
produce a long report	PRINT	/ LONG
display arcs/variables with zero flow/value	PRINT	/ ZERO
display arcs/variables with nonzero flow/value	PRINT	/ NONZERO
SHOW Statement Options:		
show problem, optimization status	SHOW	STATUS
show LP model parameters	SHOW	NETSTMT
show data sets that have been or will be created	SHOW	DATASETS
Miscellaneous Options:		
infinity value	PROC NETFLOW	INFINITY=
scale constraint row, variable column coefficients, or both	PROC NETFLOW	SCALE=
maximization instead of minimization	PROC NETFLOW	MAXIMIZE

Generalized Networks: NETFLOW Procedure

In this section we introduce how to use the NETFLOW procedure to solve *generalized* network programming problems.

What Is a Generalized Network?

It is well known that in a pure network the sum of flows entering an arc is equal to the sum of flows leaving it. However, in a generalized network there may be a gain or a loss as flow traverses an arc. Each arc has a multiplier to represent these gains or losses.

To illustrate what is meant, consider the network shown in Figure 5.26.

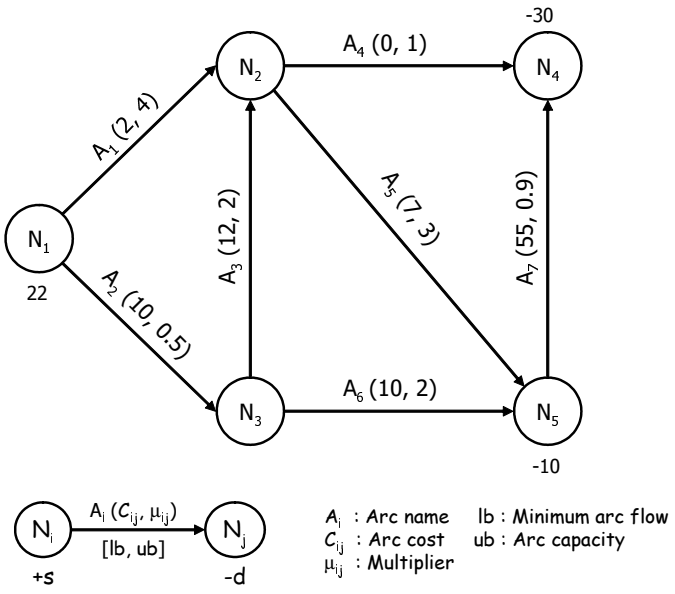


Figure 5.26. Generalized Network Example

You can think of this as a network representing a supply node (N_1), trans-shipment nodes (N_2, N_3), and demand nodes (N_4, N_5). As indicated by the legend, the number below a node represents its supdem value. Above each arc is its name, followed by the arc cost and arc multiplier in parentheses. The lower and upper bounds on flow allowed to **enter** an arc are represented in square brackets below it. When no bounds are specified (as in Figure 5.26), they are assumed to be [0, 99999999].

Now consider the node pair (N_1, N_2). The information on arc A_1 says that it costs 2 per unit of flow to traverse it, and for each unit of flow entering the arc, four units get accumulated at node N_2. The corresponding component in the objective function is two times the flow through arc A_1 leaving node N_1, not two times the flow through arc A_1 arriving at node N_2.

A commonly encountered example of a generalized network is in power generation: as electricity is transmitted over wires, there is some unavoidable loss along the way. This loss is represented by a multiplier less than 1.0.

Arc multipliers need not always be less than 1.0. For instance, in financial models, a flow through an arc could represent money in a bank account earning interest. In that case, the arc would have a multiplier greater than 1.0.

Generalized networks offer convenience when flow commodity changes. For a pure network, care must be taken to ensure the flow commodity is the same throughout the entire model. For example, in a model to determine how sugar should be grown, refined, packaged, and sold, the flow commodity might be kilograms of sugar, and all numerical parameters throughout the model (all supplies, arc costs, capacities, bounds, demands, etc.) must be in terms of kilograms of sugar. Some arcs might correspond to the movement of 5-kilogram bags of sugar. If a generalized network formulation is used, the arc that represents packaging could be given a multiplier of 0.2, so flow through arcs that convey flow corresponding to bags of sugar will have

arc costs in terms of dollars per bag, and capacities, bounds, demands, etc. in terms of number of bags.

In the following sections we describe in detail how to provide data for arc multipliers, how to deal with excess supply or demand in pure and generalized networks, how to model maximum flow problems, and how to handle networks with missing supply and demand nodes, and ranges on supply and demand.

How to Specify Data for Arc Multipliers

If you are familiar with using the NETFLOW procedure to solve pure network problems, then solving generalized network problems is fairly simple. You just need to provide the additional data for the arc multipliers. Arcs by default have a multiplier of 1.0, so you only need to provide arc multipliers that are not equal to 1.0. You can specify the arc multiplier data in either or both of the ARCDATA= and CONDATA= data sets. The procedure scans the SAS variables in the ARCDATA= data set, and if it finds a name _MULT_ (or a similar name with letters of different case), then it assumes that the SAS variable contains data for arc multipliers. CONDATA= is scanned for special type values that indicates data are arc multipliers.

The rest of this section describes the various ways in which you can specify data for the arc multipliers. The network in Figure 5.26 is used for illustration.

All Arc Multiplier Data in the ARCDATA= Data Set

You can specify all the arc multiplier data in the ARCDATA= data set. The following code creates the input SAS data sets:

```
data nodes;
   input _node_ $ _sd_ ;
datalines;
N1  22
N4 -30
N5 -10
;

data arcs;
   input _from_ $ _to_ $  _cost_ _mult_;
datalines;
N1 N2   2   4
N1 N3  10 0.5
N2 N4   0   1
N2 N5   7   3
N3 N2  12   2
N3 N5  10   2
N5 N4  55 0.9
;
```

Let us first look at the data for this problem. There is a variable named _mult_ in the ARCDATA= data set, so PROC NETFLOW assumes it represents the arc multipliers. The SAS variable _sd_ represents the supdem value of a node. A positive or missing S value indicates supply, and a negative or missing D value indicates demand.

The optimal solution can be obtained from the CONOUT= data set. Note that you need to specify the CONOUT= data set even if the network has no side constraints; you cannot use the ARCOUT= data set.

You can use the following SAS code to run PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   bytes    = 100000
   nodedata = nodes
   arcdata  = arcs
   conout   = solution;
run;
```

The optimal solution is displayed in Figure 5.27.

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	Arc capacity or Nonarc upper bound.	Arc (Nonarc) lower flow (value) bound.	_mult_
1	N1	N2	2	99999999	0	4.0
2	N3	N2	12	99999999	0	2.0
3	N1	N3	10	99999999	0	0.5
4	N2	N4	0	99999999	0	1.0
5	N5	N4	55	99999999	0	0.9
6	N2	N5	7	99999999	0	3.0
7	N3	N5	10	99999999	0	2.0

Obs	Supply of tail node.	Demand of head node.	Arc flow or Nonarc value.	Arc flow*cost, Nonarc value*objfn coef.
1	22	.	5.9999	12.000
2	.	.	3.0003	36.004
3	22	.	16.0001	160.001
4	.	30	29.9999	0.000
5	.	30	0.0001	0.004
6	.	10	0.0002	0.001
7	.	10	4.9997	49.997

Figure 5.27. Output of the Example Problem

All Arc Multiplier Data in CONDATA= Data Set

Let us now solve the same problem, but with all the arc multipliers specified in the CONDATA= data set. The CONDATA= data set can have either a sparse format or a dense format. The following code illustrates the dense format representation:

```
data arcs1b;
   input _from_ $ _to_ $  _cost_;
```

```
datalines;
N1 N2   2
N1 N3  10
N2 N4   0
N2 N5   7
N3 N2  12
N3 N5  10
N5 N4  55
;

data MUdense;
   input _type_ $ N1_N2 N1_N3 N2_N4 N2_N5 N3_N2 N3_N5 N5_N4;
datalines;
mult 4.0  0.5  1.0  0.3  2.0  2.0  0.9
;
```

You can use the following SAS code to obtain the solution:

```
proc netflow
   gennet
   nodedata = nodes
   arcdata  = arcs1b
   condata  = MUdense
   conout   = soln1b;
run;
```

Note that a new option, GENNET, has been specified in the call to PROC NETFLOW. This option is necessary when the network is generalized and there are no arc multiplier data in the ARCDATA= data set. If this option is not specified, then the procedure assumes that the network is pure (without arc multipliers) and sets up the excess supply node and the excess arcs.

The sparse format representation is as follows:

```
data MUsparse;
   input _type_ $ _col_ $ _coef_;
datalines;
mult N1_N2 4.0
mult N1_N3 0.5
mult N2_N4 1.0
mult N2_N5 0.3
mult N3_N2 2.0
mult N3_N5 2.0
mult N5_N4 0.9
;
```

You can use the following SAS code to obtain the solution:

```
proc netflow
   gennet   sparsecondata
   nodedata = nodes
   arcdata  = arcs1b
   condata  = MUsparse
   conout   = soln1c;
run;
```

Note that you need to specify the SPARSECONDATA option in the call to PROC NETFLOW.

Arc Multiplier Data in Both ARCDATA= and CONDATA= Data Sets

You can also provide some multiplier data in the ARCDATA= data set, and the rest in the CONDATA= data set as follows:

```
data arcs1c;
   input _from_ $ _to_ $  _cost_ _mult_;
datalines;
N1 N2   2   4
N1 N3  10  .5
N2 N4   0   .
N2 N5   7   .
N3 N2  12   .
N3 N5  10   .
N5 N4  55   .
;

data MUdense1;
   input _type_ $ N2_N4 N2_N5 N3_N2 N3_N5 N5_N4;
datalines;
mult 1.0  0.3  2.0  2.0  0.9
;
```

The procedure merges the data when all the input data sets have been read.

Specifying Arc Multiplier Data Using a List Statement

You can also specify the name of the multiplier variable in the list statement MULT, or MULTIPLIER. For example, if the name of the variable is lossrate, then use the following:

```
proc netflow
...
;
mult lossrate;
run;
```

You may also use MULT, GAIN, or LOSS (or similar names with letters of different case) as a value of the TYPE list SAS variable.

Using the New EXCESS= Option in Pure Networks: NETFLOW Procedure

In this section we describe how to use the new EXCESS= option to solve a wide variety of problems. These include the following:

- networks with excess supply or demand
- networks containing nodes with unknown supply and demand values
- maximum flow problems
- networks with nodes having supply and demand ranges

Handling Excess Supply or Demand

The supdem value of a node can be specified in the following formats:

- in the NODEDATA= data set, using the _supdem_ or _sd_ list variable
- in the ARCDATA= data set, using the _SUPPLY_ and _DEMAND_ list variables

If there is only one supply (demand) node, then use the SOURCE= (SINK=) option to refer to it, and use the SUPPLY= (DEMAND=) option to specify its supdem value.

To ensure feasibility, there are different methods by which flow can be added to or drained from the network. This extra flow can be added to or drained from the network at either the supply or demand nodes. The new EXCESS= option is used to address such instances.

For pure networks there are two valid values that can be specified for the EXCESS= option: EXCESS=ARCS and EXCESS=SLACKS.

EXCESS=ARCS is the default value. An extra node, referred to as _EXCESS_, is added to the network and is connected to the actual network by "excess" arcs.

- If total supply exceeds total demand, then _EXCESS_ is an extra demand node with demand equal to total supply minus total demand.
  - If the THRUNET option is specified, the "excess" arcs are directed away from any actual demand node (even nodes with missing D demand) and toward _EXCESS_.
  - Or else if there are demand nodes with missing D demands, the "excess" arcs are directed away from these nodes and toward _EXCESS_.
  - Or else the "excess" arcs are directed away from the supply nodes and toward _EXCESS_.
- If the total demand exceeds the total supply, then _EXCESS_ is an extra supply node with supply equal to the total demand minus the total supply.

- If the THRUNET option is specified, the "excess" arcs are directed away from _EXCESS_ and toward any actual supply node (even nodes with missing S supply.)
- Or else if there are supply nodes with missing S supplies, the "excess" arcs are directed away from _EXCESS_ and toward these nodes.
- Or else the "excess" arcs are directed away from _EXCESS_ and toward the demand nodes.

The node _EXCESS_ and the associated arcs are created to ensure that the problem presented to the optimizer has a total supply equal to the total demand. They are neither displayed in the optimal solution nor saved in any output SAS data set.

If EXCESS=SLACKS is specified, then slack variables are created for some flow conservation constraints instead of having the node _EXCESS_ and "excess" arcs. The flow conservation constraint (which was an inequality) is now converted to an equality with the addition of the slack variable. Alternatively, you can think of these slacks as arcs with one of their end nodes missing — they are directed from a node but not toward any other node (or directed toward a node but not from any other node). Figure 5.28 presents a clearer picture of this.

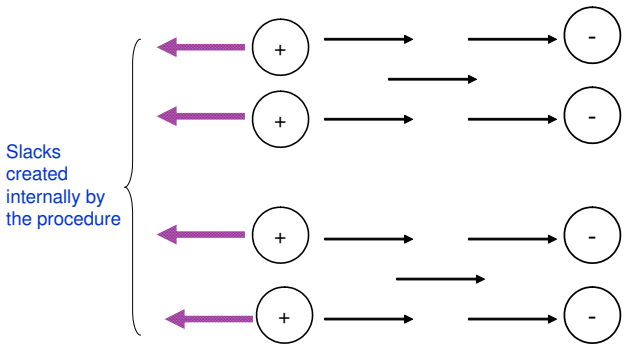


Figure 5.28. EXCESS=SLACKS, Total Supply Exceeds Total Demand, THRUNET Not Specified, No Nodes with Missing Demand

Note: When you specify EXCESS=SLACKS, the interior point solver is used. The output SAS data set needs to be specified by the CONOUT= data set, even if side constraints are not present. Also, when you specify the EXCESS=SLACKS option, the size of the model submitted to the optimizer is smaller than with EXCESS=ARCS since it no longer has the _EXCESS_ node and the excess arcs associated with it.

Handling Missing Supply and Demand Simultaneously

Another new feature in the NETFLOW procedure is that it enables you to specify a network containing both nodes with missing S supply values and nodes with missing D demand values. This feature is a powerful modeling tool, and we show in the later sections how to use it to formulate and solve maximum flow problems and network models with range constraints on supply and demand.

Whenever a network is detected to have both nodes with missing S supply values and nodes with missing D demand values, a special value of the EXCESS= option is assigned internally by the procedure; any value you specify for the EXCESS= option is overridden. The procedure solves the problem in the following manner:

- Nodes with positive (negative) supdem values supply (demand) the exact amount of flow specified.
- Nodes with missing S supply (missing D demand) values supply (demand) flow quantities that are determined by optimization.

Figure 5.29 displays how the slack variables are set up by the procedure internally. These variables are neither a part of the input data set nor displayed in any output SAS data set or printed output.

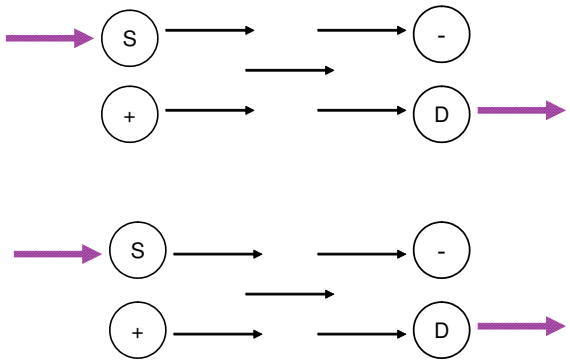


Figure 5.29. A Network with Both Missing S Supply and Missing D Demand Nodes

Maximum Flow Problems

The maximum flow problem (MFP) can be stated as follows: Given a directed graph $G = (N, A)$ with capacity $u_{ij} \geq 0$ on each arc $(i, j) \in A$, a source node s and a sink node t, find the maximum flow that can go from s to t, while obeying the flow conservation constraints at each node. You can solve such problems using the MAXFLOW option in the call to PROC NETFLOW.

Ordinarily many, if not all, arcs in an MFP network have capacities, and it is common that these arcs have zero costs. However, the NETFLOW procedure enables you to have nonzero costs to influence the optimal solution in cases where multiple maximum flow patterns are known to exist.

The following two subsections explain the role of the EXCESS= option in solving pure and generalized maximum flow problems.

The EXCESS=ARCS Option

Consider a maximum flow problem involving a pure network. Assume that you do not explicitly specify the EXCESS= option (the EXCESS=ARCS option is used by the procedure by default). The NETFLOW procedure sets up the problem in the following manner:

1. The source node is assigned a SUPDEM value equal to INFINITY−1.
2. The sink node is assigned a SUPDEM value equal to −(INFINITY−1).
3. If there is no existing arc between the source node and the sink node, an arc called the *bypass arc* directed from the source node to the sink node is added.
4. If there is an existing arc between the source node and the sink node, a dummy node is used to break up what would have been a single bypass arc: *source* —> *sink* gets transformed into two arcs, *source* —> *dummy* —> *sink*.
5. If you are maximizing, then the cost of the bypass arc(s) is equal to −1 if all other arcs have zero costs; otherwise the cost of the bypass arc(s) is equal to −(INFINITY / BYPASSDIV).
6. If you are minimizing, then the cost of the bypass arc(s) is equal to 1 if all other arcs have zero costs; otherwise the cost of the bypass arc(s) is equal to INFINITY / BYPASSDIV.

You can specify the value of the INFINITY= option in the procedure statement, or you can use the default value of 99999999. You can also specify the BYPASSDIV= option. The default value for the BYPASSDIV= option is 100.

This scenario is depicted in Figure 5.30. Since the cost of the bypass arc is unattractive, the optimization process minimizes the flow through it, thereby maximizing the flow through the real network. See the first subsection in Example 5.10 for an illustration.

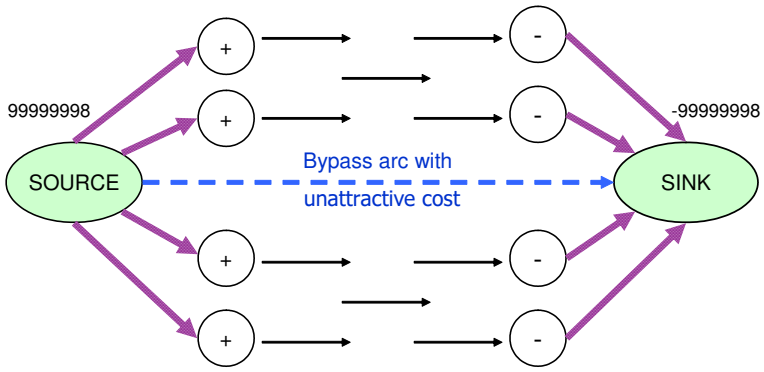


Figure 5.30. Pure Maximum Flow Problem, EXCESS=ARCS Option Specified

This method of setting up a maximum flow problem does come with a drawback. It is likely to produce incorrect results if the following occur:

- the maximum flow is greater than INFINITY−1, or
- the cost of the bypass arc is insufficiently unattractive to ensure that the entire flow traverses the real network and not through the bypass arc

Additionally, numbers of large magnitude can cause problems during optimization, including numerical instability and loss of precision. In the next section, we explain how to overcome these difficulties when solving maximum flow problems.

The EXCESS=SLACKS Option

Assume you have a pure maximum flow problem and you specify the EXCESS=SLACKS option. The NETFLOW procedure sets up the problem in the following manner:

- The source node is assigned a missing S supply value.
- The sink node is assigned a missing D demand value.

Since this network contains a node with a missing S supply value and a node with a missing D demand value, we have a situation similar to the one described in the section "Handling Missing Supply and Demand Simultaneously" on page 596. Both of these nodes have slack variables. Usually, slack variables have zero objective function coefficients, but because the MAXFLOW option is specified, one of the slack variables must be attractive enough to make it worthwhile for flow to traverse the network. Figure 5.31 presents the scenario clearly.

If you are maximizing, then the objective function coefficient of the slack variable associated with the sink node is -1 if all other arcs have zero costs. Otherwise it is $-$(INFINITY / BYPASSDIV). If you are minimizing, then the objective function coefficient of the slack variable associated with the sink node is 1 if all arcs have zero costs. Otherwise it is INFINITY / BYPASSDIV. See the second subsection in Example 5.10 for an illustration of the EXCESS=SLACKS option in pure maximum flow problems.

Note: If the MAXFLOW option is not specified, these slack variables assume zero objective function coefficients, and the MFP may not be solved properly.

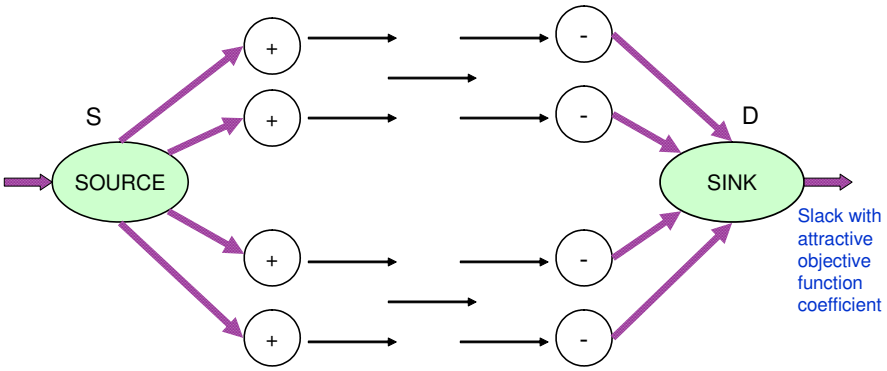


Figure 5.31. Pure Maximum Flow Problem with EXCESS=SLACKS Option Specified

When you use the MAXFLOW option, the procedure sets up the problem in such a way that maximum flow traverses the network. This enables you to transform certain types of problems into maximum flow problems. One such instance is when you have a network where the amount of flow that is supplied or demanded falls within a range of values. The following section describes how to solve such problems.

Handling Supply and Demand Ranges

Consider the scenario depicted by Figure 5.32, where the supply and demand nodes have ranges; i.e., the amounts they can supply or demand are constrained to be within certain lower and upper bounds.

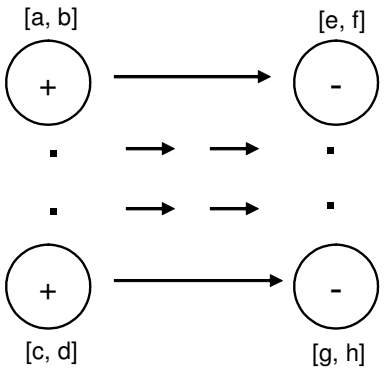


Figure 5.32. Network with Ranges on Supplies and Demands

To model this situation, you first need to add a supply node with missing S supply value (the Y node in Figure 5.33) and a demand node with missing D demand value (the Z node in Figure 5.33). The bounds on the supply and demand nodes get transformed into upper/lower bounds on the arcs that connect them to nodes Y and Z, respectively. It might be necessary to have costs for these arcs to make it worthwhile for flow to traverse them, and subsequently to traverse the actual network. In practice, these costs represent procurement costs, profit from sales, etc.

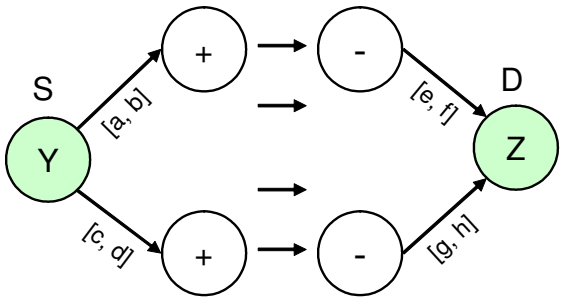


Figure 5.33. Network with Ranges of Supplies and Demands: Transformed Model

You could set up all your network models in this fashion, not only in scenarios in which there are supply and demand ranges. For instance, this modeling technique could be used under the following conditions:

- if there are no ranges
- if the network is generalized, and you do not know whether to specify EXCESS=SUPPLY or EXCESS=DEMAND

- if some of the lower bounds are zero or some capacities are infinite, in which case you simply do not specify the capacity

Using the New EXCESS= Option in Generalized Networks: NETFLOW Procedure

In this section we briefly describe how to use the new EXCESS= option in generalized networks. We provide simple scenarios to enable you to understand what happens internally in the solver when different values for the EXCESS= option are specified.

Total Supply and Total Demand: How Generalized Networks Differ from Pure Networks

For a pure network, it is easy to check for excess supply or excess demand. If the sum of positive supdem values exceeds (is less than) the absolute value of the sum of negative supdem values, then the network has excess supply (demand).

However, in a generalized network you need to specify whether the network should have excess supply or excess demand. To do that you can specify the option EXCESS=SUPPLY or EXCESS=DEMAND, respectively.

Although the total supply and total demand of a generalized network can be determined, you may not know beforehand if excess flow must be added to, removed from, or left unused by the network. For example, consider a simple network, one consisting of two nodes, A and B, connected by a single arc, A —> B. Suppose the supply of node A is 10 and the demand of node B is 30. If this is a pure network, then the network solved must be either _EXCESS_ —> A —> B if the THRUNET option is not specified and the flow through the arc between A and B is 30 units, or A —> B <— _EXCESS_ if the THRUNET option is specified and the flow through the arc from A to B is 10 units. _EXCESS_ is the name of an extra node that is set up by the procedure behind the scenes, and in both cases it would have a supply capacity of 20 units, which is the flow through the excess arc. However, if the network is generalized, and the arc from A to B has a multiplier of 3.0, then the flow through the arc from A to B would be 10 units, and the network would be feasible without any excess node and arcs. Indeed, no excess node and arcs would be created, even though total supply and total demand are unequal. Therefore, once the NETFLOW procedure detects that the network has arc multipliers that are not 1.0, it might not set up the excess node and the excess arcs.

In Example 5.11 we illustrate the use of the EXCESS= option to solve generalized networks that have total supply equal to total demand, but have arcs with varying multipliers.

In the section "Handling Missing Supply and Demand Simultaneously" on page 596, we discuss the case where a network has both nodes with missing S supply values and nodes with missing D demand values. In the next two subsections we analyze scenarios where a network has nodes with either missing S supply values or missing D demand values, but not both.

The EXCESS=SUPPLY Option

If you specify the EXCESS=SUPPLY option, then there are three possible scenarios to deal with:

Case 1: No Node with Missing D Demand, THRUNET Not Specified (see Figure 5.34)

Drain the excess supply from all supply nodes.

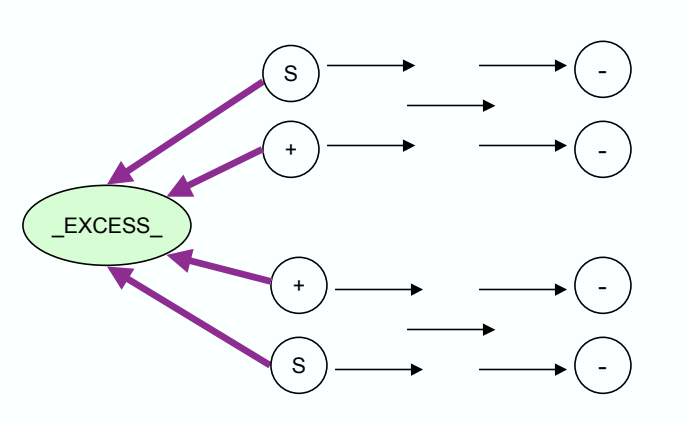


Figure 5.34. Nodes with Missing S Supply, THRUNET Specified

Case 2: Some Nodes with Missing D Demand, THRUNET Not Specified (see Figure 5.35)

Drain the excess supply from nodes that have missing D demand values. If a node has a missing D demand value, then the amount it demands is determined by optimization. For a demand node with negative supdem value, that value negated is equal to the sum of flows on all actual arcs directed toward that node.

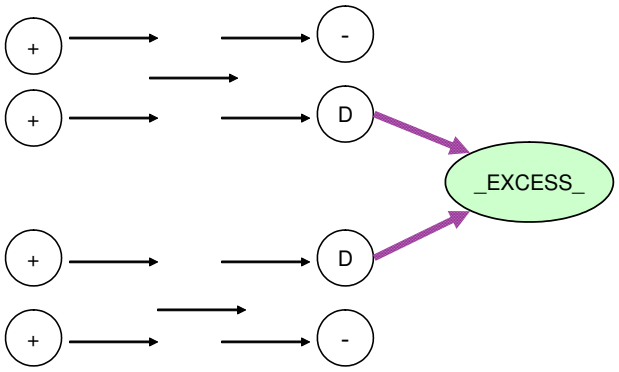


Figure 5.35. Nodes with Missing D Demand

Case 3: THRUNET Specified (see Figure 5.36)

Drain the excess supply from all demand nodes. If a node has a negative supdem value, that value negated is the lower bound on the sum of flows on

all actual arcs directed toward that node. If a node has a missing D demand value, then the amount it demands is determined by optimization.

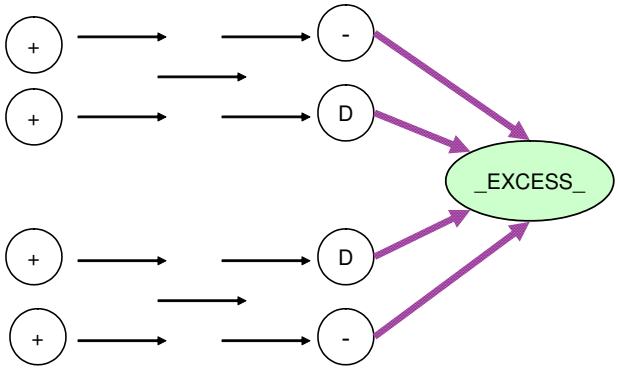


Figure 5.36. Nodes with Missing D Demand, THRUNET Specified

The EXCESS=DEMAND Option

If you specify the EXCESS=DEMAND option, then there are three possible scenarios to deal with:

Case 1: No Node with Missing S Supply, THRUNET Not Specified (see Figure 5.37)

Supply the excess demand to all demand nodes directly.

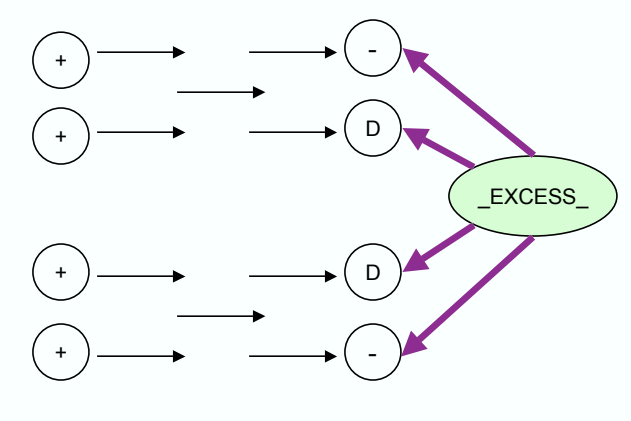


Figure 5.37. Nodes with Missing D Demand

Case 2: Some Nodes with Missing S Supply, THRUNET Not Specified (see Figure 5.38)

Supply the excess demand by the nodes that have a missing S supply value. If a node has a missing S supply value, then the amount it supplies is determined by optimization. For a supply node with a positive supdem value, that value is equal to the sum of flows on all actual arcs directed away from that node.

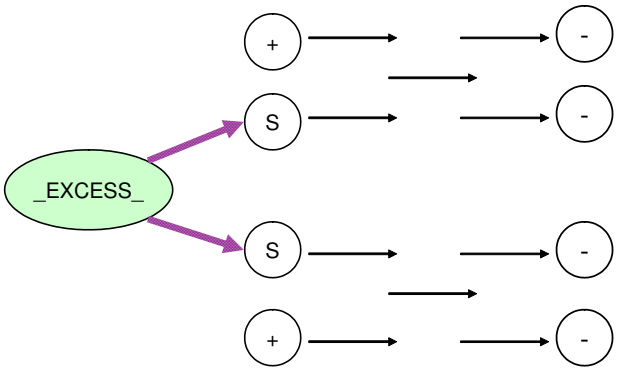


Figure 5.38. Nodes with Missing S Supply

Case 3: THRUNET Specified (see Figure 5.39)

Supply the excess demand by all supply nodes. If a node has a positive supdem value, that value is the lower bound on the sum of flows on all actual arcs directed away from that node. If a node has a missing S supply value, then the amount it supplies is determined by optimization.

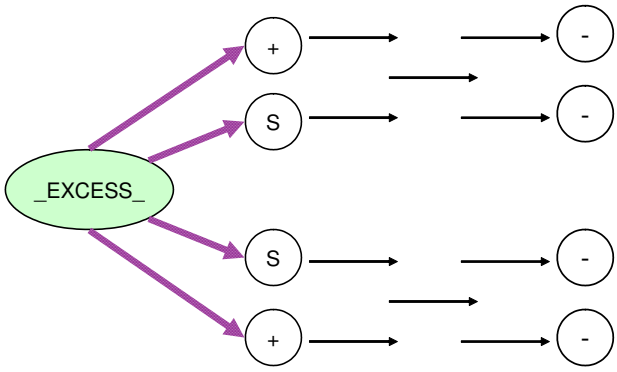


Figure 5.39. Nodes with Missing S Supply, THRUNET Specified

Examples: NETFLOW Procedure

The following examples illustrate some of the capabilities of PROC NETFLOW. These examples, together with the other SAS/OR examples, can be found in the SAS sample library.

Example 5.1. Shortest Path Problem

Whole pineapples are served in a restaurant in London. To ensure freshness, the pineapples are purchased in Hawaii and air freighted from Honolulu to Heathrow in London. The network diagram in Figure 5.40 outlines the different routes that the pineapples could take.

The cost to freight a pineapple is known for each arc. You can use PROC NETFLOW to determine what routes should be used to minimize total shipping cost. The shortest path is the least cost path that all pineapples should use. The SHORTPATH option indicates this type of network problem.

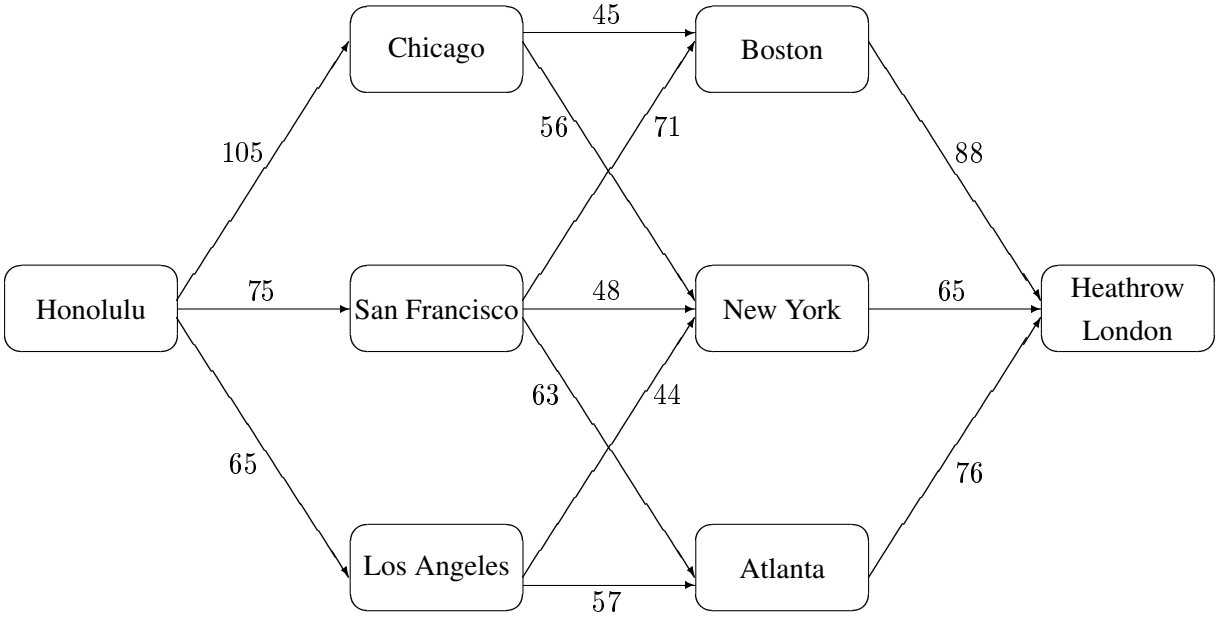


Figure 5.40. Pineapple Routes: Shortest Path Problem

The SINK= option value HEATHROW LONDON is not a valid SAS variable name so it must be enclosed in single quotes. The TAILNODE list variable is FFROM. Because the name of this variable is not _TAIL_ or _FROM_, the TAILNODE list must be specified in the PROC NETFLOW statement. The HEADNODE list must also be explicitly specified because the variable that belongs to this list does not have the name _HEAD_ or _TO_, but is TTO.

```
title 'Shortest Path Problem';
title2 'How to get Hawaiian Pineapples to a London Restaurant';
data aircost1;
   input   ffrom&$13. tto&$15. _cost_ ;
   datalines;
Honolulu       Chicago          105
Honolulu       San Francisco     75
Honolulu       Los Angeles       68
Chicago        Boston            45
Chicago        New York          56
San Francisco  Boston            71
San Francisco  New York          48
San Francisco  Atlanta           63
Los Angeles    New York          44
Los Angeles    Atlanta           57
Boston         Heathrow London   88
New York       Heathrow London   65
Atlanta        Heathrow London   76
;

proc netflow
   shortpath
   sourcenode=Honolulu
   sinknode='Heathrow London'   /* Quotes for embedded blank */
   ARCDATA=aircost1
   arcout=spath;
   tail    ffrom;
   head    tto;
   run;

proc print data=spath;
   sum _fcost_;
   run;
```

The length at optimality is written to the SAS log as

```
NOTE: Number of nodes= 8 .
NOTE: Number of arcs= 13 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 5 .
NOTE: Of these, 4 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Shortest path= 177 .
NOTE: The data set WORK.SPATH has 13 observations and 13
      variables.
```

The output data set ARCOUT=SPATH in Output 5.1.1 shows that the best route for the pineapples is from Honolulu to Los Angeles to New York to Heathrow London.

Output 5.1.1. ARCOUT=SPATH

Shortest Path Problem
How to get Hawaiian Pineapples to a London Restaurant

Obs	ffrom	tto	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_
1	San Francisco	Atlanta	63	99999999	0	.	.
2	Los Angeles	Atlanta	57	99999999	0	.	.
3	Chicago	Boston	45	99999999	0	.	.
4	San Francisco	Boston	71	99999999	0	.	.
5	Honolulu	Chicago	105	99999999	0	1	.
6	Boston	Heathrow London	88	99999999	0	.	1
7	New York	Heathrow London	65	99999999	0	.	1
8	Atlanta	Heathrow London	76	99999999	0	.	1
9	Honolulu	Los Angeles	68	99999999	0	1	.
10	Chicago	New York	56	99999999	0	.	.
11	San Francisco	New York	48	99999999	0	.	.
12	Los Angeles	New York	44	99999999	0	.	.
13	Honolulu	San Francisco	75	99999999	0	1	.

Obs	_FLOW_	_FCOST_	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_
1	0	0	13	9	3	LOWERBD NONBASIC
2	0	0	.	10	4	KEY_ARC BASIC
3	0	0	4	4	2	LOWERBD NONBASIC
4	0	0	.	5	3	KEY_ARC BASIC
5	0	0	.	1	1	KEY_ARC BASIC
6	0	0	57	11	5	LOWERBD NONBASIC
7	1	65	.	12	6	KEY_ARC BASIC
8	0	0	24	13	7	LOWERBD NONBASIC
9	1	68	.	3	1	KEY_ARC BASIC
10	0	0	49	6	2	LOWERBD NONBASIC
11	0	0	11	7	3	LOWERBD NONBASIC
12	1	44	.	8	4	KEY_ARC BASIC
13	0	0	.	2	1	KEY_ARC BASIC
		=======				
		177				

Example 5.2. Minimum Cost Flow Problem

You can continue to use the pineapple example in Example 5.1 by supposing that the airlines now stipulate that no more than 350 pineapples per week can be handled in any single leg of the journey. The restaurant uses 500 pineapples each week. How many pineapples should take each route between Hawaii and London?

You will probably have more minimum cost flow problems because they are more general than maximal flow and shortest path problems. A shortest path formulation is no longer valid because the sink node does not demand one flow unit.

All arcs have the same capacity of 350 pineapples. Because of this, the DEFCAPACITY= option can be specified in the PROC NETFLOW statement, rather than having a CAPACITY list variable in ARCDATA=aircost1. You can have a CAPACITY list variable, but the value of this variable would be 350 in all observations, so using the DEFCAPACITY= option is more convenient. You would have to use the CAPACITY list variable if arcs had differing capacities. You can use both the DEFCAPACITY= option and a CAPACITY list variable.

There is only one supply node and one demand node. These can be named in the SOURCE= and SINK= options. DEMAND=500 is specified for the restaurant demand. There is no need to specify SUPPLY=500, as this is assumed.

```
title 'Minimum Cost Flow Problem';
title2 'How to get Hawaiian Pineapples to a London Restaurant';
proc netflow
   defcapacity=350
   sourcenode='Honolulu'
   sinknode='Heathrow London'  /* Quotes for embedded blank */
   demand=500
      arcdata=aircost1
      arcout=arcout1
      nodeout=nodeout1;
         tail    ffrom;
         head    tto;
   set future1;
proc print data=arcout1; sum _fcost_;
proc print data=nodeout1;
run;
```

The following notes appear on the SAS log:

```
NOTE: SOURCENODE was assigned supply of the total
      network demand= 500 .
NOTE: Number of nodes= 8 .
NOTE: Number of supply nodes= 1 .
NOTE: Number of demand nodes= 1 .
NOTE: Total supply= 500 , total demand= 500 .
NOTE: Number of arcs= 13 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 6 .
NOTE: Of these, 4 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 93750 .
NOTE: The data set WORK.ARCOUT1 has 13 observations and
      13 variables.
NOTE: The data set WORK.NODEOUT1 has 9 observations and
      10 variables.
```

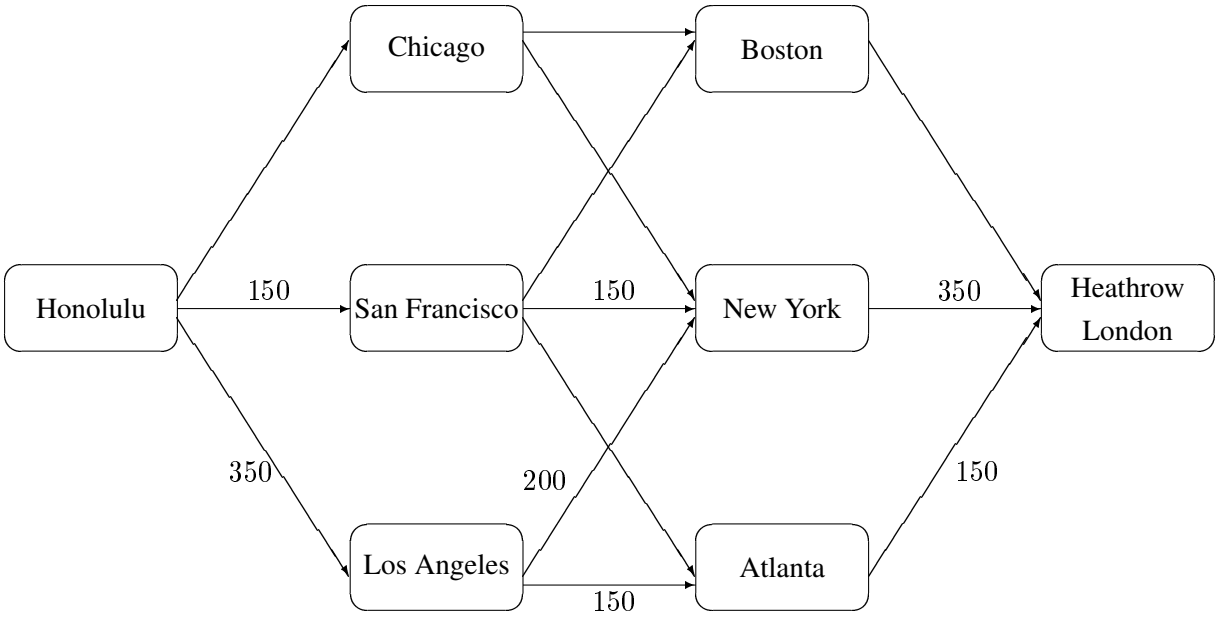


Figure 5.41. Pineapple Routes: Minimum Cost Flow Solution

The routes and numbers of pineapples in each arc can be seen in the output data set ARCOUT=arcout1 in Output 5.2.1. NODEOUT=NODEOUT1 is shown in Output 5.2.2.

Output 5.2.1. ARCOUT=ARCOUT1

Minimum Cost Flow Problem
How to get Hawaiian Pineapples to a London Restaurant

Obs	ffrom	tto	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_
1	San Francisco	Atlanta	63	350	0	.	.
2	Los Angeles	Atlanta	57	350	0	.	.
3	Chicago	Boston	45	350	0	.	.
4	San Francisco	Boston	71	350	0	.	.
5	Honolulu	Chicago	105	350	0	500	.
6	Boston	Heathrow London	88	350	0	.	500
7	New York	Heathrow London	65	350	0	.	500
8	Atlanta	Heathrow London	76	350	0	.	500
9	Honolulu	Los Angeles	68	350	0	500	.
10	Chicago	New York	56	350	0	.	.
11	San Francisco	New York	48	350	0	.	.
12	Los Angeles	New York	44	350	0	.	.
13	Honolulu	San Francisco	75	350	0	500	.

Obs	_FLOW_	_FCOST_	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_
1	0	0	2	9	3	LOWERBD NONBASIC
2	150	8550	.	10	4	KEY_ARC BASIC
3	0	0	4	4	2	LOWERBD NONBASIC
4	0	0	.	5	3	KEY_ARC BASIC
5	0	0	.	1	1	KEY_ARC BASIC
6	0	0	22	11	5	LOWERBD NONBASIC
7	350	22750	-24	12	6	UPPERBD NONBASIC
8	150	11400	.	13	7	KEY_ARC BASIC
9	350	23800	-11	3	1	UPPERBD NONBASIC
10	0	0	38	6	2	LOWERBD NONBASIC
11	150	7200	.	7	3	KEY_ARC BASIC
12	200	8800	.	8	4	KEY_ARC BASIC
13	150	11250	.	2	1	KEY_ARC BASIC
		93750				

Output 5.2.2. NODEOUT=NODEOUT1

Minimum Cost Flow Problem
How to get Hawaiian Pineapples to a London Restaurant

Obs	_NODE_	_SUPPLY_	_DUAL_	_NUMBR_	_PRED_	_TRAVS_	_SCCS_	_ARCIN_	_FLOW_	_FBQ_
1	_ROOT_	0	0	9	0	1	0	-1	81	-14
2	Atlanta	.	-136	7	4	8	2	10	150	9
3	Boston	.	-146	5	3	9	1	5	0	4
4	Chicago	.	-105	2	1	3	1	1	0	1
5	Heathrow London	-500	-212	8	7	5	1	13	150	11
6	Honolulu	500	0	1	9	2	8	-14	0	-1
7	Los Angeles	.	-79	4	6	7	3	-8	200	3
8	New York	.	-123	6	3	4	4	7	150	6
9	San Francisco	.	-75	3	1	6	6	2	150	2

Example 5.3. Using a Warm Start

Suppose that the airlines state that the freight cost per pineapple in flights that leave Chicago has been reduced by 30. How many pineapples should take each route between Hawaii and London? This example illustrates how PROC NETFLOW uses a warm start.

In Example 5.2, the RESET statement of PROC NETFLOW is used to specify FUTURE1. A NODEOUT= data set is also specified. The warm start information is saved in the arcout1 and nodeout1 data sets.

In the following DATA step, the costs, reduced costs, and flows in the arcout1 data set are saved in variables called oldcost, oldflow, and oldfc. These variables form an implicit ID list in the following PROC NETFLOW run and will appear in ARCOUT=arcout2. Thus, it is easy to compare the previous optimum and the new optimum.

```
title 'Minimum Cost Flow Problem - Warm Start';
title2 'How to get Hawaiian Pineapples to a London Restaurant';
data aircost2;
   set arcout1;
      oldcost=_cost_;
      oldflow=_flow_;
      oldfc=_fcost_;
      if ffrom='Chicago' then _cost_=_cost_-30;
proc netflow
   warm
      arcdata=aircost2
      nodedata=nodeout1
      arcout=arcout2;
         tail    ffrom;
         head    tto;
proc print data=arcout2;
```

```
   var ffrom tto _cost_ oldcost _capac_ _lo_
       _flow_ oldflow _fcost_ oldfc;
   sum _fcost_ oldfc;
run;
```

The following notes appear on the SAS log:

```
NOTE: Number of nodes= 8 .
NOTE: Number of supply nodes= 1 .
NOTE: Number of demand nodes= 1 .
NOTE: Total supply= 500 , total demand= 500 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 3 .
NOTE: Of these, 1 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= 93150 .
NOTE: The data set WORK.ARCOUT2 has 13 observations
      and 16 variables.
```

ARCOUT=arcout2 is shown in Output 5.3.1.

Output 5.3.1. ARCOUT=ARCOUT2

Minimum Cost Flow Problem - Warm Start
How to get Hawaiian Pineapples to a London Restaurant

Obs	from	to	_cost_	oldcost	_capac_	_lo_	_flow_	oldflow	_fcost_	oldfc
1	San Francisco	Atlanta	63	63	350	0	0	0	0	0
2	Los Angeles	Atlanta	57	57	350	0	0	150	0	8550
3	Chicago	Boston	15	45	350	0	150	0	2250	0
4	San Francisco	Boston	71	71	350	0	0	0	0	0
5	Honolulu	Chicago	105	105	350	0	150	0	15750	0
6	Boston	Heathrow London	88	88	350	0	150	0	13200	0
7	New York	Heathrow London	65	65	350	0	350	350	22750	22750
8	Atlanta	Heathrow London	76	76	350	0	0	150	0	11400
9	Honolulu	Los Angeles	68	68	350	0	350	350	23800	23800
10	Chicago	New York	26	56	350	0	0	0	0	0
11	San Francisco	New York	48	48	350	0	0	150	0	7200
12	Los Angeles	New York	44	44	350	0	350	200	15400	8800
13	Honolulu	San Francisco	75	75	350	0	0	150	0	11250
									93150	93750

Example 5.4. Production, Inventory, Distribution Problem

Example 5.4 through Example 5.8 use data from a company that produces two sizes of televisions in order to illustrate variations in the use the NETFLOW procedure. The company makes televisions with a diagonal screen measurement of either 19 inches or 25 inches. These televisions are made between March and May at both of the company's two factories. Each factory has a limit on the total number of televisions of each screen dimension that can be made during those months.

The televisions are distributed to one of two shops, stored at the factory where they were made and sold later, or shipped to the other factory. Some sets can be used to

fill backorders from the previous months. Each shop demands a number of each type of TV for the months of March through May. The following network in Figure 5.42 illustrates the model. Arc costs can be interpreted as production costs, storage costs, backorder penalty costs, inter-factory transportation costs, and sales profits. The arcs can have capacities and lower flow bounds.

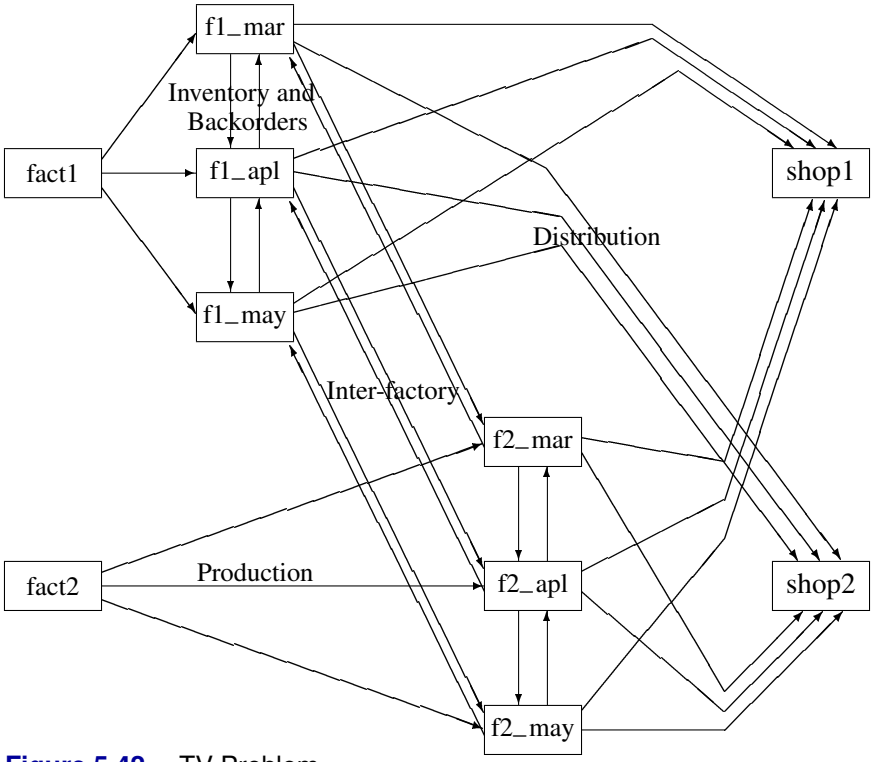


Figure 5.42. TV Problem

There are two similarly structured networks, one for the 19-inch televisions and the other for the 25-inch screen TVs. The minimum cost production, inventory, and distribution plan for both TV types can be determined in the same run of PROC NETFLOW. To ensure that node names are unambiguous, the names of nodes in the 19-inch network have suffix _1, and the node names in the 25-inch network have suffix _2.

The FUTURE1 option is specified because further processing could be required. Information concerning an optimal solution is retained so it can be used to warm start later optimizations. Warm start information is mostly in variables named _NNUMB_, _PRED_, _TRAV_, _SCESS_, _ARCID_, and _FBQ_ and in observations for nodes named _EXCESS_ and _ROOT_, that are in the NODEOUT=NODE2 output data set. (PROC NETFLOW uses similar devices to store warm start information in the DUALOUT= data set when the FUTURE2 option is specified.) Variables _ANUMB_ and _TNUMB_ and observations for arcs directed from or toward a node called _EXCESS_ are present in ARCOUT=arc1. (PROC NETFLOW uses similar devices to store warm start information in the CONOUT= data set when the FUTURE2 option is specified.)

The following code shows how to save the problem data in data sets and solve the model with PROC NETFLOW.

```
title 'Minimum Cost Flow problem';
title2 'Production Planning/Inventory/Distribution';
data node0;
   input _node_ $ _supdem_ ;
   datalines;
fact1_1   1000
fact2_1    850
fact1_2   1000
fact2_2   1500
shop1_1   -900
shop2_1   -900
shop1_2   -900
shop2_2  -1450
;

data arc0;
   input _tail_ $ _head_ $ _cost_ _capac_ _lo_ diagonal factory
         key_id $10. mth_made $ _name_&$17. ;
   datalines;
fact1_1  f1_mar_1  127.9  500 50 19 1 production March prod f1 19 mar
fact1_1  f1_apr_1   78.6  600 50 19 1 production April prod f1 19 apl
fact1_1  f1_may_1   95.1  400 50 19 1 production May   .
f1_mar_1 f1_apr_1   15     50  . 19 1 storage    March .
f1_apr_1 f1_may_1   12     50  . 19 1 storage    April .
f1_apr_1 f1_mar_1   28     20  . 19 1 backorder  April back f1 19 apl
f1_may_1 f1_apr_1   28     20  . 19 1 backorder  May   back f1 19 may
f1_mar_1 f2_mar_1   11      .  . 19 . f1_to_2    March .
f1_apr_1 f2_apr_1   11      .  . 19 . f1_to_2    April .
f1_may_1 f2_may_1   16      .  . 19 . f1_to_2    May   .
f1_mar_1 shop1_1  -327.65 250  . 19 1 sales      March .
f1_apr_1 shop1_1  -300    250  . 19 1 sales      April .
f1_may_1 shop1_1  -285    250  . 19 1 sales      May   .
f1_mar_1 shop2_1  -362.74 250  . 19 1 sales      March .
f1_apr_1 shop2_1  -300    250  . 19 1 sales      April .
f1_may_1 shop2_1  -245    250  . 19 1 sales      May   .
fact2_1  f2_mar_1   88.0  450 35 19 2 production March prod f2 19 mar
fact2_1  f2_apr_1   62.4  480 35 19 2 production April prod f2 19 apl
fact2_1  f2_may_1  133.8  250 35 19 2 production May   .
f2_mar_1 f2_apr_1   18     30  . 19 2 storage    March .
f2_apr_1 f2_may_1   20     30  . 19 2 storage    April .
f2_apr_1 f2_mar_1   17     15  . 19 2 backorder  April back f2 19 apl
f2_may_1 f2_apr_1   25     15  . 19 2 backorder  May   back f2 19 may
f2_mar_1 f1_mar_1   10     40  . 19 . f2_to_1    March .
f2_apr_1 f1_apr_1   11     40  . 19 . f2_to_1    April .
f2_may_1 f1_may_1   13     40  . 19 . f2_to_1    May   .
f2_mar_1 shop1_1  -297.4  250  . 19 2 sales      March .
f2_apr_1 shop1_1  -290    250  . 19 2 sales      April .
f2_may_1 shop1_1  -292    250  . 19 2 sales      May   .
f2_mar_1 shop2_1  -272.7  250  . 19 2 sales      March .
f2_apr_1 shop2_1  -312    250  . 19 2 sales      April .
f2_may_1 shop2_1  -299    250  . 19 2 sales      May   .
fact1_2  f1_mar_2  217.9  400 40 25 1 production March prod f1 25 mar
fact1_2  f1_apr_2  174.5  550 50 25 1 production April prod f1 25 apl
fact1_2  f1_may_2  133.3  350 40 25 1 production May   .
f1_mar_2 f1_apr_2   20     40  . 25 1 storage    March .
f1_apr_2 f1_may_2   18     40  . 25 1 storage    April .
f1_apr_2 f1_mar_2   32     30  . 25 1 backorder  April back f1 25 apl
f1_may_2 f1_apr_2   41     15  . 25 1 backorder  May   back f1 25 may
f1_mar_2 f2_mar_2   23      .  . 25 . f1_to_2    March .
```

```
f1_apr_2 f2_apr_2   23      .   . 25 . f1_to_2    April .
f1_may_2 f2_may_2   26      .   . 25 . f1_to_2    May   .
f1_mar_2 shop1_2  -559.76   .   . 25 1 sales      March .
f1_apr_2 shop1_2  -524.28   .   . 25 1 sales      April .
f1_may_2 shop1_2  -475.02   .   . 25 1 sales      May   .
f1_mar_2 shop2_2  -623.89   .   . 25 1 sales      March .
f1_apr_2 shop2_2  -549.68   .   . 25 1 sales      April .
f1_may_2 shop2_2  -460.00   .   . 25 1 sales      May   .
fact2_2  f2_mar_2  182.0  650 35 25 2 production March prod f2 25 mar
fact2_2  f2_apr_2  196.7  680 35 25 2 production April prod f2 25 apl
fact2_2  f2_may_2  201.4  550 35 25 2 production May   .
f2_mar_2 f2_apr_2   28     50   . 25 2 storage    March .
f2_apr_2 f2_may_2   38     50   . 25 2 storage    April .
f2_apr_2 f2_mar_2   31     15   . 25 2 backorder  April back f2 25 apl
f2_may_2 f2_apr_2   54     15   . 25 2 backorder  May   back f2 25 may
f2_mar_2 f1_mar_2   20     25   . 25 . f2_to_1    March .
f2_apr_2 f1_apr_2   21     25   . 25 . f2_to_1    April .
f2_may_2 f1_may_2   43     25   . 25 . f2_to_1    May   .
f2_mar_2 shop1_2  -567.83 500   . 25 2 sales      March .
f2_apr_2 shop1_2  -542.19 500   . 25 2 sales      April .
f2_may_2 shop1_2  -461.56 500   . 25 2 sales      May   .
f2_mar_2 shop2_2  -542.83 500   . 25 2 sales      March .
f2_apr_2 shop2_2  -559.19 500   . 25 2 sales      April .
f2_may_2 shop2_2  -489.06 500   . 25 2 sales      May   .
;

proc netflow
       nodedata=node0
       arcdata=arc0;
   set future1
       nodeout=node2
       arcout=arc1;
proc print data=arc1; sum _fcost_;
proc print data=node2;
run;
```

The following notes appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 74 .
NOTE: Of these, 1 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= -1281110.35 .
NOTE: The data set WORK.ARC1 has 68 observations and
      18 variables.
NOTE: The data set WORK.NODE2 has 22 observations and
      10 variables.
```

The solution is given in the NODEOUT=node2 and ARCOUT=arc1 data sets. In the ARCOUT= data set, shown in Output 5.4.1 and Output 5.4.2, the variables diagonal, factory, key_id, and mth_made form an implicit ID list. The diagonal variable

has one of two values, 19 or 25. factory also has one of two values, 1 or 2, to denote the factory where either production or storage occurs, from where TVs are either sold to shops or satisfy backorders. PRODUCTION, STORAGE, SALES, and BACKORDER are values of the key_id variable.

Other values of this variable, F1_TO_2 and F2_TO_1, are used when flow through arcs represents the transportation of TVs between factories. The mth_made variable has values MARCH, APRIL, and MAY, the months when TVs that are modeled as flow through an arc were made (assuming that no televisions are stored for more than one month and none manufactured in May are used to fill March backorders).

These ID variables can be used after the PROC NETFLOW run to produce reports and perform analysis on particular parts of the company's operation. For example, reports can be generated for production numbers for each factory; optimal sales figures for each shop; and how many TVs should be stored, used to fill backorders, sent to the other factory, or any combination of these, for TVs with a particular screen, those produced in a particular month, or both.

Output 5.4.1. ARCOUT=ARC1

Minimum Cost Flow problem
Production Planning/Inventory/Distribution

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_name_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	fact1_1	_EXCESS_	0.00	99999999	0		1000	200	5	0.00
2	fact2_1	_EXCESS_	0.00	99999999	0		850	200	45	0.00
3	fact1_2	_EXCESS_	0.00	99999999	0		1000	200	10	0.00
4	fact2_2	_EXCESS_	0.00	99999999	0		1500	200	140	0.00
5	fact1_1	f1_apr_1	78.60	600	50	prod f1 19 apl	1000	.	600	47160.00
6	f1_mar_1	f1_apr_1	15.00	50	0		.	.	0	0.00
7	f1_may_1	f1_apr_1	28.00	20	0	back f1 19 may	.	.	0	0.00
8	f2_apr_1	f1_apr_1	11.00	40	0		.	.	0	0.00
9	fact1_2	f1_apr_2	174.50	550	50	prod f1 25 apl	1000	.	550	95975.00
10	f1_mar_2	f1_apr_2	20.00	40	0		.	.	0	0.00
11	f1_may_2	f1_apr_2	41.00	15	0	back f1 25 may	.	.	15	615.00
12	f2_apr_2	f1_apr_2	21.00	25	0		.	.	0	0.00
13	fact1_1	f1_mar_1	127.90	500	50	prod f1 19 mar	1000	.	345	44125.50
14	f1_apr_1	f1_mar_1	28.00	20	0	back f1 19 apl	.	.	20	560.00
15	f2_mar_1	f1_mar_1	10.00	40	0		.	.	40	400.00
16	fact1_2	f1_mar_2	217.90	400	40	prod f1 25 mar	1000	.	400	87160.00
17	f1_apr_2	f1_mar_2	32.00	30	0	back f1 25 apl	.	.	30	960.00
18	f2_mar_2	f1_mar_2	20.00	25	0		.	.	25	500.00
19	fact1_1	f1_may_1	95.10	400	50		1000	.	50	4755.00
20	f1_apr_1	f1_may_1	12.00	50	0		.	.	50	600.00
21	f2_may_1	f1_may_1	13.00	40	0		.	.	0	0.00
22	fact1_2	f1_may_2	133.30	350	40		1000	.	40	5332.00
23	f1_apr_2	f1_may_2	18.00	40	0		.	.	0	0.00
24	f2_may_2	f1_may_2	43.00	25	0		.	.	0	0.00
25	f1_apr_1	f2_apr_1	11.00	99999999	0		.	.	30	330.00
26	fact2_1	f2_apr_1	62.40	480	35	prod f2 19 apl	850	.	480	29952.00
27	f2_mar_1	f2_apr_1	18.00	30	0		.	.	0	0.00
28	f2_may_1	f2_apr_1	25.00	15	0	back f2 19 may	.	.	0	0.00
29	f1_apr_2	f2_apr_2	23.00	99999999	0		.	.	0	0.00
30	fact2_2	f2_apr_2	196.70	680	35	prod f2 25 apl	1500	.	680	133756.00
31	f2_mar_2	f2_apr_2	28.00	50	0		.	.	0	0.00
32	f2_may_2	f2_apr_2	54.00	15	0	back f2 25 may	.	.	15	810.00
33	f1_mar_1	f2_mar_1	11.00	99999999	0		.	.	0	0.00
34	fact2_1	f2_mar_1	88.00	450	35	prod f2 19 mar	850	.	290	25520.00
35	f2_apr_1	f2_mar_1	17.00	15	0	back f2 19 apl	.	.	0	0.00
36	f1_mar_2	f2_mar_2	23.00	99999999	0		.	.	0	0.00
37	fact2_2	f2_mar_2	182.00	650	35	prod f2 25 mar	1500	.	645	117390.00
38	f2_apr_2	f2_mar_2	31.00	15	0	back f2 25 apl	.	.	0	0.00
39	f1_may_1	f2_may_1	16.00	99999999	0		.	.	100	1600.00
40	fact2_1	f2_may_1	133.80	250	35		850	.	35	4683.00
41	f2_apr_1	f2_may_1	20.00	30	0		.	.	15	300.00
42	f1_may_2	f2_may_2	26.00	99999999	0		.	.	0	0.00
43	fact2_2	f2_may_2	201.40	550	35		1500	.	35	7049.00
44	f2_apr_2	f2_may_2	38.00	50	0		.	.	0	0.00
45	f1_mar_1	shop1_1	-327.65	250	0		.	900	155	-50785.75
46	f1_apr_1	shop1_1	-300.00	250	0		.	900	250	-75000.00
47	f1_may_1	shop1_1	-285.00	250	0		.	900	0	0.00
48	f2_mar_1	shop1_1	-297.40	250	0		.	900	250	-74350.00
49	f2_apr_1	shop1_1	-290.00	250	0		.	900	245	-71050.00
50	f2_may_1	shop1_1	-292.00	250	0		.	900	0	0.00
51	f1_mar_2	shop1_2	-559.76	99999999	0		.	900	0	0.00
52	f1_apr_2	shop1_2	-524.28	99999999	0		.	900	0	0.00
53	f1_may_2	shop1_2	-475.02	99999999	0		.	900	25	-11875.50
54	f2_mar_2	shop1_2	-567.83	500	0		.	900	500	-283915.00
55	f2_apr_2	shop1_2	-542.19	500	0		.	900	375	-203321.25
56	f2_may_2	shop1_2	-461.56	500	0		.	900	0	0.00
57	f1_mar_1	shop2_1	-362.74	250	0		.	900	250	-90685.00
58	f1_apr_1	shop2_1	-300.00	250	0		.	900	250	-75000.00
59	f1_may_1	shop2_1	-245.00	250	0		.	900	0	0.00
60	f2_mar_1	shop2_1	-272.70	250	0		.	900	0	0.00
61	f2_apr_1	shop2_1	-312.00	250	0		.	900	250	-78000.00
62	f2_may_1	shop2_1	-299.00	250	0		.	900	150	-44850.00
63	f1_mar_2	shop2_2	-623.89	99999999	0		.	1450	455	-283869.95
64	f1_apr_2	shop2_2	-549.68	99999999	0		.	1450	535	-294078.80
65	f1_may_2	shop2_2	-460.00	99999999	0		.	1450	0	0.00
66	f2_mar_2	shop2_2	-542.83	500	0		.	1450	120	-65139.60
67	f2_apr_2	shop2_2	-559.19	500	0		.	1450	320	-178940.80
68	f2_may_2	shop2_2	-489.06	500	0		.	1450	20	-9781.20
										-1281110.35

Output 5.4.2. ARCOUT=ARC1 (continued)

Obs	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_	diagonal	factory	key_id	mth_made
1	.	65	1	KEY_ARC BASIC	.	.		
2	.	66	10	KEY_ARC BASIC	.	.		
3	.	67	11	KEY_ARC BASIC	.	.		
4	.	68	20	KEY_ARC BASIC	.	.		
5	-0.650	4	1	UPPERBD NONBASIC	19	1	production	April
6	63.650	5	2	LOWERBD NONBASIC	19	1	storage	March
7	43.000	6	4	LOWERBD NONBASIC	19	1	backorder	May
8	22.000	7	6	LOWERBD NONBASIC	19	.	f2_to_1	April
9	-14.350	36	11	UPPERBD NONBASIC	25	1	production	April
10	94.210	37	12	LOWERBD NONBASIC	25	1	storage	March
11	-16.660	38	14	UPPERBD NONBASIC	25	1	backorder	May
12	30.510	39	16	LOWERBD NONBASIC	25	.	f2_to_1	April
13	.	1	1	KEY_ARC BASIC	19	1	production	March
14	-20.650	2	3	UPPERBD NONBASIC	19	1	backorder	April
15	-29.900	3	5	UPPERBD NONBASIC	19	.	f2_to_1	March
16	-45.160	33	11	UPPERBD NONBASIC	25	1	production	March
17	-42.210	34	13	UPPERBD NONBASIC	25	1	backorder	April
18	-61.060	35	15	UPPERBD NONBASIC	25	.	f2_to_1	March
19	0.850	8	1	LOWERBD NONBASIC	19	1	production	May
20	-3.000	9	3	UPPERBD NONBASIC	19	1	storage	April
21	29.000	10	7	LOWERBD NONBASIC	19	.	f2_to_1	May
22	2.110	40	11	LOWERBD NONBASIC	25	1	production	May
23	75.660	41	13	LOWERBD NONBASIC	25	1	storage	April
24	40.040	42	17	LOWERBD NONBASIC	25	.	f2_to_1	May
25	.	14	3	KEY_ARC BASIC	19	.	f1_to_2	April
26	-27.850	15	10	UPPERBD NONBASIC	19	2	production	April
27	15.750	16	5	LOWERBD NONBASIC	19	2	storage	March
28	45.000	17	7	LOWERBD NONBASIC	19	2	backorder	May
29	13.490	46	13	LOWERBD NONBASIC	25	.	f1_to_2	April
30	-1.660	47	20	UPPERBD NONBASIC	25	2	production	April
31	11.640	48	15	LOWERBD NONBASIC	25	2	storage	March
32	-16.130	49	17	UPPERBD NONBASIC	25	2	backorder	May
33	50.900	11	2	LOWERBD NONBASIC	19	.	f1_to_2	March
34	.	12	10	KEY_ARC BASIC	19	2	production	March
35	19.250	13	6	LOWERBD NONBASIC	19	2	backorder	April
36	104.060	43	12	LOWERBD NONBASIC	25	.	f1_to_2	March
37	.	44	20	KEY_ARC BASIC	25	2	production	March
38	47.360	45	16	LOWERBD NONBASIC	25	2	backorder	April
39	.	18	4	KEY_ARC BASIC	19	.	f1_to_2	May
40	23.550	19	10	LOWERBD NONBASIC	19	2	production	May
41	.	20	6	KEY_ARC BASIC	19	2	storage	April
42	28.960	50	14	LOWERBD NONBASIC	25	.	f1_to_2	May
43	73.170	51	20	LOWERBD NONBASIC	25	2	production	May
44	108.130	52	16	LOWERBD NONBASIC	25	2	storage	April
45	.	21	2	KEY_ARC BASIC	19	1	sales	March
46	-21.000	22	3	UPPERBD NONBASIC	19	1	sales	April
47	9.000	23	4	LOWERBD NONBASIC	19	1	sales	May
48	-9.650	24	5	UPPERBD NONBASIC	19	2	sales	March
49	.	25	6	KEY_ARC BASIC	19	2	sales	April
50	18.000	26	7	LOWERBD NONBASIC	19	2	sales	May
51	47.130	53	12	LOWERBD NONBASIC	25	1	sales	March
52	8.400	54	13	LOWERBD NONBASIC	25	1	sales	April
53	.	55	14	KEY_ARC BASIC	25	1	sales	May
54	-42.000	56	15	UPPERBD NONBASIC	25	2	sales	March
55	.	57	16	KEY_ARC BASIC	25	2	sales	April
56	10.500	58	17	LOWERBD NONBASIC	25	2	sales	May
57	-46.090	27	2	UPPERBD NONBASIC	19	1	sales	March
58	-32.000	28	3	UPPERBD NONBASIC	19	1	sales	April
59	38.000	29	4	LOWERBD NONBASIC	19	1	sales	May
60	4.050	30	5	LOWERBD NONBASIC	19	2	sales	March
61	-33.000	31	6	UPPERBD NONBASIC	19	2	sales	April
62	.	32	7	KEY_ARC BASIC	19	2	sales	May
63	.	59	12	KEY_ARC BASIC	25	1	sales	March
64	.	60	13	KEY_ARC BASIC	25	1	sales	April
65	32.020	61	14	LOWERBD NONBASIC	25	1	sales	May
66	.	62	15	KEY_ARC BASIC	25	2	sales	March
67	.	63	16	KEY_ARC BASIC	25	2	sales	April
68	.	64	17	KEY_ARC BASIC	25	2	sales	May

Output 5.4.3. NODEOUT=NODE2

Minimum Cost Flow problem
Production Planning/Inventory/Distribution

Obs	_node_	_sup_	_DUAL_	_NUMBED	_PRED	_TRAV	_SCESS	_ARCID	_FLOW	_FBQ_
1	_ROOT_	238	0.00	22	0	8	0	3	166	-69
2	_EXCESS_	-200	-100000198.75	21	1	11	13	65	5	65
3	f1_apr_1	.	-100000278.00	3	6	7	1	-14	30	4
4	f1_apr_2	.	-100000387.60	13	19	17	1	-60	535	36
5	f1_mar_1	.	-100000326.65	2	8	1	15	-21	155	1
6	f1_mar_2	.	-100000461.81	12	19	13	1	-59	455	33
7	f1_may_1	.	-100000293.00	4	7	2	1	-18	100	8
8	f1_may_2	.	-100000329.94	14	18	12	1	-55	25	40
9	f2_apr_1	.	-100000289.00	6	8	3	5	-25	245	14
10	f2_apr_2	.	-100000397.11	16	19	18	3	-63	320	46
11	f2_mar_1	.	-100000286.75	5	10	22	1	12	255	11
12	f2_mar_2	.	-100000380.75	15	20	19	8	44	610	43
13	f2_may_1	.	-100000309.00	7	6	9	3	20	15	18
14	f2_may_2	.	-100000326.98	17	19	10	1	-64	20	50
15	fact1_1	1000	-100000198.75	1	2	21	14	-1	295	-1
16	fact1_2	1000	-100000198.75	11	21	20	1	-67	10	-33
17	fact2_1	850	-100000198.75	10	21	5	2	-66	45	-33
18	fact2_2	1500	-100000198.75	20	21	15	9	-68	140	-65
19	shop1_1	-900	-99999999.00	8	22	6	21	0	0	21
20	shop1_2	-900	-99999854.92	18	16	14	2	57	375	53
21	shop2_1	-900	-100000010.00	9	7	4	1	32	150	27
22	shop2_2	-1450	-99999837.92	19	15	16	7	62	120	59

Example 5.5. Using an Unconstrained Solution Warm Start

This example examines the effect of changing some of the arc costs. The backorder penalty costs are increased by twenty percent. The sales profit of 25-inch TVs sent to the shops in May is increased by thirty units. The backorder penalty costs of 25-inch TVs manufactured in May for April consumption is decreased by thirty units. The production cost of 19- and 25-inch TVs made in May are decreased by five units and twenty units, respectively. How does the optimal solution of the network after these arc cost alterations compare with the optimum of the original network? If you want to use the warm start facilities of PROC NETFLOW to solve this undefined problem, specify the WARM option. Notice that the FUTURE1 option was specified in the last PROC NETFLOW run.

The following SAS statements produce the new NODEOUT= and ARCOUT= data sets.

```
title 'Minimum Cost Flow problem- Unconstrained Warm Start';
title2 'Production Planning/Inventory/Distribution';
data arc2;
   set arc1;
   oldcost=_cost_;
   oldfc=_fcost_;
```

```
   oldflow=_flow_;
   if key_id='backorder'
      then _cost_=_cost_*1.2;
      else if _tail_='f2_may_2' then _cost_=_cost_-30;
   if key_id='production' & mth_made='May' then
      if diagonal=19 then _cost_=_cost_-5;
                     else _cost_=_cost_-20;

proc netflow
   warm future1
   nodedata=node2
   arcdata=arc2
   nodeout=node3
   arcout=arc3;

proc print data=arc3 (drop = _status_ _rcost_);
   var _tail_ _head_ _capac_ _lo_ _supply_ _demand_ _name_
       _cost_ _flow_ _fcost_ oldcost oldflow oldfc
       diagonal factory key_id mth_made _anumb_ _tnumb_;
       /* to get this variable order */
           sum oldfc _fcost_;
proc print data=node3;
   run;
```

The following notes appear on the SAS log:

```
NOTE: Number of nodes= 21 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 5 .
NOTE: The greater of total supply and total
      demand= 4350 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 8 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= -1285086.45 .
NOTE: The data set WORK.ARC3 has 68 observations and
      21 variables.
NOTE: The data set WORK.NODE3 has 22 observations and
      10 variables.
```

The solution is displayed in Output 5.5.1 and Output 5.5.2. The associated NODEOUT data set is in Output 5.5.3.

Output 5.5.1. ARCOUT=ARC3

Minimum Cost Flow problem- Unconstrained Warm Start
Production Planning/Inventory/Distribution

Obs	_tail_	_head_	_capac_	_lo_	_SUPPLY_	_DEMAND_	_name_	_cost_	_FLOW_	_FCOST_
1	fact1_1	_EXCESS_	99999999	0	1000	200		0.00	5	0.00
2	fact2_1	_EXCESS_	99999999	0	850	200		0.00	45	0.00
3	fact1_2	_EXCESS_	99999999	0	1000	200		0.00	0	0.00
4	fact2_2	_EXCESS_	99999999	0	1500	200		0.00	150	0.00
5	fact1_1	f1_apr_1	600	50	1000	.	prod f1 19 apl	78.60	540	42444.00
6	f1_mar_1	f1_apr_1	50	0	.	.		15.00	0	0.00
7	f1_may_1	f1_apr_1	20	0	.	.	back f1 19 may	33.60	0	0.00
8	f2_apr_1	f1_apr_1	40	0	.	.		11.00	0	0.00
9	fact1_2	f1_apr_2	550	50	1000	.	prod f1 25 apl	174.50	250	43625.00
10	f1_mar_2	f1_apr_2	40	0	.	.		20.00	0	0.00
11	f1_may_2	f1_apr_2	15	0	.	.	back f1 25 may	49.20	15	738.00
12	f2_apr_2	f1_apr_2	25	0	.	.		21.00	0	0.00
13	fact1_1	f1_mar_1	500	50	1000	.	prod f1 19 mar	127.90	340	43486.00
14	f1_apr_1	f1_mar_1	20	0	.	.	back f1 19 apl	33.60	20	672.00
15	f2_mar_1	f1_mar_1	40	0	.	.		10.00	40	400.00
16	fact1_2	f1_mar_2	400	40	1000	.	prod f1 25 mar	217.90	400	87160.00
17	f1_apr_2	f1_mar_2	30	0	.	.	back f1 25 apl	38.40	30	1152.00
18	f2_mar_2	f1_mar_2	25	0	.	.		20.00	25	500.00
19	fact1_1	f1_may_1	400	50	1000	.		90.10	115	10361.50
20	f1_apr_1	f1_may_1	50	0	.	.		12.00	0	0.00
21	f2_may_1	f1_may_1	40	0	.	.		13.00	0	0.00
22	fact1_2	f1_may_2	350	40	1000	.		113.30	350	39655.00
23	f1_apr_2	f1_may_2	40	0	.	.		18.00	0	0.00
24	f2_may_2	f1_may_2	25	0	.	.		13.00	0	0.00
25	f1_apr_1	f2_apr_1	99999999	0	.	.		11.00	20	220.00
26	fact2_1	f2_apr_1	480	35	850	.	prod f2 19 apl	62.40	480	29952.00
27	f2_mar_1	f2_apr_1	30	0	.	.		18.00	0	0.00
28	f2_may_1	f2_apr_1	15	0	.	.	back f2 19 may	30.00	0	0.00
29	f1_apr_2	f2_apr_2	99999999	0	.	.		23.00	0	0.00
30	fact2_2	f2_apr_2	680	35	1500	.	prod f2 25 apl	196.70	680	133756.00
31	f2_mar_2	f2_apr_2	50	0	.	.		28.00	0	0.00
32	f2_may_2	f2_apr_2	15	0	.	.	back f2 25 may	64.80	0	0.00
33	f1_mar_1	f2_mar_1	99999999	0	.	.		11.00	0	0.00
34	fact2_1	f2_mar_1	450	35	850	.	prod f2 19 mar	88.00	290	25520.00
35	f2_apr_1	f2_mar_1	15	0	.	.	back f2 19 apl	20.40	0	0.00
36	f1_mar_2	f2_mar_2	99999999	0	.	.		23.00	0	0.00
37	fact2_2	f2_mar_2	650	35	1500	.	prod f2 25 mar	182.00	635	115570.00
38	f2_apr_2	f2_mar_2	15	0	.	.	back f2 25 apl	37.20	0	0.00
39	f1_may_1	f2_may_1	99999999	0	.	.		16.00	115	1840.00
40	fact2_1	f2_may_1	250	35	850	.		128.80	35	4508.00
41	f2_apr_1	f2_may_1	30	0	.	.		20.00	0	0.00
42	f1_may_2	f2_may_2	99999999	0	.	.		26.00	335	8710.00
43	fact2_2	f2_may_2	550	35	1500	.		181.40	35	6349.00
44	f2_apr_2	f2_may_2	50	0	.	.		38.00	0	0.00
45	f1_mar_1	shop1_1	250	0	.	900		-327.65	150	-49147.50
46	f1_apr_1	shop1_1	250	0	.	900		-300.00	250	-75000.00
47	f1_may_1	shop1_1	250	0	.	900		-285.00	0	0.00
48	f2_mar_1	shop1_1	250	0	.	900		-297.40	250	-74350.00
49	f2_apr_1	shop1_1	250	0	.	900		-290.00	250	-72500.00
50	f2_may_1	shop1_1	250	0	.	900		-292.00	0	0.00
51	f1_mar_2	shop1_2	99999999	0	.	900		-559.76	0	0.00
52	f1_apr_2	shop1_2	99999999	0	.	900		-524.28	0	0.00
53	f1_may_2	shop1_2	99999999	0	.	900		-475.02	0	0.00
54	f2_mar_2	shop1_2	500	0	.	900		-567.83	500	-283915.00
55	f2_apr_2	shop1_2	500	0	.	900		-542.19	400	-216876.00
56	f2_may_2	shop1_2	500	0	.	900		-491.56	0	0.00
57	f1_mar_1	shop2_1	250	0	.	900		-362.74	250	-90685.00
58	f1_apr_1	shop2_1	250	0	.	900		-300.00	250	-75000.00
59	f1_may_1	shop2_1	250	0	.	900		-245.00	0	0.00
60	f2_mar_1	shop2_1	250	0	.	900		-272.70	0	0.00
61	f2_apr_1	shop2_1	250	0	.	900		-312.00	250	-78000.00
62	f2_may_1	shop2_1	250	0	.	900		-299.00	150	-44850.00
63	f1_mar_2	shop2_2	99999999	0	.	1450		-623.89	455	-283869.95
64	f1_apr_2	shop2_2	99999999	0	.	1450		-549.68	235	-129174.80
65	f1_may_2	shop2_2	99999999	0	.	1450		-460.00	0	0.00
66	f2_mar_2	shop2_2	500	0	.	1450		-542.83	110	-59711.30
67	f2_apr_2	shop2_2	500	0	.	1450		-559.19	280	-156573.20
68	f2_may_2	shop2_2	500	0	.	1450		-519.06	370	-192052.20
										-1285086.45

Output 5.5.2. ARCOUT=ARC3 (continued)

Obs	oldcost	oldflow	oldfc	diagonal	factory	key_id	mth_made	_ANUMB_	_TNUMB_
1	0.00	5	0.00	.	.			65	1
2	0.00	45	0.00	.	.			66	10
3	0.00	10	0.00	.	.			67	11
4	0.00	140	0.00	.	.			68	20
5	78.60	600	47160.00	19	1	production	April	4	1
6	15.00	0	0.00	19	1	storage	March	5	2
7	28.00	0	0.00	19	1	backorder	May	6	4
8	11.00	0	0.00	19	.	f2_to_1	April	7	6
9	174.50	550	95975.00	25	1	production	April	36	11
10	20.00	0	0.00	25	1	storage	March	37	12
11	41.00	15	615.00	25	1	backorder	May	38	14
12	21.00	0	0.00	25	.	f2_to_1	April	39	16
13	127.90	345	44125.50	19	1	production	March	1	1
14	28.00	20	560.00	19	1	backorder	April	2	3
15	10.00	40	400.00	19	.	f2_to_1	March	3	5
16	217.90	400	87160.00	25	1	production	March	33	11
17	32.00	30	960.00	25	1	backorder	April	34	13
18	20.00	25	500.00	25	.	f2_to_1	March	35	15
19	95.10	50	4755.00	19	1	production	May	8	1
20	12.00	50	600.00	19	1	storage	April	9	3
21	13.00	0	0.00	19	.	f2_to_1	May	10	7
22	133.30	40	5332.00	25	1	production	May	40	11
23	18.00	0	0.00	25	1	storage	April	41	13
24	43.00	0	0.00	25	.	f2_to_1	May	42	17
25	11.00	30	330.00	19	.	f1_to_2	April	14	3
26	62.40	480	29952.00	19	2	production	April	15	10
27	18.00	0	0.00	19	2	storage	March	16	5
28	25.00	0	0.00	19	2	backorder	May	17	7
29	23.00	0	0.00	25	.	f1_to_2	April	46	13
30	196.70	680	133756.00	25	2	production	April	47	20
31	28.00	0	0.00	25	2	storage	March	48	15
32	54.00	15	810.00	25	2	backorder	May	49	17
33	11.00	0	0.00	19	.	f1_to_2	March	11	2
34	88.00	290	25520.00	19	2	production	March	12	10
35	17.00	0	0.00	19	2	backorder	April	13	6
36	23.00	0	0.00	25	.	f1_to_2	March	43	12
37	182.00	645	117390.00	25	2	production	March	44	20
38	31.00	0	0.00	25	2	backorder	April	45	16
39	16.00	100	1600.00	19	.	f1_to_2	May	18	4
40	133.80	35	4683.00	19	2	production	May	19	10
41	20.00	15	300.00	19	2	storage	April	20	6
42	26.00	0	0.00	25	.	f1_to_2	May	50	14
43	201.40	35	7049.00	25	2	production	May	51	20
44	38.00	0	0.00	25	2	storage	April	52	16
45	-327.65	155	-50785.75	19	1	sales	March	21	2
46	-300.00	250	-75000.00	19	1	sales	April	22	3
47	-285.00	0	0.00	19	1	sales	May	23	4
48	-297.40	250	-74350.00	19	2	sales	March	24	5
49	-290.00	245	-71050.00	19	2	sales	April	25	6
50	-292.00	0	0.00	19	2	sales	May	26	7
51	-559.76	0	0.00	25	1	sales	March	53	12
52	-524.28	0	0.00	25	1	sales	April	54	13
53	-475.02	25	-11875.50	25	1	sales	May	55	14
54	-567.83	500	-283915.00	25	2	sales	March	56	15
55	-542.19	375	-203321.25	25	2	sales	April	57	16
56	-461.56	0	0.00	25	2	sales	May	58	17
57	-362.74	250	-90685.00	19	1	sales	March	27	2
58	-300.00	250	-75000.00	19	1	sales	April	28	3
59	-245.00	0	0.00	19	1	sales	May	29	4
60	-272.70	0	0.00	19	2	sales	March	30	5
61	-312.00	250	-78000.00	19	2	sales	April	31	6
62	-299.00	150	-44850.00	19	2	sales	May	32	7
63	-623.89	455	-283869.95	25	1	sales	March	59	12
64	-549.68	535	-294078.80	25	1	sales	April	60	13
65	-460.00	0	0.00	25	1	sales	May	61	14
66	-542.83	120	-65139.60	25	2	sales	March	62	15
67	-559.19	320	-178940.80	25	2	sales	April	63	16
68	-489.06	20	-9781.20	25	2	sales	May	64	17
			==========						
			-1281110.35						

Output 5.5.3. NODEOUT=NODE3

Minimum Cost Flow problem- Unconstrained Warm Start
Production Planning/Inventory/Distribution

Obs	_node_	_supdem_	_DUAL_	_NUMBED_	_PREDAV_	_TRASS_	_SCARCID_	_FLOW_	_FBQ_
1	_ROOT_	238	0.00	22	0	8	0	3	166	-69
2	_EXCESS_	-200	-100000198.75	21	1	20	13	65	5	65
3	f1_apr_1	.	-100000277.35	3	1	6	2	4	490	4
4	f1_apr_2	.	-100000387.60	13	19	11	2	-60	235	36
5	f1_mar_1	.	-100000326.65	2	8	1	20	-21	150	1
6	f1_mar_2	.	-100000461.81	12	19	13	1	-59	455	33
7	f1_may_1	.	-100000288.85	4	1	7	3	8	65	8
8	f1_may_2	.	-100000330.98	14	17	10	1	-50	335	40
9	f2_apr_1	.	-100000288.35	6	3	4	1	14	20	14
10	f2_apr_2	.	-100000397.11	16	19	18	2	-63	280	46
11	f2_mar_1	.	-100000286.75	5	10	22	1	12	255	11
12	f2_mar_2	.	-100000380.75	15	20	19	9	44	600	43
13	f2_may_1	.	-100000304.85	7	4	9	2	18	115	18
14	f2_may_2	.	-100000356.98	17	19	14	2	-64	370	50
15	fact1_1	1000	-100000198.75	1	2	3	19	-1	290	-1
16	fact1_2	1000	-100000213.10	11	13	17	1	-36	200	-33
17	fact2_1	850	-100000198.75	10	21	5	2	-66	45	-33
18	fact2_2	1500	-100000198.75	20	21	15	10	-68	150	-65
19	shop1_1	-900	-99999999.00	8	22	2	21	0	0	21
20	shop1_2	-900	-99999854.92	18	16	12	1	57	400	53
21	shop2_1	-900	-100000005.85	9	7	21	1	32	150	27
22	shop2_2	-1450	-99999837.92	19	15	16	8	62	110	59

Example 5.6. Adding Side Constraints, Using a Warm Start

The manufacturer of Gizmo chips, which are parts needed to make televisions, can supply only 2600 chips to factory 1 and 3750 chips to factory 2 in time for production in each of the months of March and April. However, Gizmo chips will not be in short supply in May. Three chips are required to make each 19-inch TV while the 25-inch TVs require four chips each. To limit the production of televisions produced at factory 1 in March so that the TVs have the correct number of chips, a side constraint called FACT1 MAR GIZMO is used. The form of this constraint is

```
3 * prod f1 19 mar   +   4 * prod f1 25 mar   <=   2600
```

"prod f1 19 mar" is the name of the arc directed from the node fact1_1 toward node f1_mar_1 and, in the previous constraint, designates the flow assigned to this arc. The ARCDATA= and ARCOUT= data sets have arc names in a variable called _name_.

The other side constraints (shown below) are called FACT2 MAR GIZMO , FACT1 APL GIZMO, and FACT2 APL GIZMO.

```
3 * prod f2 19 mar   +   4 * prod f2 25 mar   <=   3750
3 * prod f1 19 apl   +   4 * prod f1 25 apl   <=   2600
3 * prod f2 19 apl   +   4 * prod f2 25 apl   <=   3750
```

To maintain customer goodwill, the total number of backorders is not to exceed 50 sets. The side constraint TOTAL BACKORDER that models this restriction is:

```
back f1 19 apl  +  back f1 25 apl  +
back f2 19 apl  +  back f2 25 apl  +
back f1 19 may  +  back f1 25 may  +
back f2 19 may  +  back f2 25 may   <=   50
```

The sparse CONDATA= data set format is used. All side constraints are less than or equal type. Because this is the default type value for the DEFCONTYPE= option, type information is not necessary in the following CONDATA=CON3. Also, DEFCONTYPE= <= does not have to be specified in the PROC NETFLOW statement that follows. Notice that the _column_ variable value CHIP/BO LIMIT indicates that an observation of the CON3 data set contains rhs information. Therefore, specify RHSOBS='CHIP/BO LIMIT'.

```
title 'Adding Side Constraints and Using a Warm Start';
title2 'Production Planning/Inventory/Distribution';
data con3;
   input _column_ &$14. _row_ &$15. _coef_ ;
   datalines;
prod f1 19 mar  FACT1 MAR GIZMO  3
prod f1 25 mar  FACT1 MAR GIZMO  4
CHIP/BO LIMIT   FACT1 MAR GIZMO  2600
prod f2 19 mar  FACT2 MAR GIZMO  3
prod f2 25 mar  FACT2 MAR GIZMO  4
CHIP/BO LIMIT   FACT2 MAR GIZMO  3750
prod f1 19 apl  FACT1 APL GIZMO  3
prod f1 25 apl  FACT1 APL GIZMO  4
CHIP/BO LIMIT   FACT1 APL GIZMO  2600
prod f2 19 apl  FACT2 APL GIZMO  3
prod f2 25 apl  FACT2 APL GIZMO  4
CHIP/BO LIMIT   FACT2 APL GIZMO  3750
back f1 19 apl  TOTAL BACKORDER  1
back f1 25 apl  TOTAL BACKORDER  1
back f2 19 apl  TOTAL BACKORDER  1
back f2 25 apl  TOTAL BACKORDER  1
back f1 19 may  TOTAL BACKORDER  1
back f1 25 may  TOTAL BACKORDER  1
back f2 19 may  TOTAL BACKORDER  1
back f2 25 may  TOTAL BACKORDER  1
CHIP/BO LIMIT   TOTAL BACKORDER  50
;
```

The four pairs of data sets that follow can be used as ARCDATA= and NODEDATA= data sets in the following PROC NETFLOW run. The set used depends on which cost information the arcs are to have and whether a warm start is to be used.

```
ARCDATA=arc0    NODEDATA=node0
ARCDATA=arc1    NODEDATA=node2
ARCDATA=arc2    NODEDATA=node2
ARCDATA=arc3    NODEDATA=node3
```

arc0, node0, arc1, and node2 were created in Example 5.4. The first two data sets are the original input data sets. arc1 and node2 were the ARCOUT= and NODEOUT= data sets of a PROC NETFLOW run with FUTURE1 specified. Now, if you use arc1 and node2 as the ARCDATA= data set and NODEDATA= data set in a PROC NETFLOW run, you can specify WARM, as these data sets contain additional information describing a warm start.

In Example 5.5, arc2 was created by modifying arc1 to reflect different arc costs. arc2 and node2 can also be used as the ARCDATA= and NODEDATA= data sets in a PROC NETFLOW run. Again, specify WARM, as these data sets contain additional information describing a warm start. This start, however, contains the optimal basis using the original costs.

If you are going to continue optimization using the changed arc costs, it is probably best to use arc3 and node3 as the ARCDATA= and NODEDATA= data sets. These data sets, created in Example 5.6 by PROC NETFLOW when the FUTURE1 option was specified, contain an optimal basis that can be used as a warm start.

PROC NETFLOW is used to find the changed cost network solution that obeys the chip limit and backorder side constraints. The FUTURE2 option is specified in case further processing is required. An explicit ID list has also been specified so that the variables oldcost, oldfc and oldflow do not appear in the subsequent output data sets.

```
proc netflow
   nodedata=node3 arcdata=arc3 warm
   condata=con3 sparsecondata rhsobs='CHIP/BO LIMIT'
   future2 dualout=dual4 conout=con4;
      id diagonal factory key_id mth_made;
proc print data=con4;
   sum _fcost_;
proc print data=dual4;
   run;
```

The following messages appear on the SAS log:

```
NOTE: The following 3 variables in ARCDATA do not belong to
      any SAS variable list. These will be ignored.
      oldcost
      oldfc
      oldflow
NOTE: Number of nodes= 21 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 5 .
NOTE: The greater of total supply and total demand= 4350 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 1 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= -1285086.45 .
NOTE: Number of <= side constraints= 5 .
NOTE: Number of == side constraints= 0 .
```

```
NOTE: Number of >= side constraints= 0 .
NOTE: Number of arc and nonarc variable side constraint
      coefficients= 16 .
NOTE: Number of iterations, optimizing with constraints= 10 .
NOTE: Of these, 0 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= -1282708.625 .
NOTE: The data set WORK.CON4 has 68 observations and 18
      variables.
NOTE: The data set WORK.DUAL4 has 27 observations and 14
      variables.
```

Output 5.6.1. CONOUT=CON4

Adding Side Constraints and Using a Warm Start
Production Planning/Inventory/Distribution

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_name_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	fact1_1	_EXCESS_	0.00	99999999	0		1000	200	5.000	0.00
2	fact2_1	_EXCESS_	0.00	99999999	0		850	200	45.000	0.00
3	fact1_2	_EXCESS_	0.00	99999999	0		1000	200	0.000	0.00
4	fact2_2	_EXCESS_	0.00	99999999	0		1500	200	150.000	0.00
5	fact1_1	f1_apr_1	78.60	600	50	prod f1 19 apl	1000	.	533.333	41920.00
6	f1_mar_1	f1_apr_1	15.00	50	0		.	.	0.000	0.00
7	f1_may_1	f1_apr_1	33.60	20	0	back f1 19 may	.	.	0.000	0.00
8	f2_apr_1	f1_apr_1	11.00	40	0		.	.	0.000	0.00
9	fact1_2	f1_apr_2	174.50	550	50	prod f1 25 apl	1000	.	250.000	43625.00
10	f1_mar_2	f1_apr_2	20.00	40	0		.	.	0.000	0.00
11	f1_may_2	f1_apr_2	49.20	15	0	back f1 25 may	.	.	0.000	0.00
12	f2_apr_2	f1_apr_2	21.00	25	0		.	.	0.000	0.00
13	fact1_1	f1_mar_1	127.90	500	50	prod f1 19 mar	1000	.	333.333	42633.33
14	f1_apr_1	f1_mar_1	33.60	20	0	back f1 19 apl	.	.	20.000	672.00
15	f2_mar_1	f1_mar_1	10.00	40	0		.	.	40.000	400.00
16	fact1_2	f1_mar_2	217.90	400	40	prod f1 25 mar	1000	.	400.000	87160.00
17	f1_apr_2	f1_mar_2	38.40	30	0	back f1 25 apl	.	.	30.000	1152.00
18	f2_mar_2	f1_mar_2	20.00	25	0		.	.	25.000	500.00
19	fact1_1	f1_may_1	90.10	400	50		1000	.	128.333	11562.83
20	f1_apr_1	f1_may_1	12.00	50	0		.	.	0.000	0.00
21	f2_may_1	f1_may_1	13.00	40	0		.	.	0.000	0.00
22	fact1_2	f1_may_2	113.30	350	40		1000	.	350.000	39655.00
23	f1_apr_2	f1_may_2	18.00	40	0		.	.	0.000	0.00
24	f2_may_2	f1_may_2	13.00	25	0		.	.	0.000	0.00
25	f1_apr_1	f2_apr_1	11.00	99999999	0		.	.	13.333	146.67
26	fact2_1	f2_apr_1	62.40	480	35	prod f2 19 apl	850	.	480.000	29952.00
27	f2_mar_1	f2_apr_1	18.00	30	0		.	.	0.000	0.00
28	f2_may_1	f2_apr_1	30.00	15	0	back f2 19 may	.	.	0.000	0.00
29	f1_apr_2	f2_apr_2	23.00	99999999	0		.	.	0.000	0.00
30	fact2_2	f2_apr_2	196.70	680	35	prod f2 25 apl	1500	.	577.500	113594.25
31	f2_mar_2	f2_apr_2	28.00	50	0		.	.	0.000	0.00
32	f2_may_2	f2_apr_2	64.80	15	0	back f2 25 may	.	.	0.000	0.00
33	f1_mar_1	f2_mar_1	11.00	99999999	0		.	.	0.000	0.00
34	fact2_1	f2_mar_1	88.00	450	35	prod f2 19 mar	850	.	290.000	25520.00
35	f2_apr_1	f2_mar_1	20.40	15	0	back f2 19 apl	.	.	0.000	0.00
36	f1_mar_2	f2_mar_2	23.00	99999999	0		.	.	0.000	0.00
37	fact2_2	f2_mar_2	182.00	650	35	prod f2 25 mar	1500	.	650.000	118300.00
38	f2_apr_2	f2_mar_2	37.20	15	0	back f2 25 apl	.	.	0.000	0.00
39	f1_may_1	f2_may_1	16.00	99999999	0		.	.	115.000	1840.00
40	fact2_1	f2_may_1	128.80	250	35		850	.	35.000	4508.00
41	f2_apr_1	f2_may_1	20.00	30	0		.	.	0.000	0.00
42	f1_may_2	f2_may_2	26.00	99999999	0		.	.	350.000	9100.00
43	fact2_2	f2_may_2	181.40	550	35		1500	.	122.500	22221.50
44	f2_apr_2	f2_may_2	38.00	50	0		.	.	0.000	0.00
45	f1_mar_1	shop1_1	-327.65	250	0		.	900	143.333	-46963.17
46	f1_apr_1	shop1_1	-300.00	250	0		.	900	250.000	-75000.00
47	f1_may_1	shop1_1	-285.00	250	0		.	900	13.333	-3800.00
48	f2_mar_1	shop1_1	-297.40	250	0		.	900	250.000	-74350.00
49	f2_apr_1	shop1_1	-290.00	250	0		.	900	243.333	-70566.67
50	f2_may_1	shop1_1	-292.00	250	0		.	900	0.000	0.00
51	f1_mar_2	shop1_2	-559.76	99999999	0		.	900	0.000	0.00
52	f1_apr_2	shop1_2	-524.28	99999999	0		.	900	0.000	0.00
53	f1_may_2	shop1_2	-475.02	99999999	0		.	900	0.000	0.00
54	f2_mar_2	shop1_2	-567.83	500	0		.	900	500.000	-283915.00
55	f2_apr_2	shop1_2	-542.19	500	0		.	900	400.000	-216876.00
56	f2_may_2	shop1_2	-491.56	500	0		.	900	0.000	0.00
57	f1_mar_1	shop2_1	-362.74	250	0		.	900	250.000	-90685.00
58	f1_apr_1	shop2_1	-300.00	250	0		.	900	250.000	-75000.00
59	f1_may_1	shop2_1	-245.00	250	0		.	900	0.000	0.00
60	f2_mar_1	shop2_1	-272.70	250	0		.	900	0.000	0.00
61	f2_apr_1	shop2_1	-312.00	250	0		.	900	250.000	-78000.00
62	f2_may_1	shop2_1	-299.00	250	0		.	900	150.000	-44850.00
63	f1_mar_2	shop2_2	-623.89	99999999	0		.	1450	455.000	-283869.95
64	f1_apr_2	shop2_2	-549.68	99999999	0		.	1450	220.000	-120929.60
65	f1_may_2	shop2_2	-460.00	99999999	0		.	1450	0.000	0.00
66	f2_mar_2	shop2_2	-542.83	500	0		.	1450	125.000	-67853.75
67	f2_apr_2	shop2_2	-559.19	500	0		.	1450	177.500	-99256.23
68	f2_may_2	shop2_2	-519.06	500	0		.	1450	472.500	-245255.85
										==========
										-1282708.63

Output 5.6.2. CONOUT=CON4 (continued)

Obs	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_	diagonal	factory	key_id	mth_made
1	.	65	1	KEY_ARC BASIC	.	.		
2	.	66	10	KEY_ARC BASIC	.	.		
3	30.187	67	11	LOWERBD NONBASIC	.	.		
4	.	68	20	KEY_ARC BASIC	.	.		
5	.	4	1	KEY_ARC BASIC	19	1	production	April
6	63.650	5	2	LOWERBD NONBASIC	19	1	storage	March
7	47.020	6	4	LOWERBD NONBASIC	19	1	backorder	May
8	22.000	7	6	LOWERBD NONBASIC	19	.	f2_to_1	April
9	.	36	11	KEY_ARC BASIC	25	1	production	April
10	94.210	37	12	LOWERBD NONBASIC	25	1	storage	March
11	.	38	14	NONKEY ARC BASIC	25	1	backorder	May
12	30.510	39	16	LOWERBD NONBASIC	25	.	f2_to_1	April
13	.	1	1	KEY_ARC BASIC	19	1	production	March
14	-7.630	2	3	UPPERBD NONBASIC	19	1	backorder	April
15	-34.750	3	5	UPPERBD NONBASIC	19	.	f2_to_1	March
16	-31.677	33	11	UPPERBD NONBASIC	25	1	production	March
17	-28.390	34	13	UPPERBD NONBASIC	25	1	backorder	April
18	-61.060	35	15	UPPERBD NONBASIC	25	.	f2_to_1	March
19	.	8	1	KEY_ARC BASIC	19	1	production	May
20	6.000	9	3	LOWERBD NONBASIC	19	1	storage	April
21	29.000	10	7	LOWERBD NONBASIC	19	.	f2_to_1	May
22	-11.913	40	11	UPPERBD NONBASIC	25	1	production	May
23	74.620	41	13	LOWERBD NONBASIC	25	1	storage	April
24	39.000	42	17	LOWERBD NONBASIC	25	.	f2_to_1	May
25	.	14	3	KEY_ARC BASIC	19	.	f1_to_2	April
26	-14.077	15	10	UPPERBD NONBASIC	19	2	production	April
27	10.900	16	5	LOWERBD NONBASIC	19	2	storage	March
28	48.420	17	7	LOWERBD NONBASIC	19	2	backorder	May
29	13.490	46	13	LOWERBD NONBASIC	25	.	f1_to_2	April
30	.	47	20	KEY_ARC BASIC	25	2	production	April
31	11.640	48	15	LOWERBD NONBASIC	25	2	storage	March
32	32.090	49	17	LOWERBD NONBASIC	25	2	backorder	May
33	55.750	11	2	LOWERBD NONBASIC	19	.	f1_to_2	March
34	.	12	10	KEY_ARC BASIC	19	2	production	March
35	34.920	13	6	LOWERBD NONBASIC	19	2	backorder	April
36	104.060	43	12	LOWERBD NONBASIC	25	.	f1_to_2	March
37	-23.170	44	20	UPPERBD NONBASIC	25	2	production	March
38	60.980	45	16	LOWERBD NONBASIC	25	2	backorder	April
39	.	18	4	KEY_ARC BASIC	19	.	f1_to_2	May
40	22.700	19	10	LOWERBD NONBASIC	19	2	production	May
41	9.000	20	6	LOWERBD NONBASIC	19	2	storage	April
42	.	50	14	KEY_ARC BASIC	25	.	f1_to_2	May
43	.	51	20	NONKEY ARC BASIC	25	2	production	May
44	78.130	52	16	LOWERBD NONBASIC	25	2	storage	April
45	.	21	2	KEY_ARC BASIC	19	1	sales	March
46	-21.000	22	3	UPPERBD NONBASIC	19	1	sales	April
47	.	23	4	NONKEY ARC BASIC	19	1	sales	May
48	-14.500	24	5	UPPERBD NONBASIC	19	2	sales	March
49	.	25	6	NONKEY ARC BASIC	19	2	sales	April
50	9.000	26	7	LOWERBD NONBASIC	19	2	sales	May
51	47.130	53	12	LOWERBD NONBASIC	25	1	sales	March
52	8.400	54	13	LOWERBD NONBASIC	25	1	sales	April
53	1.040	55	14	LOWERBD NONBASIC	25	1	sales	May
54	-42.000	56	15	UPPERBD NONBASIC	25	2	sales	March
55	.	57	16	KEY_ARC BASIC	25	2	sales	April
56	10.500	58	17	LOWERBD NONBASIC	25	2	sales	May
57	-37.090	27	2	UPPERBD NONBASIC	19	1	sales	March
58	-23.000	28	3	UPPERBD NONBASIC	19	1	sales	April
59	38.000	29	4	LOWERBD NONBASIC	19	1	sales	May
60	8.200	30	5	LOWERBD NONBASIC	19	2	sales	March
61	-24.000	31	6	UPPERBD NONBASIC	19	2	sales	April
62	.	32	7	KEY_ARC BASIC	19	2	sales	May
63	.	59	12	KEY_ARC BASIC	25	1	sales	March
64	.	60	13	KEY_ARC BASIC	25	1	sales	April
65	33.060	61	14	LOWERBD NONBASIC	25	1	sales	May
66	.	62	15	KEY_ARC BASIC	25	2	sales	March
67	.	63	16	KEY_ARC BASIC	25	2	sales	April
68	.	64	17	KEY_ARC BASIC	25	2	sales	May

Output 5.6.3. DUALOUT=DUAL4

Adding Side Constraints and Using a Warm Start
Production Planning/Inventory/Distribution

Obs	_node_	_supdem_	_DUAL_	_NNUMB_	_PRED_	_TRAV_	_SCESS_
1	_ROOT_	238	0.00	22	0	8	5
2	_EXCESS_	-200	-100000193.90	21	1	20	13
3	f1_apr_1	.	-100000278.00	3	1	6	2
4	f1_apr_2	.	-100000405.92	13	19	11	2
5	f1_mar_1	.	-100000326.65	2	8	1	20
6	f1_mar_2	.	-100000480.13	12	19	13	1
7	f1_may_1	.	-100000284.00	4	1	7	3
8	f1_may_2	.	-100000349.30	14	17	15	1
9	f2_apr_1	.	-100000289.00	6	3	4	1
10	f2_apr_2	.	-100000415.43	16	20	18	9
11	f2_mar_1	.	-100000281.90	5	10	3	1
12	f2_mar_2	.	-100000399.07	15	19	10	1
13	f2_may_1	.	-100000300.00	7	4	9	2
14	f2_may_2	.	-100000375.30	17	19	14	2
15	fact1_1	1000	-100000193.90	1	2	21	19
16	fact1_2	1000	-100000224.09	11	13	17	1
17	fact2_1	850	-100000193.90	10	21	5	2
18	fact2_2	1500	-100000193.90	20	21	16	10
19	shop1_1	-900	-99999999.00	8	22	2	21
20	shop1_2	-900	-99999873.24	18	16	19	1
21	shop2_1	-900	-100000001.00	9	7	22	1
22	shop2_2	-1450	-99999856.24	19	16	12	7
23		.	-1.83	2	8	.	.
24		.	-1.62	0	8	.	.
25		.	-6.21	3	17	.	.
26		.	0.00	1	1	.	1
27		.	-7.42	4	13	.	.

Obs	_ARCID_	_FLOW_	_FBQ_	_VALUE_	_RHS_	_TYPE_	_row_
1	3	166.000	-69	0	75		
2	65	5.000	65	.	.		
3	4	483.333	4	.	.		
4	-60	220.000	36	.	.		
5	-21	143.333	1	.	.		
6	-59	455.000	33	.	.		
7	8	78.333	8	.	.		
8	-50	350.000	40	.	.		
9	14	13.333	14	.	.		
10	47	542.500	46	.	.		
11	12	255.000	11	.	.		
12	-62	125.000	43	.	.		
13	18	115.000	18	.	.		
14	-64	472.500	50	.	.		
15	-1	283.333	-1	.	.		
16	-36	200.000	-33	.	.		
17	-66	45.000	-33	.	.		
18	-68	150.000	-65	.	.		
19	0	0.000	21	.	.		
20	57	400.000	53	.	.		
21	32	150.000	27	.	.		
22	63	177.500	59	.	.		
23	25	243.333	.	2600	2600	LE	FACT1 APL GIZMO
24	23	13.333	.	2600	2600	LE	FACT1 MAR GIZMO
25	51	87.500	.	3750	3750	LE	FACT2 APL GIZMO
26	.	280.000	.	3470	3750	LE	FACT2 MAR GIZMO
27	38	0.000	.	50	50	LE	TOTAL BACKORDER

Example 5.7. Using a Constrained Solution Warm Start

Suppose the 25-inch screen TVs produced at factory 1 in May can be sold at either shop with an increased profit of 40 dollars each. What is the new optimal solution? Because only arc costs have been changed, information about the present solution in DUALOUT=dual4 and CONOUT=con4 can be used as a warm start in the following PROC NETFLOW run. It is still necessary to specify CONDATA=con3 SPARSECONDATA RHSOBS='CHIP/BO LIMIT', since the CONDATA= data set is always read.

```
title 'Using a Constrained Solution Warm Start';
title2 'Production Planning/Inventory/Distribution';
data new_con4;
   set con4;
   oldcost=_cost_;
   oldflow=_flow_;
   oldfc=_fcost_;
   if _tail_='f1_may_2'
      & (_head_='shop1_2' | _head_='shop2_2')
      then _cost_=_cost_-40;
   run;

proc netflow
   warm
   arcdata=new_con4
   dualin=dual4
   condata=con3
   sparsecondata
   rhsobs='CHIP/BO LIMIT'
   dualout=dual5
   conout=con5;
   run;

proc print data=con5 (drop = _status_ _rcost_);
   var _tail_ _head_ _capac_ _lo_ _supply_ _demand_ _name_
       _cost_ _flow_ _fcost_ oldcost oldflow oldfc
       diagonal factory key_id mth_made _anumb_ _tnumb_;
          /* to get this variable order */
           sum oldfc _fcost_;
   run;

proc print data=dual5;
   run;
```

The following messages appear on the SAS log:

```
NOTE: The following 1 variables in NODEDATA do not belong to
      any SAS variable list. These will be ignored.
      _VALUE_
NOTE: Number of nodes= 21 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 5 .
```

```
NOTE: The greater of total supply and total demand= 4350 .
NOTE: Number of <= side constraints= 5 .
NOTE: Number of == side constraints= 0 .
NOTE: Number of >= side constraints= 0 .
NOTE: Number of arc and nonarc variable side constraint
      coefficients= 16 .
NOTE: Number of iterations, optimizing with constraints= 7 .
NOTE: Of these, 1 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= -1295661.8 .
NOTE: The data set WORK.CON5 has 64 observations and 21
      variables.
NOTE: The data set WORK.DUAL5 has 25 observations and 14
      variables.
```

Output 5.7.1. CONOUT=CON5

Using a Constrained Solution Warm Start
Production Planning/Inventory/Distribution

Obs	_tail_	_head_	_capac_	_lo_	_SUPPLY_	_DEMAND_	_name_	_cost_	_FLOW_	_FCOST_
1	fact1_1	f1_apr_1	600	50	1000	.	prod f1 19 apl	78.60	533.333	41920.00
2	f1_mar_1	f1_apr_1	50	0	.	.		15.00	0.000	0.00
3	f1_may_1	f1_apr_1	20	0	.	.	back f1 19 may	33.60	0.000	0.00
4	f2_apr_1	f1_apr_1	40	0	.	.		11.00	0.000	0.00
5	fact1_2	f1_apr_2	550	50	1000	.	prod f1 25 apl	174.50	250.000	43625.00
6	f1_mar_2	f1_apr_2	40	0	.	.		20.00	0.000	0.00
7	f1_may_2	f1_apr_2	15	0	.	.	back f1 25 may	49.20	0.000	0.00
8	f2_apr_2	f1_apr_2	25	0	.	.		21.00	0.000	0.00
9	fact1_1	f1_mar_1	500	50	1000	.	prod f1 19 mar	127.90	333.333	42633.33
10	f1_apr_1	f1_mar_1	20	0	.	.	back f1 19 apl	33.60	20.000	672.00
11	f2_mar_1	f1_mar_1	40	0	.	.		10.00	40.000	400.00
12	fact1_2	f1_mar_2	400	40	1000	.	prod f1 25 mar	217.90	400.000	87160.00
13	f1_apr_2	f1_mar_2	30	0	.	.	back f1 25 apl	38.40	30.000	1152.00
14	f2_mar_2	f1_mar_2	25	0	.	.		20.00	25.000	500.00
15	fact1_1	f1_may_1	400	50	1000	.		90.10	128.333	11562.83
16	f1_apr_1	f1_may_1	50	0	.	.		12.00	0.000	0.00
17	f2_may_1	f1_may_1	40	0	.	.		13.00	0.000	0.00
18	fact1_2	f1_may_2	350	40	1000	.		113.30	350.000	39655.00
19	f1_apr_2	f1_may_2	40	0	.	.		18.00	0.000	0.00
20	f2_may_2	f1_may_2	25	0	.	.		13.00	0.000	0.00
21	f1_apr_1	f2_apr_1	99999999	0	.	.		11.00	13.333	146.67
22	fact2_1	f2_apr_1	480	35	850	.	prod f2 19 apl	62.40	480.000	29952.00
23	f2_mar_1	f2_apr_1	30	0	.	.		18.00	0.000	0.00
24	f2_may_1	f2_apr_1	15	0	.	.	back f2 19 may	30.00	0.000	0.00
25	f1_apr_2	f2_apr_2	99999999	0	.	.		23.00	0.000	0.00
26	fact2_2	f2_apr_2	680	35	1500	.	prod f2 25 apl	196.70	550.000	108185.00
27	f2_mar_2	f2_apr_2	50	0	.	.		28.00	0.000	0.00
28	f2_may_2	f2_apr_2	15	0	.	.	back f2 25 may	64.80	0.000	0.00
29	f1_mar_1	f2_mar_1	99999999	0	.	.		11.00	0.000	0.00
30	fact2_1	f2_mar_1	450	35	850	.	prod f2 19 mar	88.00	290.000	25520.00
31	f2_apr_1	f2_mar_1	15	0	.	.	back f2 19 apl	20.40	0.000	0.00
32	f1_mar_2	f2_mar_2	99999999	0	.	.		23.00	0.000	0.00
33	fact2_2	f2_mar_2	650	35	1500	.	prod f2 25 mar	182.00	650.000	118300.00
34	f2_apr_2	f2_mar_2	15	0	.	.	back f2 25 apl	37.20	0.000	0.00
35	f1_may_1	f2_may_1	99999999	0	.	.		16.00	115.000	1840.00
36	fact2_1	f2_may_1	250	35	850	.		128.80	35.000	4508.00
37	f2_apr_1	f2_may_1	30	0	.	.		20.00	0.000	0.00
38	f1_may_2	f2_may_2	99999999	0	.	.		26.00	0.000	0.00
39	fact2_2	f2_may_2	550	35	1500	.		181.40	150.000	27210.00
40	f2_apr_2	f2_may_2	50	0	.	.		38.00	0.000	0.00
41	f1_mar_1	shop1_1	250	0	.	900		-327.65	143.333	-46963.17
42	f1_apr_1	shop1_1	250	0	.	900		-300.00	250.000	-75000.00
43	f1_may_1	shop1_1	250	0	.	900		-285.00	13.333	-3800.00
44	f2_mar_1	shop1_1	250	0	.	900		-297.40	250.000	-74350.00
45	f2_apr_1	shop1_1	250	0	.	900		-290.00	243.333	-70566.67
46	f2_may_1	shop1_1	250	0	.	900		-292.00	0.000	0.00
47	f1_mar_2	shop1_2	99999999	0	.	900		-559.76	0.000	0.00
48	f1_apr_2	shop1_2	99999999	0	.	900		-524.28	0.000	0.00
49	f1_may_2	shop1_2	99999999	0	.	900		-515.02	350.000	-180257.00
50	f2_mar_2	shop1_2	500	0	.	900		-567.83	500.000	-283915.00
51	f2_apr_2	shop1_2	500	0	.	900		-542.19	50.000	-27109.50
52	f2_may_2	shop1_2	500	0	.	900		-491.56	0.000	0.00
53	f1_mar_1	shop2_1	250	0	.	900		-362.74	250.000	-90685.00
54	f1_apr_1	shop2_1	250	0	.	900		-300.00	250.000	-75000.00
55	f1_may_1	shop2_1	250	0	.	900		-245.00	0.000	0.00
56	f2_mar_1	shop2_1	250	0	.	900		-272.70	0.000	0.00
57	f2_apr_1	shop2_1	250	0	.	900		-312.00	250.000	-78000.00
58	f2_may_1	shop2_1	250	0	.	900		-299.00	150.000	-44850.00
59	f1_mar_2	shop2_2	99999999	0	.	1450		-623.89	455.000	-283869.95
60	f1_apr_2	shop2_2	99999999	0	.	1450		-549.68	220.000	-120929.60
61	f1_may_2	shop2_2	99999999	0	.	1450		-500.00	0.000	0.00
62	f2_mar_2	shop2_2	500	0	.	1450		-542.83	125.000	-67853.75
63	f2_apr_2	shop2_2	500	0	.	1450		-559.19	500.000	-279595.00
64	f2_may_2	shop2_2	500	0	.	1450		-519.06	150.000	-77859.00
										==========
										-1295661.80

Output 5.7.2. CONOUT=CON5 (continued)

Obs	oldcost	oldflow	oldfc	diagonal	factory	key_id	mth_made	_ANUMB_	_TNUMB_
1	78.60	533.333	41920.00	19	1	production	April	4	1
2	15.00	0.000	0.00	19	1	storage	March	5	2
3	33.60	0.000	0.00	19	1	backorder	May	6	4
4	11.00	0.000	0.00	19	.	f2_to_1	April	7	6
5	174.50	250.000	43625.00	25	1	production	April	36	11
6	20.00	0.000	0.00	25	1	storage	March	37	12
7	49.20	0.000	0.00	25	1	backorder	May	38	14
8	21.00	0.000	0.00	25	.	f2_to_1	April	39	16
9	127.90	333.333	42633.33	19	1	production	March	1	1
10	33.60	20.000	672.00	19	1	backorder	April	2	3
11	10.00	40.000	400.00	19	.	f2_to_1	March	3	5
12	217.90	400.000	87160.00	25	1	production	March	33	11
13	38.40	30.000	1152.00	25	1	backorder	April	34	13
14	20.00	25.000	500.00	25	.	f2_to_1	March	35	15
15	90.10	128.333	11562.83	19	1	production	May	8	1
16	12.00	0.000	0.00	19	1	storage	April	9	3
17	13.00	0.000	0.00	19	.	f2_to_1	May	10	7
18	113.30	350.000	39655.00	25	1	production	May	40	11
19	18.00	0.000	0.00	25	1	storage	April	41	13
20	13.00	0.000	0.00	25	.	f2_to_1	May	42	17
21	11.00	13.333	146.67	19	.	f1_to_2	April	14	3
22	62.40	480.000	29952.00	19	2	production	April	15	10
23	18.00	0.000	0.00	19	2	storage	March	16	5
24	30.00	0.000	0.00	19	2	backorder	May	17	7
25	23.00	0.000	0.00	25	.	f1_to_2	April	46	13
26	196.70	577.500	113594.25	25	2	production	April	47	20
27	28.00	0.000	0.00	25	2	storage	March	48	15
28	64.80	0.000	0.00	25	2	backorder	May	49	17
29	11.00	0.000	0.00	19	.	f1_to_2	March	11	2
30	88.00	290.000	25520.00	19	2	production	March	12	10
31	20.40	0.000	0.00	19	2	backorder	April	13	6
32	23.00	0.000	0.00	25	.	f1_to_2	March	43	12
33	182.00	650.000	118300.00	25	2	production	March	44	20
34	37.20	0.000	0.00	25	2	backorder	April	45	16
35	16.00	115.000	1840.00	19	.	f1_to_2	May	18	4
36	128.80	35.000	4508.00	19	2	production	May	19	10
37	20.00	0.000	0.00	19	2	storage	April	20	6
38	26.00	350.000	9100.00	25	.	f1_to_2	May	50	14
39	181.40	122.500	22221.50	25	2	production	May	51	20
40	38.00	0.000	0.00	25	2	storage	April	52	16
41	-327.65	143.333	-46963.17	19	1	sales	March	21	2
42	-300.00	250.000	-75000.00	19	1	sales	April	22	3
43	-285.00	13.333	-3800.00	19	1	sales	May	23	4
44	-297.40	250.000	-74350.00	19	2	sales	March	24	5
45	-290.00	243.333	-70566.67	19	2	sales	April	25	6
46	-292.00	0.000	0.00	19	2	sales	May	26	7
47	-559.76	0.000	0.00	25	1	sales	March	53	12
48	-524.28	0.000	0.00	25	1	sales	April	54	13
49	-475.02	0.000	0.00	25	1	sales	May	55	14
50	-567.83	500.000	-283915.00	25	2	sales	March	56	15
51	-542.19	400.000	-216876.00	25	2	sales	April	57	16
52	-491.56	0.000	0.00	25	2	sales	May	58	17
53	-362.74	250.000	-90685.00	19	1	sales	March	27	2
54	-300.00	250.000	-75000.00	19	1	sales	April	28	3
55	-245.00	0.000	0.00	19	1	sales	May	29	4
56	-272.70	0.000	0.00	19	2	sales	March	30	5
57	-312.00	250.000	-78000.00	19	2	sales	April	31	6
58	-299.00	150.000	-44850.00	19	2	sales	May	32	7
59	-623.89	455.000	-283869.95	25	1	sales	March	59	12
60	-549.68	220.000	-120929.60	25	1	sales	April	60	13
61	-460.00	0.000	0.00	25	1	sales	May	61	14
62	-542.83	125.000	-67853.75	25	2	sales	March	62	15
63	-559.19	177.500	-99256.23	25	2	sales	April	63	16
64	-519.06	472.500	-245255.85	25	2	sales	May	64	17
			==========						
			-1282708.63						

Output 5.7.3. DUALOUT=DUAL5

Using a Constrained Solution Warm Start
Production Planning/Inventory/Distribution

Obs	_node_	_supdem_	_DUAL_	_NNUMB_	_PRED_	_TRAV_	_SCESS_
1	f1_apr_1	.	-100000278.00	3	1	6	2
2	f1_apr_2	.	-100000405.92	13	19	11	2
3	f1_mar_1	.	-100000326.65	2	8	1	20
4	f1_mar_2	.	-100000480.13	12	19	13	1
5	f1_may_1	.	-100000284.00	4	1	7	3
6	f1_may_2	.	-100000363.43	14	18	10	1
7	f2_apr_1	.	-100000289.00	6	3	4	1
8	f2_apr_2	.	-100000390.60	16	20	18	3
9	f2_mar_1	.	-100000281.90	5	10	3	1
10	f2_mar_2	.	-100000399.07	15	19	16	1
11	f2_may_1	.	-100000300.00	7	4	9	2
12	f2_may_2	.	-100000375.30	17	20	19	6
13	fact1_1	1000	-100000193.90	1	2	21	19
14	fact1_2	1000	-100000224.09	11	13	15	1
15	fact2_1	850	-100000193.90	10	21	5	2
16	fact2_2	1500	-100000193.90	20	21	17	10
17	shop1_1	-900	-99999999.00	8	22	2	21
18	shop1_2	-900	-99999848.41	18	16	14	2
19	shop2_1	-900	-100000001.00	9	7	22	1
20	shop2_2	-1450	-99999856.24	19	17	12	5
21		.	-1.83	2	8	.	.
22		.	-1.62	0	8	.	.
23		.	0.00	3	3	.	3
24		.	0.00	1	1	.	1
25		.	0.00	4	4	.	4

Obs	_ARCID_	_FLOW_	_FBQ_	_VALUE_	_RHS_	_TYPE_	_row_
1	4	483.333	4	.	.		
2	-60	220.000	36	.	.		
3	-21	143.333	1	.	.		
4	-59	455.000	33	.	.		
5	8	78.333	8	.	.		
6	-55	350.000	40	.	.		
7	14	13.333	14	.	.		
8	47	515.000	46	.	.		
9	12	255.000	11	.	.		
10	-62	125.000	43	.	.		
11	18	115.000	18	.	.		
12	51	115.000	50	.	.		
13	-1	283.333	-1	.	.		
14	-36	200.000	-33	.	.		
15	-66	45.000	-33	.	.		
16	-68	150.000	-65	.	.		
17	0	0.000	21	.	.		
18	57	50.000	53	.	.		
19	32	150.000	27	.	.		
20	64	150.000	59	.	.		
21	25	243.333	.	2600	2600	LE	FACT1 APL GIZMO
22	23	13.333	.	2600	2600	LE	FACT1 MAR GIZMO
23	.	110.000	.	3640	3750	LE	FACT2 APL GIZMO
24	.	280.000	.	3470	3750	LE	FACT2 MAR GIZMO
25	.	0.000	.	50	50	LE	TOTAL BACKORDER

Example 5.8. Nonarc Variables in the Side Constraints

Notice in DUALOUT=dual5 from Example 5.7 the FACT2 MAR GIZMO constraint (observation 24) has a _VALUE_ of 3470, which is not equal to the _RHS_ of this constraint. Not all of the 3750 chips that can be supplied to factory 2 for March production are used. It is suggested that all the possible chips be obtained in March and those not used be saved for April production. Because chips must be kept in an air-controlled environment, it costs 1 dollar to store each chip purchased in March until April. The maximum number of chips that can be stored in this environment at each factory is 150. In addition, a search of the parts inventory at factory 1 turned up 15 chips available for their March production.

Nonarc variables are used in the side constraints that handle the limitations of supply of Gizmo chips. A nonarc variable called "f1 unused chips" has as a value the number of chips that are not used at factory 1 in March. Another nonarc variable, "f2 unused chips", has as a value the number of chips that are not used at factory 2 in March. "f1 chips from mar" has as a value the number of chips left over from March used for production at factory 1 in April. Similarly, "f2 chips from mar" has as a value the number of chips left over from March used for April production at factory 2 in April. The last two nonarc variables have objective function coefficients of 1 and upper bounds of 150. The Gizmo side constraints are

```
3*prod f1 19 mar + 4*prod f1 25 mar + f1 unused chips = 2615
3*prod f2 19 apl + 4*prod f2 25 apl + f2 unused chips = 3750
3*prod f1 19 apl + 4*prod f1 25 apl - f1 chips from mar = 2600
3*prod f2 19 apl + 4*prod f2 25 apl - f2 chips from mar = 3750
f1 unused chips + f2 unused chips -
f1 chips from mar - f2 chips from mar >= 0
```

The last side constraint states that the number of chips not used in March is not less than the number of chips left over from March and used in April. Here, this constraint is called CHIP LEFTOVER.

The following SAS code creates a new data set containing constraint data. It seems that most of the constraints are now equalities, so you specify DEFCONTYPE=EQ in the PROC NETFLOW statements from now on and provide constraint type data for constraints that are not "equal to" type, using the default TYPEOBS value _TYPE_ as the _COLUMN_ variable value to indicate observations that contain constraint type data. Also, from now on, the default RHSOBS value is used.

```
title 'Nonarc Variables in the Side Constraints';
title2 'Production Planning/Inventory/Distribution';
data con6;
   input _column_ &$17. _row_ &$15. _coef_ ;
   datalines;
prod f1 19 mar     FACT1 MAR GIZMO  3
prod f1 25 mar     FACT1 MAR GIZMO  4
f1 unused chips    FACT1 MAR GIZMO  1
_RHS_              FACT1 MAR GIZMO  2615
prod f2 19 mar     FACT2 MAR GIZMO  3
prod f2 25 mar     FACT2 MAR GIZMO  4
f2 unused chips    FACT2 MAR GIZMO  1
_RHS_              FACT2 MAR GIZMO  3750
prod f1 19 apl     FACT1 APL GIZMO  3
prod f1 25 apl     FACT1 APL GIZMO  4
f1 chips from mar  FACT1 APL GIZMO  -1
_RHS_              FACT1 APL GIZMO  2600
prod f2 19 apl     FACT2 APL GIZMO  3
prod f2 25 apl     FACT2 APL GIZMO  4
f2 chips from mar  FACT2 APL GIZMO  -1
_RHS_              FACT2 APL GIZMO  3750
f1 unused chips    CHIP LEFTOVER    1
f2 unused chips    CHIP LEFTOVER    1
```

```
f1 chips from mar  CHIP LEFTOVER    -1
f2 chips from mar  CHIP LEFTOVER    -1
_TYPE_             CHIP LEFTOVER    1
back f1 19 apl     TOTAL BACKORDER  1
back f1 25 apl     TOTAL BACKORDER  1
back f2 19 apl     TOTAL BACKORDER  1
back f2 25 apl     TOTAL BACKORDER  1
back f1 19 may     TOTAL BACKORDER  1
back f1 25 may     TOTAL BACKORDER  1
back f2 19 may     TOTAL BACKORDER  1
back f2 25 may     TOTAL BACKORDER  1
_TYPE_             TOTAL BACKORDER  -1
_RHS_              TOTAL BACKORDER  50
;
```

The nonarc variables "f1 chips from mar" and "f2 chips from mar" have objective function coefficients of 1 and upper bounds of 150. There are various ways in which this information can be furnished to PROC NETFLOW. If there were a TYPE list variable in the CONDATA= data set, observations could be in the form

```
_COLUMN_           _TYPE_  _ROW_ _COEF_
f1 chips from mar  objfn      .      1
f1 chips from mar  upperbd    .    150
f2 chips from mar  objfn      .      1
f2 chips from mar  upperbd    .    150
```

It is desirable to assign ID list variable values to all the nonarc variables:

```
data arc6;
   set con5;
   drop oldcost oldfc oldflow _flow_ _fcost_ _status_ _rcost_;
data arc6_b;
   input _name_ &$17. _cost_ _capac_ factory key_id $ ;
   datalines;
f1 unused chips    .    . 1 chips
f2 unused chips    .    . 2 chips
f1 chips from mar  1 150 1 chips
f2 chips from mar  1 150 2 chips
;

proc append
   base=arc6 data=arc6_b;
proc netflow
   nodedata=node0 arcdata=arc6
   condata=con6 defcontype=eq sparsecondata
   dualout=dual7 conout=con7;
   run;
print nonarcs/short;
```

The following messages appear on the SAS log:

```
NOTE: Number of nodes= 20 .
NOTE: Number of supply nodes= 4 .
NOTE: Number of demand nodes= 4 .
NOTE: Total supply= 4350 , total demand= 4150 .
NOTE: Number of arcs= 64 .
NOTE: Number of nonarc variables= 4 .
NOTE: Number of iterations performed (neglecting any
      constraints)= 70 .
NOTE: Of these, 1 were degenerate.
NOTE: Optimum (neglecting any constraints) found.
NOTE: Minimal total cost= -1295730.8 .
NOTE: Number of <= side constraints= 1 .
NOTE: Number of == side constraints= 4 .
NOTE: Number of >= side constraints= 1 .
NOTE: Number of arc and nonarc variable side constraint
      coefficients= 24 .
NOTE: Number of iterations, optimizing with constraints= 13 .
NOTE: Of these, 3 were degenerate.
NOTE: Optimum reached.
NOTE: Minimal total cost= -1295542.742 .
NOTE: The data set WORK.CON7 has 68 observations and 18
      variables.
NOTE: The data set WORK.DUAL7 has 26 observations and 14
      variables.
```

The output in Output 5.8.1 is produced by

print *nonarcs/short;*

Output 5.8.1. Output of PRINT NONARCS/SHORT;

Nonarc Variables in the Side Constraints
Production Planning/Inventory/Distribution

The NETFLOW Procedure

_N_	_name_	_cost_	_capac_	_lo_	_VALUE_
1	f1 chips from mar	1	150	0	20
2	f1 unused chips	0	99999999	0	0
3	f2 chips from mar	1	150	0	0
4	f2 unused chips	0	99999999	0	280

The optimal solution data sets, CONOUT=CON7 in Output 5.8.2 and Output 5.8.3 and DUALOUT=DUAL7 in Output 5.8.4 follow.

```
proc print data=con7;
   sum _fcost_;
proc print data=dual7;
```

Output 5.8.2. CONOUT=CON7

Nonarc Variables in the Side Constraints
Production Planning/Inventory/Distribution

Obs	_tail_	_head_	_cost_	_capac_	_lo_	_name_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	fact1_1	f1_apr_1	78.60	600	50	prod f1 19 apl	1000	.	540.000	42444.00
2	f1_mar_1	f1_apr_1	15.00	50	0		.	.	0.000	0.00
3	f1_may_1	f1_apr_1	33.60	20	0	back f1 19 may	.	.	0.000	0.00
4	f2_apr_1	f1_apr_1	11.00	40	0		.	.	0.000	0.00
5	fact1_2	f1_apr_2	174.50	550	50	prod f1 25 apl	1000	.	250.000	43625.00
6	f1_mar_2	f1_apr_2	20.00	40	0		.	.	0.000	0.00
7	f1_may_2	f1_apr_2	49.20	15	0	back f1 25 may	.	.	0.000	0.00
8	f2_apr_2	f1_apr_2	21.00	25	0		.	.	25.000	525.00
9	fact1_1	f1_mar_1	127.90	500	50	prod f1 19 mar	1000	.	338.333	43272.83
10	f1_apr_1	f1_mar_1	33.60	20	0	back f1 19 apl	.	.	20.000	672.00
11	f2_mar_1	f1_mar_1	10.00	40	0		.	.	40.000	400.00
12	fact1_2	f1_mar_2	217.90	400	40	prod f1 25 mar	1000	.	400.000	87160.00
13	f1_apr_2	f1_mar_2	38.40	30	0	back f1 25 apl	.	.	30.000	1152.00
14	f2_mar_2	f1_mar_2	20.00	25	0		.	.	25.000	500.00
15	fact1_1	f1_may_1	90.10	400	50		1000	.	116.667	10511.67
16	f1_apr_1	f1_may_1	12.00	50	0		.	.	0.000	0.00
17	f2_may_1	f1_may_1	13.00	40	0		.	.	0.000	0.00
18	fact1_2	f1_may_2	113.30	350	40		1000	.	350.000	39655.00
19	f1_apr_2	f1_may_2	18.00	40	0		.	.	0.000	0.00
20	f2_may_2	f1_may_2	13.00	25	0		.	.	0.000	0.00
21	f1_apr_1	f2_apr_1	11.00	99999999	0		.	.	20.000	220.00
22	fact2_1	f2_apr_1	62.40	480	35	prod f2 19 apl	850	.	480.000	29952.00
23	f2_mar_1	f2_apr_1	18.00	30	0		.	.	0.000	0.00
24	f2_may_1	f2_apr_1	30.00	15	0	back f2 19 may	.	.	0.000	0.00
25	f1_apr_2	f2_apr_2	23.00	99999999	0		.	.	0.000	0.00
26	fact2_2	f2_apr_2	196.70	680	35	prod f2 25 apl	1500	.	577.500	113594.25
27	f2_mar_2	f2_apr_2	28.00	50	0		.	.	0.000	0.00
28	f2_may_2	f2_apr_2	64.80	15	0	back f2 25 may	.	.	0.000	0.00
29	f1_mar_1	f2_mar_1	11.00	99999999	0		.	.	0.000	0.00
30	fact2_1	f2_mar_1	88.00	450	35	prod f2 19 mar	850	.	290.000	25520.00
31	f2_apr_1	f2_mar_1	20.40	15	0	back f2 19 apl	.	.	0.000	0.00
32	f1_mar_2	f2_mar_2	23.00	99999999	0		.	.	0.000	0.00
33	fact2_2	f2_mar_2	182.00	650	35	prod f2 25 mar	1500	.	650.000	118300.00
34	f2_apr_2	f2_mar_2	37.20	15	0	back f2 25 apl	.	.	0.000	0.00
35	f1_may_1	f2_may_1	16.00	99999999	0		.	.	115.000	1840.00
36	fact2_1	f2_may_1	128.80	250	35		850	.	35.000	4508.00
37	f2_apr_1	f2_may_1	20.00	30	0		.	.	0.000	0.00
38	f1_may_2	f2_may_2	26.00	99999999	0		.	.	0.000	0.00
39	fact2_2	f2_may_2	181.40	550	35		1500	.	122.500	22221.50
40	f2_apr_2	f2_may_2	38.00	50	0		.	.	0.000	0.00
41	f1_mar_1	shop1_1	-327.65	250	0		.	900	148.333	-48601.42
42	f1_apr_1	shop1_1	-300.00	250	0		.	900	250.000	-75000.00
43	f1_may_1	shop1_1	-285.00	250	0		.	900	1.667	-475.00
44	f2_mar_1	shop1_1	-297.40	250	0		.	900	250.000	-74350.00
45	f2_apr_1	shop1_1	-290.00	250	0		.	900	250.000	-72500.00
46	f2_may_1	shop1_1	-292.00	250	0		.	900	0.000	0.00
47	f1_mar_2	shop1_2	-559.76	99999999	0		.	900	0.000	0.00
48	f1_apr_2	shop1_2	-524.28	99999999	0		.	900	0.000	0.00
49	f1_may_2	shop1_2	-515.02	99999999	0		.	900	347.500	-178969.45
50	f2_mar_2	shop1_2	-567.83	500	0		.	900	500.000	-283915.00
51	f2_apr_2	shop1_2	-542.19	500	0		.	900	52.500	-28464.98
52	f2_may_2	shop1_2	-491.56	500	0		.	900	0.000	0.00
53	f1_mar_1	shop2_1	-362.74	250	0		.	900	250.000	-90685.00
54	f1_apr_1	shop2_1	-300.00	250	0		.	900	250.000	-75000.00
55	f1_may_1	shop2_1	-245.00	250	0		.	900	0.000	0.00
56	f2_mar_1	shop2_1	-272.70	250	0		.	900	0.000	0.00
57	f2_apr_1	shop2_1	-312.00	250	0		.	900	250.000	-78000.00
58	f2_may_1	shop2_1	-299.00	250	0		.	900	150.000	-44850.00
59	f1_mar_2	shop2_2	-623.89	99999999	0		.	1450	455.000	-283869.95
60	f1_apr_2	shop2_2	-549.68	99999999	0		.	1450	245.000	-134671.60
61	f1_may_2	shop2_2	-500.00	99999999	0		.	1450	2.500	-1250.00
62	f2_mar_2	shop2_2	-542.83	500	0		.	1450	125.000	-67853.75
63	f2_apr_2	shop2_2	-559.19	500	0		.	1450	500.000	-279595.00
64	f2_may_2	shop2_2	-519.06	500	0		.	1450	122.500	-63584.85
65			1.00	150	0	f1 chips from mar	.	.	20.000	20.00
66			0.00	99999999	0	f1 unused chips	.	.	0.000	0.00
67			1.00	150	0	f2 chips from mar	.	.	0.000	0.00
68			0.00	99999999	0	f2 unused chips	.	.	280.000	0.00
										==========
										-1295542.74

Output 5.8.3. CONOUT=CON7 (continued)

Obs	_RCOST_	_ANUMB_	_TNUMB_	_STATUS_	diagonal	factory	key_id	mth_made
1	.	1	1	KEY_ARC BASIC	19	1	production	April
2	66.150	2	3	LOWERBD NONBASIC	19	1	storage	March
3	42.580	3	4	LOWERBD NONBASIC	19	1	backorder	May
4	22.000	4	5	LOWERBD NONBASIC	19	.	f2_to_1	April
5	.	15	6	KEY_ARC BASIC	25	1	production	April
6	94.210	16	8	LOWERBD NONBASIC	25	1	storage	March
7	.	17	9	NONKEY ARC BASIC	25	1	backorder	May
8	-1.510	18	10	UPPERBD NONBASIC	25	.	f2_to_1	April
9	.	5	1	KEY_ARC BASIC	19	1	production	March
10	-17.070	6	2	UPPERBD NONBASIC	19	1	backorder	April
11	-34.750	7	11	UPPERBD NONBASIC	19	.	f2_to_1	March
12	-28.343	19	6	UPPERBD NONBASIC	25	1	production	March
13	-35.330	20	7	UPPERBD NONBASIC	25	1	backorder	April
14	-61.060	21	12	UPPERBD NONBASIC	25	.	f2_to_1	March
15	.	8	1	KEY_ARC BASIC	19	1	production	May
16	3.500	9	2	LOWERBD NONBASIC	19	1	storage	April
17	29.000	10	13	LOWERBD NONBASIC	19	.	f2_to_1	May
18	-15.520	22	6	UPPERBD NONBASIC	25	1	production	May
19	67.680	23	7	LOWERBD NONBASIC	25	1	storage	April
20	32.060	24	14	LOWERBD NONBASIC	25	.	f2_to_1	May
21	.	11	2	KEY_ARC BASIC	19	.	f1_to_2	April
22	-35.592	12	15	UPPERBD NONBASIC	19	2	production	April
23	13.400	13	11	LOWERBD NONBASIC	19	2	storage	March
24	43.980	14	13	LOWERBD NONBASIC	19	2	backorder	May
25	45.510	25	7	LOWERBD NONBASIC	25	.	f1_to_2	April
26	.	26	16	KEY_ARC BASIC	25	2	production	April
27	43.660	27	12	LOWERBD NONBASIC	25	2	storage	March
28	57.170	28	14	LOWERBD NONBASIC	25	2	backorder	May
29	55.750	29	3	LOWERBD NONBASIC	19	.	f1_to_2	March
30	.	30	15	KEY_ARC BASIC	19	2	production	March
31	25.480	31	5	LOWERBD NONBASIC	19	2	backorder	April
32	104.060	32	8	LOWERBD NONBASIC	25	.	f1_to_2	March
33	-23.170	33	16	UPPERBD NONBASIC	25	2	production	March
34	22.020	34	10	LOWERBD NONBASIC	25	2	backorder	April
35	.	35	4	KEY_ARC BASIC	19	.	f1_to_2	May
36	22.700	36	15	LOWERBD NONBASIC	19	2	production	May
37	6.500	37	5	LOWERBD NONBASIC	19	2	storage	April
38	6.940	38	9	LOWERBD NONBASIC	25	.	f1_to_2	May
39	.	39	16	KEY_ARC BASIC	25	2	production	May
40	46.110	40	10	LOWERBD NONBASIC	25	2	storage	April
41	.	41	3	KEY_ARC BASIC	19	1	sales	March
42	-23.500	42	2	UPPERBD NONBASIC	19	1	sales	April
43	.	43	4	NONKEY ARC BASIC	19	1	sales	May
44	-14.500	44	11	UPPERBD NONBASIC	19	2	sales	March
45	-2.500	45	5	UPPERBD NONBASIC	19	2	sales	April
46	9.000	46	13	LOWERBD NONBASIC	19	2	sales	May
47	79.150	47	8	LOWERBD NONBASIC	25	1	sales	March
48	40.420	48	7	LOWERBD NONBASIC	25	1	sales	April
49	.	49	9	KEY_ARC BASIC	25	1	sales	May
50	-9.980	50	12	UPPERBD NONBASIC	25	2	sales	March
51	.	51	10	KEY_ARC BASIC	25	2	sales	April
52	42.520	52	14	LOWERBD NONBASIC	25	2	sales	May
53	-37.090	53	3	UPPERBD NONBASIC	19	1	sales	March
54	-25.500	54	2	UPPERBD NONBASIC	19	1	sales	April
55	38.000	55	4	LOWERBD NONBASIC	19	1	sales	May
56	8.200	56	11	LOWERBD NONBASIC	19	2	sales	March
57	-26.500	57	5	UPPERBD NONBASIC	19	2	sales	April
58	.	58	13	KEY_ARC BASIC	19	2	sales	May
59	.	59	8	KEY_ARC BASIC	25	1	sales	March
60	.	60	7	KEY_ARC BASIC	25	1	sales	April
61	.	61	9	NONKEY ARC BASIC	25	1	sales	May
62	.	62	12	KEY_ARC BASIC	25	2	sales	March
63	-32.020	63	10	UPPERBD NONBASIC	25	2	sales	April
64	.	64	14	KEY_ARC BASIC	25	2	sales	May
65	.	-2	.	NONKEY BASIC	.	1	chips	
66	1.617	0	.	LOWERBD NONBASIC	.	1	chips	
67	2.797	-3	.	LOWERBD NONBASIC	.	2	chips	
68	.	-1	.	NONKEY BASIC	.	2	chips	

Output 5.8.4. DUALOUT=DUAL7

Nonarc Variables in the Side Constraints
Production Planning/Inventory/Distribution

Obs	_node_	_supdem_	_DUAL_	_NNUMB_	_PRED_	_TRAV_	_SCESS_
1	f1_apr_1	.	-100000275.50	2	1	5	2
2	f1_apr_2	.	-100000405.92	7	20	6	2
3	f1_mar_1	.	-100000326.65	3	17	1	20
4	f1_mar_2	.	-100000480.13	8	20	7	1
5	f1_may_1	.	-100000284.00	4	1	13	3
6	f1_may_2	.	-100000356.24	9	18	2	1
7	f2_apr_1	.	-100000286.50	5	2	4	1
8	f2_apr_2	.	-100000383.41	10	16	18	3
9	f2_mar_1	.	-100000281.90	11	15	16	1
10	f2_mar_2	.	-100000399.07	12	20	10	1
11	f2_may_1	.	-100000300.00	13	4	19	2
12	f2_may_2	.	-100000375.30	14	16	20	6
13	fact1_1	1000	-100000193.90	1	3	21	19
14	fact1_2	1000	-100000227.42	6	7	12	1
15	fact2_1	850	-100000193.90	15	21	11	2
16	fact2_2	1500	-100000193.90	16	21	14	10
17	shop1_1	-900	-99999999.00	17	22	3	21
18	shop1_2	-900	-99999841.22	18	10	9	2
19	shop2_1	-900	-100000001.00	19	13	22	1
20	shop2_2	-1450	-99999856.24	20	14	8	5
21		.	0.00	4	4	.	4
22		.	-1.00	2	2	.	.
23		.	-1.62	0	17	.	.
24		.	1.80	3	20	.	.
25		.	0.00	1	1	.	.
26		.	-0.48	5	7	.	.

Obs	_ARCID_	_FLOW_	_FBQ_	_VALUE_	_RHS_	_TYPE_	_row_
1	1	490.000	1	.	.		
2	-60	245.000	15	.	.		
3	-41	148.333	5	.	.		
4	-59	455.000	19	.	.		
5	8	66.667	8	.	.		
6	-49	347.500	22	.	.		
7	11	20.000	11	.	.		
8	26	542.500	25	.	.		
9	30	255.000	29	.	.		
10	-62	125.000	32	.	.		
11	35	115.000	35	.	.		
12	39	87.500	38	.	.		
13	-5	288.333	-1	.	.		
14	-15	200.000	-15	.	.		
15	-67	45.000	-41	.	.		
16	-68	150.000	-41	.	.		
17	0	0.000	41	.	.		
18	51	52.500	47	.	.		
19	58	150.000	53	.	.		
20	64	122.500	59	.	.		
21	.	260.000	.	260	0	GE	CHIP LEFTOVER
22	-2	20.000	.	2600	2600	EQ	FACT1 APL GIZMO
23	43	1.667	.	2615	2615	EQ	FACT1 MAR GIZMO
24	61	2.500	.	3750	3750	EQ	FACT2 APL GIZMO
25	-1	280.000	.	3750	3750	EQ	FACT2 MAR GIZMO
26	17	0.000	.	50	50	LE	TOTAL BACKORDER

The optimal value of the nonarc variable “f2 unused chips” is 280. This means that although there are 3750 chips that can be used at factory 2 in March, only 3470 are used. As the optimal value of “f1 unused chips” is zero, all chips available for production in March at factory 1 are used. The nonarc variable “f2 chips from mar” also has zero optimal value. This means that the April production at factory 2 does not need any chips that could have been held in inventory since March. However, the nonarc variable “f1 chips from mar” has value of 20. Thus, 3490 chips should be ordered for factory 2 in March. Twenty of these chips should be held in inventory until April, then sent to factory 1.

Example 5.9. Pure Networks: Using the EXCESS= Option

In this example we illustrate the use of the EXCESS= option for various scenarios in pure networks. Consider a simple network as shown in Figure 5.43. The positive numbers on the nodes correspond to supply and the negative numbers correspond to demand. The numbers on the arcs indicate costs.

Transportation Problem, Total Supply < Total Demand

We first analyze a simple transportation problem where total demand exceeds total supply, as seen in Figure 5.43. The EXCESS=SLACKS option is illustrated first.

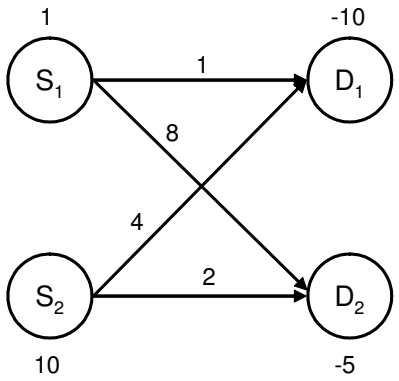


Figure 5.43. Transportation Problem

The following SAS code creates the input data sets.

```
data parcs;
   input _from_ $ _to_ $ _cost_;
datalines;
s1 d1 1
s1 d2 8
s2 d1 4
s2 d2 2
;

data SleD;
   input _node_ $ _sd_;
datalines;
s1   1
s2  10
d1 -10
d2  -5
;
```

You can solve the problem using the following call to PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   excess   = slacks
   arcdata  = parcs
   nodedata = SleD
   conout   = solex1;
run;
```

Since the EXCESS=SLACKS option is specified, the interior point method is used for optimization. Accordingly, the CONOUT= data set is specified. The optimal solution is displayed in Output 5.9.1.

Output 5.9.1. Supply < Demand

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	1	10	0.99914	0.9991
2	s2	d1	4	99999999	0	10	10	5.00746	20.0298
3	s1	d2	8	99999999	0	1	5	0.00086	0.0069
4	s2	d2	2	99999999	0	10	5	4.99254	9.9851

The solution with the THRUNET option specified is displayed in Output 5.9.2.

Output 5.9.2. Supply < Demand, THRUNET Specified

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	1	10	4.99891	4.9989
2	s2	d1	4	99999999	0	10	10	5.00109	20.0044
3	s1	d2	8	99999999	0	1	5	0.00019	0.0015
4	s2	d2	2	99999999	0	10	5	4.99981	9.9996

Note: If you want to use the network simplex solver instead, you need to specify the EXCESS=ARCS option and, accordingly, the ARCOUT= data set.

Missing D Demand

As shown in Figure 5.44, node D_1 has a missing D demand value.

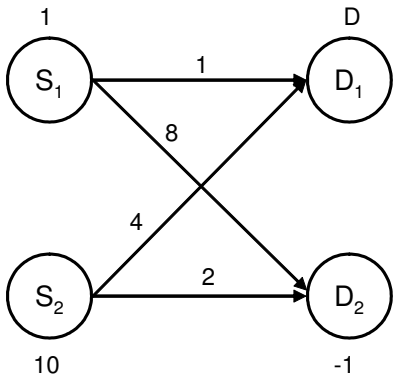


Figure 5.44. Missing D Demand

The following code creates the node data set:

```
data node_missingD1;
   input _node_ $ _sd_;
   missing D;
datalines;
s1   1
s2  10
d1   D
d2  -1
;
```

You can use the following call to PROC NETFLOW to solve the problem:

```
title1 'The NETFLOW Procedure';
proc netflow
   excess   = slacks
   arcdata  = parcs
   nodedata = node_missingD1
   conout   = solex1b;
run;
```

The optimal solution is displayed in Output 5.9.3. As you can see, the flow balance at nodes with nonmissing supdem values is maintained. In other words, if a node has a nonmissing supply (demand) value, then the sum of flows out of (into) that node is equal to its supdem value.

Output 5.9.3. THRUNET Not Specified

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	1	D	1	1
2	s2	d1	4	99999999	0	10	D	9	36
3	s1	d2	8	99999999	0	1	1	0	0
4	s2	d2	2	99999999	0	10	1	1	2

Missing D Demand, THRUNET Specified

Consider the previous example, but with the THRUNET option specified.

```
title1 'The NETFLOW Procedure';
proc netflow
   thrunet
   excess   = slacks
   arcdata  = parcs
   nodedata = node_missingD1
   conout   = solex1c;
run;
```

The optimal solution is displayed in Output 5.9.4. By specifying the THRUNET option, we have actually obtained the minimum-cost flow through the network, while maintaining flow balance at the nodes with nonmissing supply values.

Output 5.9.4. Missing D Demand, THRUNET Specified

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	1	D	0.99976	0.9998
2	s2	d1	4	99999999	0	10	D	0.00084	0.0034
3	s1	d2	8	99999999	0	1	1	0.00024	0.0019
4	s2	d2	2	99999999	0	10	1	9.99916	19.9983

Note: The case with missing S supply values is similar to the case with missing D demand values.

Example 5.10. Maximum Flow Problem

Consider the maximum flow problem depicted in Figure 5.45. The maximum flow between nodes S and T is to be determined. The minimum arc flow and arc capacities are specified as lower and upper bounds in square brackets, respectively.

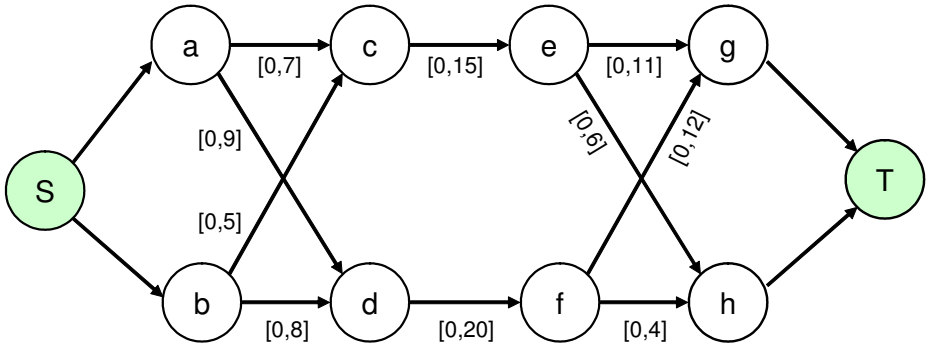


Figure 5.45. Maximum Flow Problem Example

You can solve the problem using either EXCESS=ARCS or EXCESS=SLACKS. Consider using the EXCESS=ARCS option first. You can use the following SAS code to create the input data set:

```
data arcs;
   input _from_ $ _to_ $ _cost_ _capac_;
datalines;
S a  .  .
S b  .  .
a c  1  7
b c  2  9
a d  3  5
b d  4  8
c e  5 15
d f  6 20
e g  7 11
f g  8  6
e h  9 12
f h 10  4
g T  .  .
h T  .  .
;
```

You can use the following call to PROC NETFLOW to solve the problem:

```
title1 'The NETFLOW Procedure';
proc netflow
   intpoint
   maxflow
   excess = arcs
   arcdata = arcs
   source  = S    sink = T
   conout  = gout3;
run;
```

With the EXCESS=ARCS option specified, the problem gets transformed internally to the one depicted in Figure 5.46. Note that there is an additional arc from the source node to the sink node.

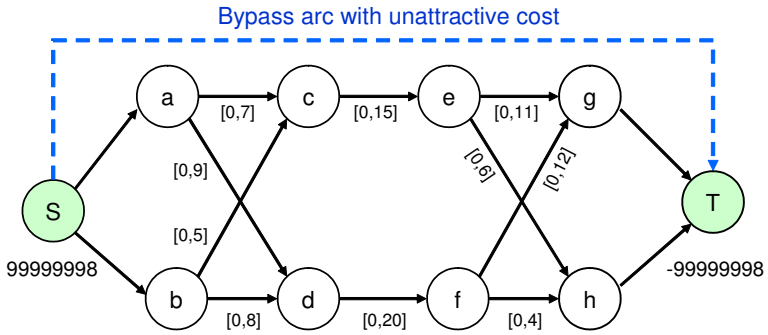


Figure 5.46. Maximum Flow Problem, EXCESS=ARCS Option Specified

The output SAS data set is displayed in Output 5.10.1.

Output 5.10.1. Maximum Flow Problem, EXCESS=ARCS Option Specified

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_capac_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	g	T	0	99999999	0	.	99999998	16.9996	0.0000
2	h	T	0	99999999	0	.	99999998	8.0004	0.0000
3	S	a	0	99999999	0	99999998	.	11.9951	0.0000
4	S	b	0	99999999	0	99999998	.	13.0049	0.0000
5	a	c	1	7	0	.	.	6.9952	6.9952
6	b	c	2	9	0	.	.	8.0048	16.0097
7	a	d	3	5	0	.	.	4.9999	14.9998
8	b	d	4	8	0	.	.	5.0001	20.0002
9	c	e	5	15	0	.	.	15.0000	75.0000
10	d	f	6	20	0	.	.	10.0000	60.0000
11	e	g	7	11	0	.	.	10.9996	76.9975
12	f	g	8	6	0	.	.	6.0000	48.0000
13	e	h	9	12	0	.	.	4.0004	36.0032
14	f	h	10	4	0	.	.	4.0000	40.0000

You can solve the same maximum flow problem, but this time with EXCESS=SLACKS specified. The SAS code is as follows:

```
title1 'The NETFLOW Procedure';
proc netflow
   intpoint
   excess  = slacks
   arcdata = arcs
   source  = S    sink = T
   maxflow
   conout  = gout3b;
run;
```

With the EXCESS=SLACKS option specified, the problem gets transformed internally to the one depicted in Figure 5.47. Note that the source node and sink node each have a single-ended "excess" arc attached to them.

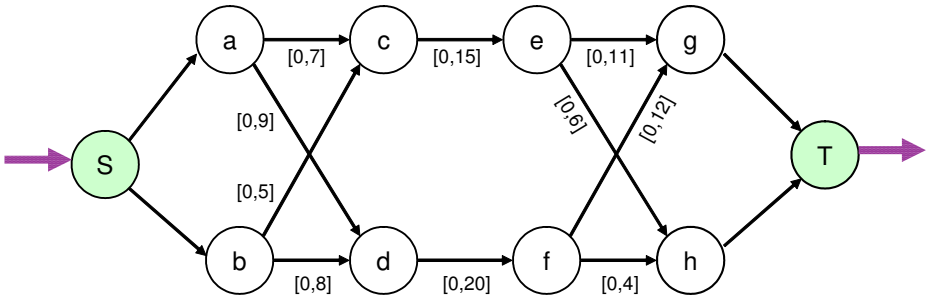


Figure 5.47. Maximum Flow Problem, EXCESS=SLACKS Option Specified

The solution, as displayed in Output 5.10.2, is the same as before. Note that the _SUPPLY_ value of the source node Y has changed from 99999998 to missing S, and the _DEMAND_ value of the sink node Z has changed from −99999998 to missing D.

Output 5.10.2. Maximal Flow Problem, EXCESS=SLACKS Option Specified

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_capac_	_LO_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	g	T	0	99999999	0	.	D	16.9993	0.0000
2	h	T	0	99999999	0	.	D	8.0007	0.0000
3	S	a	0	99999999	0	S	.	11.9867	0.0000
4	S	b	0	99999999	0	S	.	13.0133	0.0000
5	a	c	1	7	0	.	.	6.9868	6.9868
6	b	c	2	9	0	.	.	8.0132	16.0264
7	a	d	3	5	0	.	.	4.9999	14.9998
8	b	d	4	8	0	.	.	5.0001	20.0002
9	c	e	5	15	0	.	.	15.0000	75.0000
10	d	f	6	20	0	.	.	10.0000	60.0000
11	e	g	7	11	0	.	.	10.9993	76.9953
12	f	g	8	6	0	.	.	6.0000	48.0000
13	e	h	9	12	0	.	.	4.0007	36.0061
14	f	h	10	4	0	.	.	4.0000	40.0000

Example 5.11. Generalized Networks: Using the EXCESS= Option

For generalized networks you can specify either EXCESS=SUPPLY or EXCESS=DEMAND to indicate which nodal flow conservation constraints have slack variables associated with them. The default option is EXCESS=NONE.

Using the EXCESS=SUPPLY Option

Consider the simple network shown in Figure 5.48. As you can see, the sum of positive supdem values (35) is equal to the absolute sum of the negative ones. However, the arcs connecting the supply and demand nodes have varying arc multipliers. Let us now solve the problem using the EXCESS=SUPPLY option.

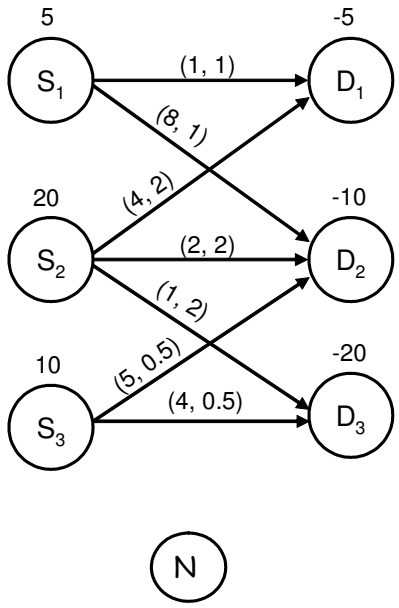


Figure 5.48. Generalized Network: Supply = Demand

You can use the following SAS code to create the input data sets:

```
data garcs;
   input _from_ $ _to_ $ _cost_ _mult_;
datalines;
s1 d1 1 .
s1 d2 8 .
s2 d1 4 2
s2 d2 2 2
s2 d3 1 2
s3 d2 5 0.5
s3 d3 4 0.5
;

data gnodes;
   input _node_ $ _sd_ ;
datalines;
s1  5
s2  20
s3  10
d1  -5
d2  -10
d3  -20
;
```

To solve the problem, use the following call to PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   arcdata  = garcs
   nodedata = gnodes
   excess   = supply
   conout   = gnetout;
run;
```

The optimal solution is displayed in Output 5.11.1.

Output 5.11.1. Optimal Solution Obtained Using the EXCESS

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_mult_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	1.0	5	5	5	5
2	s2	d1	4	99999999	0	2.0	20	5	0	0
3	s1	d2	8	99999999	0	1.0	5	10	0	0
4	s2	d2	2	99999999	0	2.0	20	10	5	10
5	s3	d2	5	99999999	0	0.5	10	10	0	0
6	s2	d3	1	99999999	0	2.0	20	20	10	10
7	s3	d3	4	99999999	0	0.5	10	20	0	0

Note: If you do not specify the EXCESS= option, or if you specify the EXCESS=DEMAND option, the procedure will declare the problem infeasible. Therefore, in case of real-life problems, you would need to have a little more detail about how the arc multipliers end up affecting the network — whether they tend to create excess demand or excess supply.

Using the EXCESS=DEMAND Option

Consider the previous example but with a slight modification: the arcs out of node S_1 have multipliers of 0.5, and the arcs out of node S_2 have multipliers of 1. You can use the following SAS code to create the input arc data set:

```
data garcs1;
   input _from_ $ _to_ $ _cost_ _mult_;
datalines;
s1 d1 1 0.5
s1 d2 8 0.5
s2 d1 4 .
s2 d2 2 .
s2 d3 1 .
s3 d2 5 0.5
s3 d3 4 0.5
;
```

Note that the node data set remains unchanged. You can use the following call to PROC NETFLOW to solve the problem:

```
title1 'The NETFLOW Procedure';
proc netflow
   arcdata  = garcs1
   nodedata = gnodes
   excess = demand
   conout   = gnetout1;
run;
```

The optimal solution is displayed in Output 5.11.2.

Output 5.11.2. Optimal Solution Obtained Using the EXCESS

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_mult_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	s1	d1	1	99999999	0	0.5	5	5	5.0000	5.0000
2	s2	d1	4	99999999	0	1.0	20	5	0.0001	0.0004
3	s1	d2	8	99999999	0	0.5	5	10	0.0000	0.0002
4	s2	d2	2	99999999	0	1.0	20	10	4.9999	9.9998
5	s3	d2	5	99999999	0	0.5	10	10	0.0004	0.0020
6	s2	d3	1	99999999	0	1.0	20	20	15.0000	15.0000
7	s3	d3	4	99999999	0	0.5	10	20	9.9996	39.9984

Example 5.12. Generalized Networks: Maximum Flow Problem

Consider the generalized network displayed in Figure 5.49. Lower and upper bounds of the flow are displayed in parentheses above the arc, and cost and multiplier, where applicable, are indicated in square brackets below the arc.

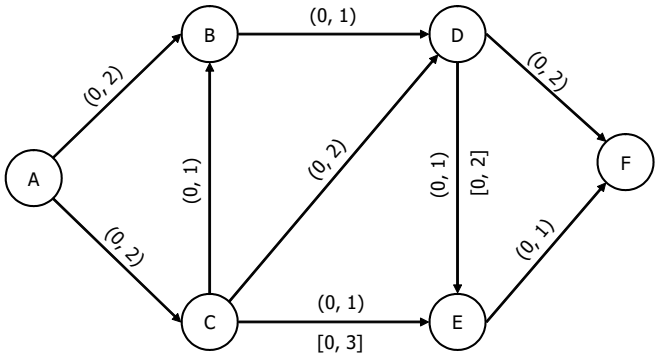


Figure 5.49. Generalized Maximum Flow Problem

You can enter the data for the problem using the following SAS code:

```
data garcsM;
   input _from_ $ _to_ $ _upper_ _mult_;
datalines;
A B 2 .
A C 2 .
C B 1 .
B D 1 .
C D 2 .
C E 1 3
D E 1 2
E F 5 .
D F 2 .
;
```

Use the following call to PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   arcdata = garcsM
   maxflow
   source  = A  sink = F
   conout  = gmfpout;
run;
```

The optimal solution is displayed in Output 5.12.1.

Output 5.12.1. Generalized Maximum Flow Problem: Optimal Solution

The NETFLOW Procedure

Obs	_from_	_to_	_COST_	_upper_	_LO_	_mult_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	A	B	0	2	0	1	S	.	0.99606	0
2	C	B	0	1	0	1	.	.	0.00202	0
3	A	C	0	2	0	1	S	.	1.99868	0
4	B	D	0	1	0	1	.	.	0.99808	0
5	C	D	0	2	0	1	.	.	0.99732	0
6	C	E	0	1	0	3	.	.	0.99934	0
7	D	E	0	1	0	2	.	.	0.99868	0
8	E	F	0	5	0	1	.	D	4.99539	0
9	D	F	0	2	0	1	.	D	0.99672	0

Example 5.13. Machine Loading Problem

Machine loading problems arise in a variety of applications. Consider a simple instance as described in Ahuja, Magnanti, and Orlin (1993). Assume you need to schedule the production of three products, $P1 - P3$, on four machines, $M1 - M4$. Suppose that machine 1 and machine 2 are each available for 40 hours and machine 3 and machine 4 are each available for 50 hours. Also, any of the machines can produce any product. The per-unit processing time and production cost for each product on each machine are indicated in Table 5.9.

Table 5.9. Processing Times and Production Costs

	M1	M2	M3	M4
p1	1	2	2	3
P2	2	3	2	1
P3	3	1	2	4

	M1	M2	M3	M4
p1	4	3	3	1
P2	0.5	2	0.5	3
P3	2	5	1	5

The problem is to satisfy the demands for the three products at minimum cost.

You can model this problem as a generalized network as shown in Figure 5.50. The network has three product nodes with demands indicated by positive supdem values and four machine nodes with availabilities (in hours) indicated by negative supdem values. The multiplier on an arc between a machine and a product indicates the hours of machine capacity needed to produce one unit of the product.

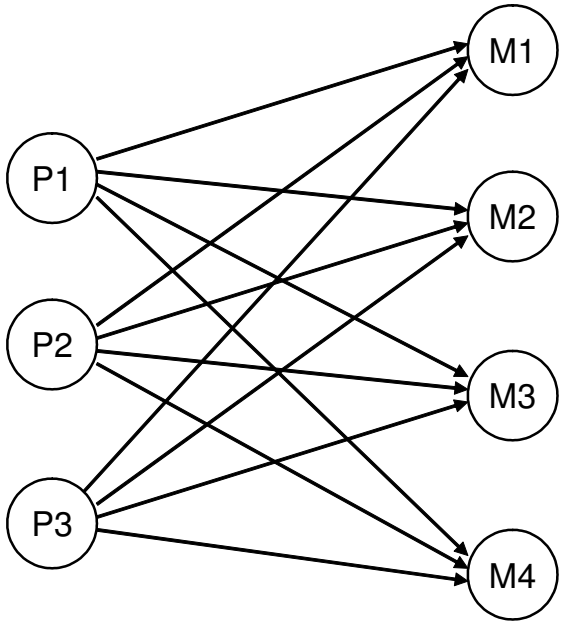


Figure 5.50. Machine Loading Problem

You can create the input data sets with the following SAS code:

```
data mlarcs;
   input _from_ $ _to_ $ _cost_ _mult_;
datalines;
P1  M1   4   .
P1  M2   3   2
P1  M3   3   2
P1  M4   1   3
P2  M1  .5   2
P2  M2   2   3
P2  M3  .5   2
P2  M4   3   1
P3  M1   2   3
P3  M2   5   .
P3  M3   1   2
P3  M4  .5   4
;

data mlnodes;
   input _node_ $ _sd_;
datalines;
P1   10
```

```
P2   5
P3   10
M1  -40
M2  -40
M3  -50
M4  -50
;
```

You can solve the problem using the following call to PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   excess   = demand
   arcdata  = mlarcs
   nodedata = mlnodes
   conout   = mlsol;
run;
```

The optimal solution, as displayed in Output 5.13.1, can be interpreted as follows:

- Product 1: 10 units on machine 4
- Product 2: 3 units on machine 1, and 2 units on machine 3
- Product 3: 5 units on machine 3, and 5 units on machine 4

Output 5.13.1. Optimum Solution to the Machine Loading Problem

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_CAPAC_	_LO_	_mult_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	P1	M1	4.0	99999999	0	1	10	40	0.00000	0.00000
2	P2	M1	0.5	99999999	0	2	5	40	3.01032	1.50516
3	P3	M1	2.0	99999999	0	3	10	40	0.00001	0.00002
4	P1	M2	3.0	99999999	0	2	10	40	0.00001	0.00004
5	P2	M2	2.0	99999999	0	3	5	40	0.00001	0.00002
6	P3	M2	5.0	99999999	0	1	10	40	0.00000	0.00000
7	P1	M3	3.0	99999999	0	2	10	50	0.00001	0.00003
8	P2	M3	0.5	99999999	0	2	5	50	1.98966	0.99483
9	P3	M3	1.0	99999999	0	2	10	50	4.99999	4.99999
10	P1	M4	1.0	99999999	0	3	10	50	9.99997	9.99997
11	P2	M4	3.0	99999999	0	1	5	50	0.00000	0.00000
12	P3	M4	0.5	99999999	0	4	10	50	5.00000	2.50000

Example 5.14. Generalized Networks: Distribution Problem

Consider a distribution problem (from Jensen and Bard 2003) with three supply plants ($S_1 - S_3$) and five demand points ($D_1 - D_5$). Further information about the problem is as follows:

S_1 To be closed. Entire inventory must be shipped or sold to scrap. The scrap value is \$5 per unit.

S_2 Maximum production of 300 units with manufacturing cost of \$10 per unit.

S_3 The production in regular time amounts to 200 units and must be shipped. An additional 100 units can be produced using overtime at \$14 per unit.

D_1 Fixed demand of 200 units must be met.

D_2 Contracted demand of 300 units. An additional 100 units can be sold at \$20 per unit.

D_3 Minimum demand of 200 units. An additional 100 units can be sold at \$20 per unit. Additional units can be procured from D_4 at \$4 per unit. There is a 5% "shipping loss" on the arc connecting these two nodes.

D_4 Fixed demand of 150 units must be met.

D_5 100 units left over from previous shipments. No firm demand, but up to 250 units can be sold at \$25 per unit.

Additionally, there is a 5% "shipping loss" on each of the arcs between supply and demand nodes.

You can model this scenario as a generalized network. Since there are both fixed and varying supply and demand supdem values, you can transform this to a case where you need to address missing supply and demand simultaneously. As seen from Figure 5.51, we have added two artificial nodes, Y and Z, with missing S supply value and missing D demand value, respectively. The extra production capability is depicted by arcs from node Y to the corresponding supply nodes, and the extra revenue generation capability of the demand points (and scrap revenue for S_1) is depicted by arcs to node Z.

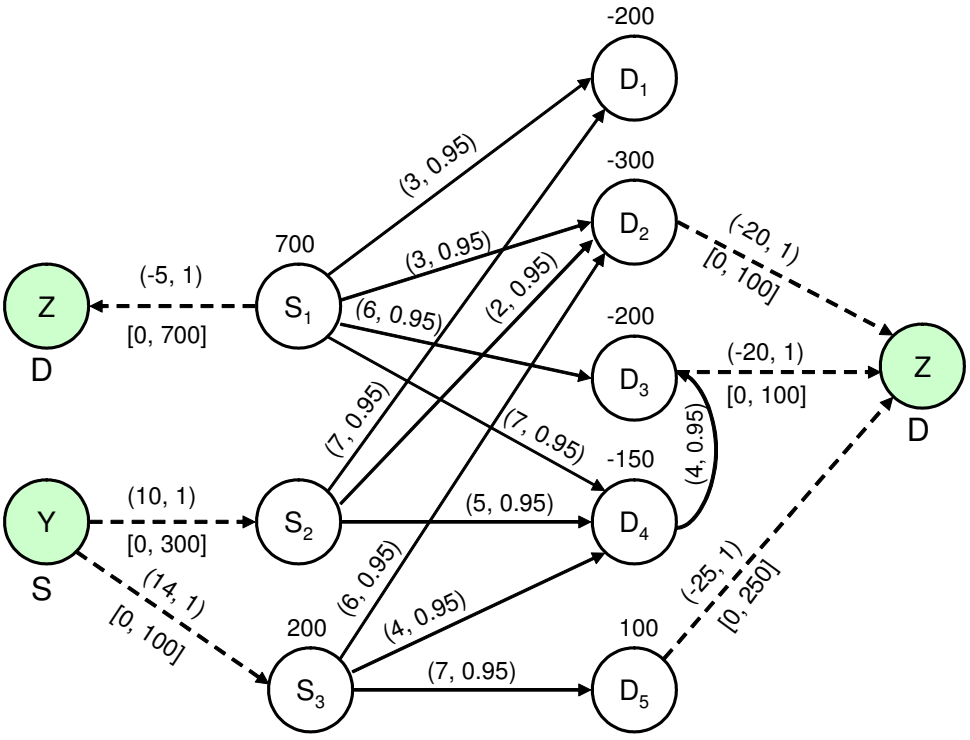


Figure 5.51. Distribution Problem

The following SAS data set has the complete information about the arc costs, multipliers, and node supdem values:

```
data dnodes;
   input _node_ $ _sd_ ;
   missing S D;
datalines;
S1   700
S2     0
S3   200
D1  -200
D2  -300
D3  -200
D4  -150
D5   100
Y      S
Z      D
;

data darcs;
   input _from_ $ _to_ $ _cost_ _capac_ _mult_;
datalines;
S1  D1    3   200  0.95
S1  D2    3   200  0.95
S1  D3    6   200  0.95
```

```
S1  D4    7   200  0.95
S2  D1    7   200  0.95
S2  D2    2   200  0.95
S2  D4    5   200  0.95
S3  D2    6   200  0.95
S3  D4    4   200  0.95
S3  D5    7   200  0.95
D4  D3    4   200  0.95
Y   S2   10   300  .
Y   S3   14   100  .
S1  Z    -5   700  .
D2  Z   -20   100  .
D3  Z   -20   100  .
D5  Z   -25   250  .
;
```

You can solve this problem by using the following call to PROC NETFLOW:

```
title1 'The NETFLOW Procedure';
proc netflow
   nodedata = dnodes
   arcdata  = darcs
   conout   = dsol;
run;
```

The optimal solution is displayed in Output 5.14.1.

Output 5.14.1. Distribution Problem: Optimal Solution

The NETFLOW Procedure

Obs	_from_	_to_	_cost_	_capac_	_LO_	_mult_	_SUPPLY_	_DEMAND_	_FLOW_	_FCOST_
1	S1	D1	3	200	0	0.95	700	200	200.000	600.00
2	S2	D1	7	200	0	0.95	.	200	10.526	73.68
3	S1	D2	3	200	0	0.95	700	300	200.000	600.00
4	S2	D2	2	200	0	0.95	.	300	200.000	400.00
5	S3	D2	6	200	0	0.95	200	300	21.053	126.32
6	S1	D3	6	200	0	0.95	700	200	200.000	1200.00
7	D4	D3	4	200	0	0.95	.	200	10.526	42.11
8	S1	D4	7	200	0	0.95	700	150	100.000	700.00
9	S2	D4	5	200	0	0.95	.	150	47.922	239.61
10	S3	D4	4	200	0	0.95	200	150	21.053	84.21
11	S3	D5	7	200	0	0.95	200	.	157.895	1105.26
12	Y	S2	10	300	0	1.00	S	.	258.449	2584.49
13	Y	S3	14	100	0	1.00	S	.	0.000	0.00
14	S1	Z	-5	700	0	1.00	700	D	0.000	0.00
15	D2	Z	-20	100	0	1.00	.	D	100.000	-2000.00
16	D3	Z	-20	100	0	1.00	.	D	0.000	-0.00
17	D5	Z	-25	250	0	1.00	100	D	250.000	-6250.00
										========
										-494.32

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Chapter 6
The OPTMODEL Procedure

Chapter Contents

Chapter 6
The OPTMODEL Procedure

Overview: OPTMODEL Procedure

The OPTMODEL procedure comprises the powerful OPTMODEL modeling language and state-of-the-art solvers for several classes of mathematical programming problems. The problems and their solvers are listed in Table 6.1.

Table 6.1. Solvers in PROC OPTMODEL

Problem	Solver
Linear Programming	**LP**
Mixed Integer Linear Programming	**MILP**
Quadratic Programming	**QP** (experimental)
Nonlinear Programming, Unconstrained	**NLPU**
General Nonlinear Programming	**NLPC**
General Nonlinear Programming	**SQP**
General Nonlinear Programming	**IPNLP** (experimental)

The OPTMODEL modeling language provides a modeling environment tailored to building, solving, and maintaining optimization models. This makes the process of translating the symbolic formulation of an optimization model into OPTMODEL virtually transparent since the modeling language mimics the symbolic algebra of the formulation as closely as possible. The OPTMODEL language also streamlines and simplifies the critical process of populating optimization models with data from SAS data sets. All of this transparency produces models that are more easily inspected for completeness and correctness, more easily corrected, and more easily modified, whether through structural changes or through the substitution of new data for old.

In addition to invoking optimization solvers directly with PROC OPTMODEL as already mentioned, you can use the OPTMODEL language purely as a modeling facility. You can save optimization models built with the OPTMODEL language in SAS data sets that can be submitted to other optimization procedures in SAS/OR. In general, the OPTMODEL language serves as a common point of access for many of the optimization capabilities of SAS/OR, whether providing both modeling and solver access or acting as a modeling interface for other optimization procedures.

For details and examples of the problems addressed and corresponding solvers, please see the dedicated chapters in this book. This chapter aims to give you a comprehensive understanding of the OPTMODEL procedure by discussing the framework provided by the OPTMODEL modeling language.

The OPTMODEL modeling language features automatic differentiation, advanced flow control, optimization-oriented syntax (parameters, variables, arrays, constraints, objective functions), dynamic model generation, model-data separation, and transparent access to SAS data sets.

Getting Started: OPTMODEL Procedure

Optimization or mathematical programming is a search for a maximum or minimum of an *objective function* (also called a *cost function*), where search variables are restricted to particular constraints. Constraints are said to define a *feasible region* (see Figure 6.1).

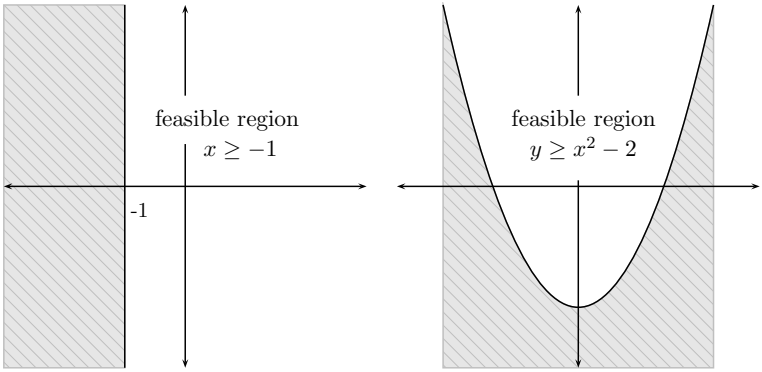


Figure 6.1. Examples of Feasible Regions

A more rigorous general formulation of such problems is as follows.

Let

$$f : S \to \mathbb{R}$$

be a real-valued function. Find x^* such that

- $x^* \in S$
- $f(x^*) \leq f(x), \quad \forall x \in S$

Note that the formulation is for the minimum of f and that the maximum of f is simply the negation of the minimum of $-f$.

Here, function f is the *objective function*, and the variable in the objective function is called the optimization variable (or decision variable). S is the *feasible region*. Typically S is a subset of the Euclidean space $\mathbb{R}^n$ specified by the set of *constraints*, which are often a set of equalities ($=$) or inequalities ($\leq, \geq$) that every element in S is required to satisfy simultaneously. For the special case where $S = \mathbb{R}^n$, the problem is an *unconstrained optimization*. An element x of S is called a *feasible solution* to the optimization problem, and the value $f(x)$ is called the *objective value*. A feasible solution x^* that minimizes the objective function is called an *optimal solution* to the optimization problem, and the corresponding objective value is called the *optimal value*.

In mathematics, special notation is used to denote an optimization problem. Generally, we can write an optimization problem as follows:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in S \end{aligned}$$

Normally, an empty body of constraint (the part after "subject to") implies that the optimization is unconstrained (i.e., the feasible region is the whole space $\mathbb{R}^n$). The optimal solution (x^*) is denoted as

$$x^* = \underset{x \in S}{\operatorname{argmin}} f(x)$$

The optimal value ($f(x^*)$) is denoted as

$$f(x^*) = \min_{x \in S} f(x)$$

Optimization problems can be classified by the forms (linear, quadratic, nonlinear, etc.) of the functions in the objective and constraints. For example, a problem is said to be *linearly constrained* if the functions in the constraints are linear. A *linear programming* problem is a linearly constrained problem with a linear objective function. A nonlinear programming problem occurs where some function in the objective or constraints is nonlinear, etc.

An Unconstrained Optimization Example

An unconstrained optimization problem formulation is simply

$$\text{minimize} \quad f(x)$$

For example, suppose you wanted to find the minimum value of this polynomial:

$$z(x, y) = x^2 - x - 2y - xy + y^2$$

You can compactly specify and solve the optimization problem by using the OPTMODEL modeling language. Here is the program:

```
/* invoke procedure */
proc optmodel;
   var x, y;  /* declare variables */

   /* objective function */
   min z=x**2 - x - 2*y - x*y + y**2;

   /* now run the solver */
   solve;

   print x y;
   quit;
```

This program produces the output in Figure 6.2.

```
                The OPTMODEL Procedure

                   Problem Summary

Objective Sense             Minimization
Objective Function                     z
Objective Type                 Quadratic

Number of Variables                    2
Bounded Above                          0
Bounded Below                          0
Bounded Below and Above                0
Free                                   2
Fixed                                  0

Number of Constraints                  0


                The OPTMODEL Procedure

                   Solution Summary

   Solver                        L-BFGS
   Objective Function                 z
   Solution Status              Optimal
   Objective Value              -2.3333
   Iterations                         1

   Optimality Error       1.3718109E-7


                     x           y

                1.3333      1.6667
```

Figure 6.2. Optimizing a Simple Polynomial

In PROC OPTMODEL you specify the mathematical formulas that describe the behavior of the optimization problem that you want to solve. In the preceding example there were two independent variables in the polynomial, x and y. These are the *optimization variables* of the problem. In PROC OPTMODEL you declare optimization variables with the VAR statement. The formula that defines the quantity that you are seeking to optimize is called the *objective function*, or *objective*. The solver varies the values of the optimization variables when searching for an optimal value for the objective.

In the preceding example the objective function is named z, declared with the MIN statement. The keyword MIN is an abbreviation for MINIMIZE. The expression that follows the equal sign (=) in the MIN statement defines the function to be minimized in terms of the optimization variables.

The VAR and MIN statements are just two of the many available PROC OPTMODEL declaration and programming statements. PROC OPTMODEL processes all such statements interactively, meaning that each statement is processed as soon as it is

complete.

After PROC OPTMODEL has completed processing of declaration and programming statements, it processes the SOLVE statement, which submits the problem to a solver and prints a summary of the results. The PRINT statement displays the optimal values of the optimization variables x and y found by the solver.

It is worth noting that PROC OPTMODEL does not use a RUN statement but instead operates on an interactive basis throughout. You can continue to interact with PROC OPTMODEL even after invoking a solver. For example, you could modify the problem and issue another SOLVE statement (see the section "Model Update" on page 769).

The Rosenbrock Problem

You can use parameters to produce a clear formulation of a problem. Consider the Rosenbrock problem:

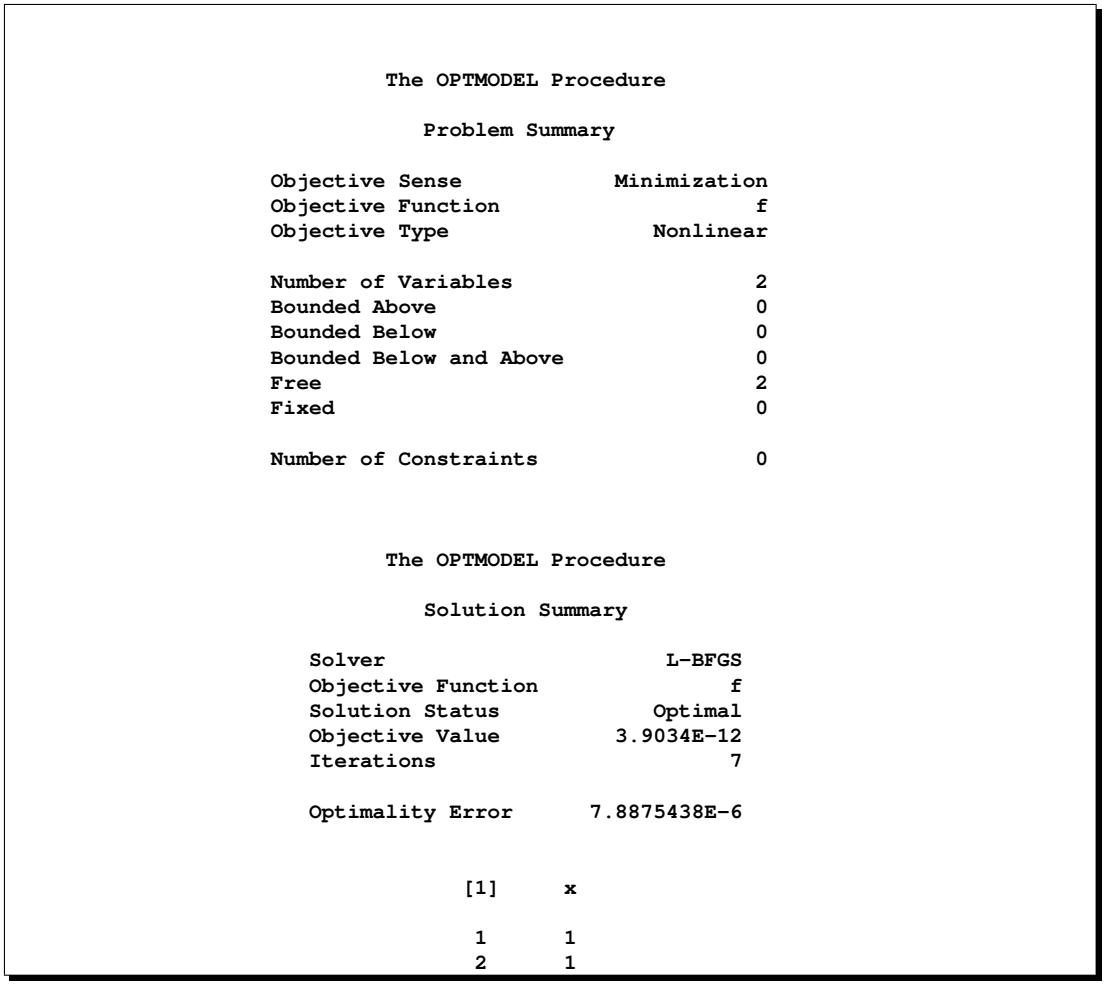
$$\text{minimize } f(x_1, x_2) = \alpha\,(x_2 - x_1^2)^2 + (1 - x_1)^2$$

where $\alpha = 100$ is a parameter (constant), x_1 and x_2 are optimization variables (whose values are to be determined), and $f(x_1, x_2)$ is an objective function.

Here is a PROC OPTMODEL program that solves the Rosenbrock problem:

```
proc optmodel;

   number alpha = 100; /* declare parameter */
   var x {1..2};       /* declare variables */

   /* objective function */
   min f = alpha*(x[2] - x[1]**2)**2 +
           (1 - x[1])**2;

   /* now run the solver */
   solve;

   print x;
   quit;
```

The PROC OPTMODEL output is shown in Figure 6.3.



The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Nonlinear
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	2
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	L-BFGS
Objective Function	f
Solution Status	Optimal
Objective Value	3.9034E-12
Iterations	7
Optimality Error	7.8875438E-6

[1]	x
1	1
2	1

Figure 6.3. Rosenbrock Function Results

A Transportation Problem

You can easily translate the symbolic formulation of a problem into the OPTMODEL procedure. Let's consider the transportation problem, which is mathematically modeled as the following linear programming problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i \in O, j \in D} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j \in D} x_{ij} = a_i, \quad \forall i \in O \qquad \text{(SUPPLY)} \\
& \sum_{i \in O} x_{ij} = b_j, \quad \forall j \in D \qquad \text{(DEMAND)} \\
& x_{ij} \geq 0, \quad \forall (i,j) \in O \times D
\end{aligned}
$$

where O is the set of origins, D is the set of destinations, c_{ij} is the cost to transport one unit from i to j, a_i is the supply of origin i, b_j is the demand of destination j, and x_{ij} is the decision variable for the amount of shipment from i to j.

Here is a very simple example. The cities in the set O of origins are Detroit and Pittsburgh. The cities in the set D of destinations are Boston and New York. The cost matrix, supply, and demand are shown in Table 6.2.

Table 6.2. A Transportation Problem

	Boston	New York	Supply
Detroit	30	20	200
Pittsburgh	40	10	100
Demand	150	150	

The problem is compactly and clearly formulated and solved by using the OPTMODEL procedure with the following code:

```
proc optmodel;

   /* specify parameters */
   set O={'Detroit','Pittsburgh'};
   set D={'Boston','New York'};
   number c{O,D}=[30 20
                  40 10];
   number a{O}=[200 100];
   number b{D}=[150 150];

   /* model description */
   var x{O,D} >= 0;
   min total_cost = sum{i in O, j in D}c[i,j]*x[i,j];
   constraint supply{i in O}: sum{j in D}x[i,j]=a[i];
   constraint demand{j in D}: sum{i in O}x[i,j]=b[j];

   /* solve and output */
```

```
solve;
print x;
```

The output is shown in Figure 6.4.

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	total_cost
Objective Type	Linear
Number of Variables	4
Bounded Above	0
Bounded Below	4
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	4
Linear LE (<=)	0
Linear EQ (=)	4
Linear GE (>=)	0
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	Dual Simplex
Objective Function	total_cost
Solution Status	Optimal
Objective Value	6500
Iterations	0
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0

x

	Boston	New York
Detroit	150	50
Pittsburgh	0	100

Figure 6.4. Solution to the Transportation Problem

OPTMODEL Modeling Language: Basic Concepts

As seen from the examples in the previous section, a PROC OPTMODEL model consists of one or more declarations of variables, objectives, constraints, and parameters, as well as possibly intermixed programming statements, which use the components that are created by the declarations. The declarations define the mathematical form of the problem to solve. The programming statements define data values, invoke the solver, or print the results. This section describes some basic concepts, such as variables, indices, etc., which are used in the section "Syntax: OPTMODEL Procedure" on page 681.

Named Parameters

In the example described in the section "An Unconstrained Optimization Example" on page 665, all the numeric constants that describe the behavior of the objective function were specified directly in the objective expression. This is a valid way to formulate the objective expression. However, in many cases it is inconvenient to specify the numeric constants directly. Direct specification of numeric constants can also hide the structure of the problem that is being solved. The objective expression text would need to be modified when the numeric values in the problem change. This can be very inconvenient with large models.

In PROC OPTMODEL, you can create named numeric values that behave as constants in expressions. These named values are called *parameters*. You can write an expression by using mnemonic parameter names in place of numeric literals. This produces a clearer formulation of the optimization problem. You can easily modify the values of parameters, define them in terms of other parameters, or read them from a SAS data set.

The model from this same example can be reformulated in a more general polynomial form, as follows:

```
data coeff;
   input c_xx c_x c_y c_xy c_yy;
   datalines;
   1 -1  -2 -1 1
   ;
proc optmodel;
   var x, y;
   number c_xx, c_x, c_y, c_xy, c_yy;
   read data coeff into c_xx c_x c_y c_xy c_yy;
   min z=c_xx*x**2 + c_x*x + c_y*y + c_xy*x*y + c_yy*y**2;
   solve;
```

This code reads the coefficients from a data set, COEFF. The NUMBER statement declares the parameters. The READ DATA statement reads the parameters from the data set. You can apply this model easily to coefficients that you have generated by various means.

Indexing

Many models have large numbers of variables or parameters that can be categorized into families of similar purpose or behavior. Such families of items can be compactly represented in PROC OPTMODEL by using indexing. You can use indexing to assign each item in such families to a separate value location.

PROC OPTMODEL indexing is similar to array indexing in the DATA step but more flexible. Index values can be numbers or strings, and are not required to fit into some rigid sequence. PROC OPTMODEL indexing is based on index sets, described further in the section "Index Sets" on page 680. For example, the following code declares an indexed parameter:

```
number p{1..3};
```

The construct that follows the parameter name p, "{1..3}," is a simple index set that uses a range expression (see "Range Expression" on page 734). The index set contains the numeric members 1, 2, and 3. The parameter has distinct value locations for each of the index set members. The first such location is referenced as p[1], the second as p[2], and the third as p[3].

The following code shows an example of indexing:

```
proc optmodel;
   number p{1..3};
   p[1]=5;
   p[2]=7;
   p[3]=9;
   put p[*]=;
```

The preceding code produces a line like the one shown in Figure 6.5 in the log.

```
p[1]=5 p[2]=7 p[3]=9
```

Figure 6.5. Indexed Parameter Output

Index sets can also specify local dummy parameters. A dummy parameter can be used as an operand in the expressions that are controlled by the index set. For example, the assignment statements in the preceding code could be replaced by an initialization in the parameter declaration, as follows:

```
number p{i in 1..3} init 3 + 2*i;
```

The initialization value of the parameter location p[1] is evaluated with the value of the local dummy parameter i equal to 1. So the initialization expression 3 + 2*i evaluates to 5. Similarly for location p[2], the value of i is 2 and the initialization expression evaluates to 7.

The OPTMODEL modeling language supports aggregation operators that combine values of an expression where a local dummy parameter (or parameters) ranges over the members of a set. For example, the SUM aggregation operator combines expression values by adding them together. The following code outputs 21, since p[1] + p[2] + p[3] = 5 + 7 + 9 = 21:

```
proc optmodel;
   number p{i in 1..3} init 3 + 2*i;
   put (sum{i in 1..3} p[i]);
```

Aggregation operators like SUM are especially useful in objective expressions because they can combine a large number of similar expressions into a compact representation. As an example, the following code defines a trivial least-squares problem:

```
proc optmodel;
   number n init 100000;
   var x{1..n};
   min z = sum{i in 1..n}(x[i] - log(i))**2;
   solve;
```

The objective function in this case is

$$z = \sum_{i=1}^{n} (x_i - \log i)^2$$

Effectively, the objective expression expands to the following large expression:

```
min z = (x[1] - log(1))**2
      + (x[2] - log(2))**2
      . . .
      + (x[99999] - log(99999))**2
      + (x[100000] - log(100000))**2;
```

Even though the problem has 100,000 variables, the aggregation operator SUM enables a compact objective expression.

Note: PROC OPTMODEL classifies as mathematically impure any function that returns a different value each time it is called. The RAND function, for example, falls into this category. PROC OPTMODEL disallows impure functions inside array index sets, objectives, and constraint expressions.

Types

In PROC OPTMODEL, parameters and expressions can have numeric or character values. These correspond to the elementary types named NUMBER and STRING, respectively. The NUMBER type is the same as the SAS data set numeric type. The NUMBER type includes support for missing values. The STRING type corresponds to the SAS character type, except that strings can have lengths up to a maximum of 65,534 characters (versus 32,767 for SAS character-type variables). The NUMBER and STRING types together are called the *scalar types*. You can abbreviate the type names as NUM and STR, respectively.

PROC OPTMODEL also supports set types for parameters and expressions. Sets represent collections of values of a member type, which can be a NUMBER, a STRING, or a vector of scalars (the latter is called a *tuple* and described in the following paragraphs). Members of a set all have the same member type. Members that have the same value are stored only once. For example, PROC OPTMODEL stores the set 2, 2, 2 as the set 2.

Specify a set of numbers with SET<NUMBER>. Similarly, specify a set of strings as SET<STRING>.

A set can also contain a collection of tuples, all of the same fixed length. A *tuple* is an ordered collection that contains a fixed number of elements. Each element in a tuple contains a scalar value. In PROC OPTMODEL, tuples of length 1 are equivalent to scalars. Two tuples have equal values if the elements at corresponding positions in each tuple have the same value. Within a set of tuples, the element type at a particular position in each tuple is the same for all set members. The element types are part of the set type. For example, the following statement declares parts as a set of tuples that have a string in the first element position and a number in the second element position and then initializes its elements to be <R 1>, <R 2>, <C 1>, and <C 2>.

```
set<string,number> parts = /<R 1> <R 2> <C 1> <C 2>/;
```

To create a compact model, use sets to take advantage of the structure of the problem being modeled. For example, a model might contain various values that specify attributes for each member of a group of suppliers. You could create a set that contains members representing each supplier. You can then model the attribute values by using arrays that are indexed by members of the set.

The section "Parameters" on page 675 has more details and examples.

Names

Names are used in the OPTMODEL modeling language to refer to various entities such as parameters or variables. Names must follow the usual rules for SAS names. Names can be up to 32 characters long and are not case sensitive. They must be declared before they are used.

Avoid declarations with names that begin with an underscore (_). These names can have special uses in PROC OPTMODEL.

Parameters

In the OPTMODEL modeling language, parameters are named locations that hold constant values. Parameter declarations specify the parameter type followed by a list of parameter names to declare. For example, the following code declares numeric parameters named a and b:

```
number a, b;
```

Similarly, the following code declares a set s of strings, a set n of numbers, and a set sn of tuples:

```
set<string> s;
set<number> n;
set<string, number> sn;
```

You can assign values to parameters in various ways. A parameter can be assigned a value with an assignment statement. For example, the following statement assigns a value to the parameter s, n, and sn in the preceding declaration:

```
s = {'a', 'b', 'c'};
n = {1, 2, 3};
sn = {<'a',1>, <'b',2>, <'c',3>};
```

Parameter values can also be assigned using a READ DATA statement (see the section "READ DATA Statement" on page 718).

A parameter declaration can provide an explicit value. To specify the value, follow the parameter name with an equal sign (=) and an expression. The value expression can be written in terms of other parameters. The declared parameter takes on a new value each time a parameter that is used in the expression changes. This automatic value update is shown in the following example:

```
proc optmodel;
   number pi=4*atan(1);
   number r;
   number circum=2*pi*r;
   r=1;
   put circum;       /* prints 6.2831853072 */
   r=2;
   put circum;       /* prints 12.566370614 */
```

The automatic update of parameter values makes it easy to perform "what if" analysis since, after the solver finds a solution, you can change parameters and reinvoke the solver. You can easily examine the effects of the changes on the optimal values.

If you declare a set parameter that has only the SET type specifier, then the element type is determined from the initialization expression. If the initialization expression is omitted or if the expression is an empty set, then the set type defaults to

SET<NUMBER>. For example, the following code implicitly declares s1 as a set of numbers:

```
set s1;
```

The following code declares s2 as a set of strings:

```
set s2 = {'A'};
```

You can declare an array parameter by following the parameter name with an index set specification (see the section "Index Sets" on page 680). For example, declare an array of 10 numbers as follows:

```
number c{1..10};
```

Individual locations of a parameter array can be referred to with an indexing expression. For example, you can refer to the third location of parameter c as c[3]. Array index sets *cannot* be specified using a function such as RAND that returns a different value each time it is called.

Parameter names must be declared before they are used. Nonarray names become available at the end of the parameter declaration item. Array names become available after the index set specification. The latter case permits some forms of recursion in the optional initialization expression that can be supplied for a parameter.

You do not need to assign values to parameters before they are referenced. Most information in PROC OPTMODEL is stored symbolically and resolved when necessary. Values are resolved in certain statements. For example, PROC OPTMODEL resolves a parameter used in the objective during the execution of a SOLVE statement. If no value is available during resolution, then an error is diagnosed.

Expressions

Expressions are grouped into three categories based on the types of values they can produce: logical, set, and scalar (i.e., numeric or character).

Logical expressions test for a Boolean (true or false) condition. As in the DATA step, logical operators produce a value equal to either 0 or 1. A value of 0 represents a false condition, while a value of 1 represents a true condition.

Logical expression operators are not allowed in certain contexts due to syntactic considerations. For example, in the VAR statement a logical operator might indicate the start of an option. Enclose a logical expression in parentheses to use it in such contexts. The difference is illustrated by the output of the following code (Figure 6.6), where two variables, x and y, are declared with initial values. The PRINT statement and the EXPAND statement are used to check the initial values and the variable bounds, respectively.

```
proc optmodel;
   var x init 0.5 >= 0 <= 1;
   var y init (0.5 >= 0) <= 1;
   print x y;
   expand;
```

```
                        The OPTMODEL Procedure

                                  x     y

                                0.5     1

Var x >= 0 <= 1
Var y <= 1
```

Figure 6.6. Logical Expression in the VAR Statement

Contexts that expect a logical expression also accept numeric expressions. In such cases zero or missing values are interpreted as false, while all nonzero nonmissing numeric values are interpreted as true.

Set expressions return a set value. PROC OPTMODEL supports a number of operators that create and manipulate sets. See the section "OPTMODEL Expression Extensions" on page 729 for a description of the various set expressions. *Index-set* syntax is described in the section "Index Sets" on page 680.

Scalar expressions are similar to the expressions in the DATA step except for PROC OPTMODEL extensions. PROC OPTMODEL provides an IF expression (described in the section "IF-THEN/ELSE Expression" on page 730). String lengths are assigned dynamically, so there is generally no padding or truncation of string values.

Table 6.3 shows the expression operators from lower to higher precedence (a higher precedence is given a larger number). Operators that have higher precedences are applied in compound expressions before operators that have lower precedence. The table also gives the order of evaluation that is applied when multiple operators of the same precedence are used together. Operators available in both PROC OPTMODEL and the DATA step have compatible precedences, except that in PROC OPTMODEL the NOT operator has a lower precedence than the relational operators. This means that, for example, `NOT 1 < 2` is equal to `NOT (1 < 2)` (which is 0), rather than `(NOT 1) < 2` (which is 1).

Table 6.3. Expression Operator Table

Precedence	Associativity	Operator	Alternates
logic expression operators			
1	left to right	**OR**	**| !**
2	unary	**OR{***index-set***}**	
		AND{*index-set***}**	
3	left to right	**AND**	**&**

Table 6.3. (continued)

Precedence	Associativity	Operator	Alternates
4	unary	**NOT**	**~ ^ ¬**
5	left to right	**<**	**LT**
		>	**GT**
		<=	**LE**
		>=	**GE**
		=	**EQ**
		~=	**NE ^= ¬=**
6	left to right	**IN**	
		NOT IN	
7	left to right	**WITHIN**	
		NOT WITHIN	
set expression operators			
11		**IF** *l* **THEN** *s1* **ELSE** *s2*	
12	left to right	**UNION**	
		DIFF	
		SYMDIFF	
13	unary	**UNION{***index-set***}**	
14	left to right	**INTER**	
15	unary	**INTER{***index-set***}**	
16	left to right	**CROSS**	
17	unary	**SETOF{***index-set***}**	
	right to left	**..**	**TO**
		.. *e* **BY**	**TO** *e* **BY**
scalar expression operators			
21		**IF** *l* **THEN** *e*	
		IF *l* **THEN** *e1* **ELSE** *e2*	
22	left to right	**||**	**!!**
23	left to right	**+ –**	
24	unary	**SUM{***index-set***}**	
		PROD{*index-set***}**	
		MIN{*index-set***}**	
		MAX{*index-set***}**	
25	left to right	*** /**	
26	unary	**+ –**	
	right to left	**><**	

Table 6.3. (continued)

Precedence	Associativity	Operator	Alternates
		<>	
		******	^

Primary expressions are the individual operands that are combined using the expression operators. Simple primary expressions can represent constants or named parameter and variable values. More complex primary expressions can be used to call functions or construct sets.

Table 6.4. Primary Expression Table

Expression	Description
identifier-expression	parameter/variable reference; see the section "Identifier Expressions" on page 679
name **(***arg-list***)**	function call; *arg-list* is 0 or more expressions separated by commas
n	numeric constant
. or **.***c*	missing value constant
"*string*" or '*string*'	string constant
{ *member-list* **}**	set constructor; *member-list* is 0 or more scalar expressions or tuple expressions separated by commas
{ *index-set* **}**	index set expression; returns the set of all index set members
/ *member(s)* **/**	set literal expression; compactly specifies a simple set value
(*expression* **)**	expression enclosed in parentheses
< *expr-list* **>**	tuple expression; used with set operations; contains one or more scalar expressions separated by commas

Identifier Expressions

Use an *identifier-expression* to refer to a variable, objective, constraint, or parameter location in expressions or initializations. This is the syntax for *identifier-expressions*:

name [**[** *expression-1 [, . . . expression-n]* **]** *] [* **.** *suffix]*

To refer to a location in an array, follow the array *name* with a list of scalar expressions in square brackets ([ ]). The expression values are compared to the index set that was used to declare *name*. If there is more than one expression, then the values are formed into a tuple. The expression values for a valid array location must match a member of the array's index set. For example, the following code defines a parameter array A that has two valid indices that match the tuples <1,2> and <3,4>:

```
proc optmodel;
   set<number, number> ISET = {<1,2>, <3,4>};
   number A{ISET};
   a[1,2] = 0;  /* OK */
   a[3,2] = 0;  /* invalid index */
```

The first assignment is valid with this definition of the index set, but the second fails because <3,2> is not a member of the set parameter ISET.

Specify a *suffix* to refer to auxiliary locations for variables or objectives. See the section "Suffixes" on page 759 for more information.

Function Expressions

Most functions that can be invoked from the DATA step or the %SYSFUNC macro can be used in PROC OPTMODEL expressions. Note that certain functions are specific to the DATA step and cannot be used in PROC OPTMODEL. Functions specific to the DATA step include these:

- functions in the LAG/DIF/DIM families
- functions that access the DATA step program data vector
- functions that access symbol attributes

The CALL statement can invoke SAS library subroutines. These subroutines can read and update the values of the parameters and variables that are used as arguments. See the section "CALL Statement" on page 695 for an example.

Index Sets

An index set represents a set of combinations of members from the component set expressions. The index set notation is used in PROC OPTMODEL to describe collections of valid array indices and to specify sets of values with which to perform an operation. Index sets can declare local dummy parameters and can further restrict the set of combinations by a selection expression.

In { *index-set* }, the *index-set* consists of one or more *index-set-items* that are separated by commas. Each *index-set-item* can include local dummy parameter declarations. An optional selection expression follows the list of *index-set-items*. This syntax describes an *index-set*:

index-set-item [, . . . index-set-item] [: logic-expression]

Index-set-item has these forms:

set-expression

name **IN** *set-expression*

< *name-1* **[,** *. . . name-n***]** **>** **IN** *set-expression*

Names preceding the IN keyword in *index-set-items* declare local dummy parameter names. Dummy parameters correspond to the dummy index variables in mathematical expressions. For example, the following code outputs the number 385:

```
proc optmodel;
   put (sum{i in 1..10} i**2);
```

The preceding code evaluates this summation:

$$\sum_{i=1}^{10} i^2 = 385$$

In both the code and the summation the index name is i.

The last form of *index-set-item* in the list can be modified to use the SLICE expression implicitly. See the section "More on Index Sets" on page 776 for details.

Array index sets cannot be defined using functions that return different values each time the functions are called. See the section "Indexing" on page 672 for details.

Syntax: OPTMODEL Procedure

PROC OPTMODEL statements are divided into three categories: the PROC statement, the declaration statements, and the programming statements. The PROC statement invokes the procedure and sets initial option values. The declaration statements declare optimization model components. The programming statements are used to read and write data, invoke the solver, and print results. The statements are listed in the order in which they appear in the following text, with declaration statements first.

Note: Solver specific options are described in the individual chapters corresponding to the solvers.

PROC OPTMODEL *options* **;**

Declaration Statements:
CONSTRAINT *constraints* **;**
MAX *objective* **;**
MIN *objective* **;**
NUMBER *parameter declarations* **;**
SET *[* **<** *types* **>** *]* *parameter declarations* **;**
STRING *parameter declarations* **;**
VAR *variable declarations* **;**

Programming Statements:
parameter **=** *expression* **;** *(Assignment)*

CALL *name* [**(** *expressions* **)**] **;**
CLOSEFILE *files* **;**
CONTINUE **;**
CREATE DATA *SAS-data-set FROM columns* **;**
DO **;** *statements* **;** **END** **;**
DO *variable* **=** *specifications* **;** *statements* **;** **END** **;**
DO UNTIL **(** *logic* **)** **;** *statements* **;** **END** **;**
DO WHILE **(** *logic* **)** **;** *statements* **;** **END** **;**
DROP *constraint* **;**
EXPAND *name* [**/** *options*] **;**
FILE *file* **;**
FIX *variable* [**=** *expression*] **;**
FOR **{** *index set* **}** *statement* **;**
IF *logic* **THEN** *statement* **;** [**ELSE** *statement* **;**]
LEAVE **;**
; *(Null)*
PRINT *print items* **;**
PUT *put items* **;**
READ DATA *SAS-data-set* **INTO** *columns* **;**
RESET OPTIONS *options* **;**
RESTORE *constraint* **;**
SAVE MPS *SAS-data-set* **;**
SAVE QPS *SAS-data-set* **;**
SOLVE [**WITH** *solver*] [**OBJECTIVE** *name*] [**/** *options* **;**
STOP **;**
UNFIX *variable* [**=** *expression*] **;**

Functional Summary

The statements and options available with PROC OPTMODEL are summarized by purpose in Table 6.5.

Table 6.5. Functional Summary

Description	Statement	Option
Declaration Statements:		
declare constraint	CONSTRAINT	
declare maximization objective	MAX	
declare minimization objective	MIN	
declare number type parameter	NUMBER	
declare set type parameter	SET	
declare string type parameter	STRING	
declare optimization variables	VAR	

Description	Statement	Option
Programming Statements:		
assign value to a variable or parameter	=	
invoke a library subroutine	CALL	
close opened file	CLOSEFILE	
terminate one iteration of a loop statement	CONTINUE	
create a new SAS data set and copy data into it from PROC OPTMODEL parameters and variables	CREATE DATA	
group a sequence of statements together as a single statement	DO	
execute statements repeatedly	DO (iterative)	
execute statements repeatedly until some condition is satisfied	DO UNTIL	
execute statements repeatedly while some condition is satisfied	DO WHILE	
ignore the specified constraint	DROP	
print the specified constraint, variable, or objective declaration expressions after expanding aggregation operators, etc.	EXPAND	
select a file for the PUT statement	FILE	
treat a variable as fixed in value	FIX	
execute statement repeatedly	FOR	
execute statement conditionally	IF	
terminate the execution of the entire loop body	LEAVE	
null statement	;	
output string and numeric data	PRINT	
write text data to the current output file	PUT	
read data from a SAS data set into PROC OPTMODEL parameters and variables	READ DATA	
set PROC OPTMODEL option values or restore them to their defaults	RESET OPTIONS	
add constraint that was previously dropped back into the model	RESTORE	
save the structure and coefficients for a linear programming model into a SAS data set	SAVE MPS	
save the structure and coefficients for a quadratic programming model into a SAS data set	SAVE QPS	
invoke an OPTMODEL solver	SOLVE	

Description	Statement	Option
halt the execution of all statements that contain it	STOP	
reverse the effect of FIX statement	UNFIX	
PROC OPTMODEL Options:		
accuracy for nonlinear constraints	PROC OPTMODEL	CDIGITS=
the method used to approximate numeric derivatives	PROC OPTMODEL	FD=
accuracy for the objective function	PROC OPTMODEL	FDIGITS=
pass initial values for variables to solver	PROC OPTMODEL	INITVAR/NOINITVAR
tolerance for rounding the bounds on integer and binary variables	PROC OPTMODEL	INTFUZZ=
maximum length for MPS row and column labels	PROC OPTMODEL	MAXLABLEN=
check missing values	PROC OPTMODEL	MISSCHECK/NOMISSCHECK
number of digits to display	PROC OPTMODEL	PDIGITS=
adjust how two-dimensional array is displayed	PROC OPTMODEL	PMATRIX=
type of presolve performed by OPTMODEL presolver	PROC OPTMODEL	PRESOLVER=
tolerance enabling OPTMODEL presolver to remove slightly infeasible constraints	PROC OPTMODEL	PRESTOL=
width to display numeric columns	PROC OPTMODEL	PWIDTH=
the smallest difference that is permitted by the OPTMODEL presolver between the upper and lower bounds of an unfixed variable	PROC OPTMODEL	VARFUZZ=

PROC OPTMODEL Statement

The PROC OPTMODEL statement invokes the OPTMODEL procedure. You can specify options to control how the optimization model is processed and how results are displayed. This is the syntax:

PROC OPTMODEL *[option(s)]*;

The following options can appear in the PROC OPTMODEL statement (these options can also be specified by the RESET statement).

CDIGITS=*num*

specifies the expected number of decimal digits of accuracy for nonlinear constraints. The value can be fractional. PROC OPTMODEL uses this option to choose a step length when numeric derivative approximations are required to evaluate the Jacobian

of nonlinear constraints. The default value depends on your operating environment. It is assumed that constraint values are accurate to the limits of machine precision.

See the section "Automatic Differentiation" on page 774 for more information about numeric derivative approximations.

FD=FORWARD|CENTRAL
selects the method used to approximate numeric derivatives when analytic derivatives are unavailable. Most solvers require the derivatives of the objective and constraints. The methods available are as follows:

FD=FORWARD use forward differences

FD=CENTRAL use central differences

The default value is FORWARD. See the section "Automatic Differentiation" on page 774 for more information about numeric derivative approximations.

FDIGITS=*num*
specifies the expected number of decimal digits of accuracy for the objective function. The value can be fractional. PROC OPTMODEL uses the value to choose a step length when numeric derivatives are required. The default value depends on your operating environment. It is assumed that objective function values are accurate to the limits of machine precision.

See the section "Automatic Differentiation" on page 774 for more information about numeric derivative approximations.

INITVAR | NOINITVAR
selects whether or not to pass initial values for variables to the solver when the SOLVE statement is executed. INITVAR enables the current variable values to be passed. NOINITVAR causes the solver to be invoked without any specific initial values for variables. The INITVAR option is the default.

Note that the LP and QP solvers always ignore initial values. The NLPU, NLPC, SQP, and IPNLP solvers attempt to use specified initial values. The MILP solver uses initial values only if the PRIMALIN option is specified.

INTFUZZ=*num*
specifies the tolerance for rounding the bounds on integer and binary variables to integer values. Bounds that differ from an integer by at most *num* are rounded to that integer. Otherwise lower bounds are rounded up to the next greater integer and upper bounds are rounded down to the next lesser integer. The value of *num* can range between 0 and 0.5. The default value is 1E−10.

MAXLABLEN=*num*
specifies the maximum length for MPS row and column labels. The allowed range is 8 to 256, with 32 as the default. This option can also be used to control the length of row and column names displayed by solvers, such as those found in the LP solver iteration log.

MISSCHECK | NOMISSCHECK
enables detailed checking of missing values in expressions. MISSCHECK requests

that a message be produced each time PROC OPTMODEL evaluates an arithmetic operation or built-in function that has missing value operands (except when the operation or function specifically supports missing values). The MISSCHECK option can increase processing time. NOMISSCHECK turns off this detailed reporting. NOMISSCHECK is the default.

PDIGITS=*num*

requests that the PRINT statement display *num* significant digits for numeric columns for which no format is specified. The value can range from 1 to 9. The default is 5.

PMATRIX=*num*

adjusts the density evaluation of a two-dimensional array to affect how it is displayed. The value *num* scales the total number of nonempty array elements and is used by the PRINT statement to evaluate whether a two-dimensional array is "sparse" or "dense." Tables containing a single two-dimensional array are printed in list form if they are sparse and in matrix form if they are dense. Any nonnegative value can be assigned to *num*; the default value is 1. Specifying values for the PMATRIX= option less than 1 causes the list form to be used in more cases, while specifying values greater than 1 causes the matrix form to be used in more cases. If the value is 0, then the list form is always used. See the section "PRINT Statement" on page 711 for more information.

PRESOLVER=*option*
PRESOLVER=*num*

specifies a presolve *option* or its corresponding value *num*, as listed in Table 6.6.

Table 6.6. Values for the PRESOLVER= Option

Number	Option	Description
–1	AUTOMATIC	Apply presolver using default setting.
0	NONE	Disable presolver.
1	BASIC	Perform minimal processing, only substituting fixed variables and removing empty feasible constraints.
2	MODERATE	Apply a higher level of presolve processing.
3	AGGRESSIVE	Apply the highest level of presolve processing.

The OPTMODEL presolver tightens variable bounds and eliminates redundant constraints. In general, this improves the performance of any solver. The AUTOMATIC option is intermediate between the MODERATE and AGGRESSIVE levels.

PRESTOL=*num*

provides a tolerance so that slightly infeasible constraints can be eliminated by the OPTMODEL presolver. If the magnitude of the infeasibility is no greater than num$(|\mathrm{X}|+1)$, where X is the value of the original bound, then the empty constraint is removed from the presolved problem. OPTMODEL's presolver does not print messages about infeasible constraints and variable bounds when the infeasibilty is within

the PRESTOL tolerance. The value of PRESTOL can range between 0 and 0.1; the default value is 1E−12.

PRINTLEVEL=*num*
controls the level of listing output during a SOLVE command. The Output Delivery System (ODS) tables printed at each level are listed in Table 6.7.

Table 6.7. Values for the PRINTLEVEL= Option

Number	Description
0	Disable all tables.
1	Print Problem Summary and Solution Summary.
2	Print Problem Summary, Solution Summary, Methods of Derivative Computation (for NLP solvers), Solver Options table, and Optimization Statistics.

For more details about the ODS tables produced by PROC OPTMODEL, see the section "ODS Table and Variable Names" on page 750.

PWIDTH=*num*
sets the width used by the PRINT statement to display numeric columns when no format is specified. The smallest value *num* can take is the value of the PDIGITS= option plus 7; the largest value *num* can take is 16. The default value is equal to the value of the PDIGITS= option plus 7.

VARFUZZ=*num*
specifies the smallest difference that is permitted by the OPTMODEL presolver between the upper and lower bounds of an unfixed variable. If the difference is smaller than *num*, then the variable is fixed to the average of the upper and lower bounds before it is presented to the solver. Any nonnegative value can be assigned to *num*; the default value is 0.

Declaration Statements

The declaration statements define the parameters, variables, constraints, and objectives that describe a PROC OPTMODEL optimization model. Declarations in the PROC OPTMODEL input are saved for later use. Unlike programming statements, declarations cannot be nested in other statements. Declaration statements are terminated by a semicolon.

Many declaration attributes, such as variable bounds, are defined using expressions. Expressions in declarations are handled symbolically and are resolved as needed. In particular, expressions are generally reevaluated when one of the parameter values they use has been changed.

CONSTRAINT Declaration

CONSTRAINT *constraint [, . . . constraint]* **;**

CON *constraint [, . . . constraint]* **;**

The constraint declaration defines one or more constraints on expressions in terms of the optimization variables. You can specify multiple constraint declaration statements.

Constraints can have an upper bound, a lower bound, or both bounds. The allowed forms are as follows:

[name [**{** *index-set* **}** *] :*] *expression* **=** *expression*
: declares an equality constraint or, when an *index-set* is specified, a family of equality constraints. The solver attempts to assign values to the optimization variables to make the two expressions equal.

[name [**{** *index-set* **}** *] :*] *expression relation expression*
: declares an inequality constraint that has a single upper or lower bound. When an *index-set* is specified, this declares a family of inequality constraints. *Relation* is the <= or >= operator. The solver tries to assign optimization variable values so that the left expression has a value less than or equal to (respectively, greater than or equal to) the right expression value.

[name [**{** *index-set* **}** *] :*] *bound relation body relation bound*
: declares an inequality constraint that is bounded on both sides, or *range constraint*. When an *index-set* is specified, this declares a family of range constraints. *Relation* is the <= or >= operator. The same operator must be used in both positions. The first *bound* expression defines the lower bound (respectively, upper bound). The second *bound* expression defines the upper bound (respectively, lower bound). The solver tries to assign optimization variables so that the value of the *body* expression is in the range between the upper and lower bounds.

Name defines the name for the constraint. Use the name to reference constraint attributes, such as the bounds, elsewhere in the PROC OPTMODEL model. If no name is provided, then a default name is created of the form _ACON_`[`n`]`, where n is an integer. See the section "Constraints" on page 755 for more information.

Here is a simple example that defines a constraint with a lower bound:

```
proc optmodel;
   var x, y;
   number low;
   con a: x+y >= low;
```

The following example adds an upper bound:

```
var x, y;
number low;
con a: low <= x+y <= low+10;
```

Indexed families of constraints can be defined by specifying an *index-set* after the name. Any dummy parameters that are declared in the *index-set* can be referenced in the expressions that define the constraint. A particular member of a indexed family can be specified using an *identifier-expression* with a bracketed index list, in the same fashion as array parameters and variables. For example, the following code creates an indexed family of constraints named incr:

```
proc optmodel;
   number n;
   var x{1..n}
   /* require nondecreasing x values */
   con incr{i in 1..n-1}: x[i+1] >= x[i];
```

The CON statement in the example creates constraints incr[1] through incr[$n-1$].

Constraint expressions cannot be defined using functions that return different values each time they are called. See the section "Indexing" on page 672 for details.

MAX and MIN Objective Declarations

MAX *name* **=** *expression* **;**

MIN *name* **=** *expression* **;**

The MAX or MIN declaration specifies an objective for the solver. The *name* names the objective function for later reference. The solver maximizes an objective specified with the MAX keyword and minimizes an objective specified with the MIN keyword. An objective is not allowed to have the same name as a parameter or variable. Multiple objectives are permitted, but the solver processes only one objective at a time.

Expression specifies the numeric function to maximize or minimize in terms of the optimization variables.

When used in an expression, an objective name refers to the current value of the named objective function. The value of unsuffixed objective names can depend on the value of optimization variables, so objective names cannot be used in constant expressions such as variable bounds. You can reference objective names in objective or constraint expressions. For example, the following code declares two objective names, q and l, which are immediately referred to in the objective declaration of z and the declarations of the constraints.

```
proc optmodel;
   var x, y;
   min q=(x+y)**2;
   max l=x+2*y;
   min z=q+l;
   con c1: q<=4;
   con c2: l>=2;
```

Objectives cannot be defined using functions that return different values each time they are called. See the section "Indexing" on page 672 for details.

NUMBER, STRING, and SET Parameter Declarations

NUMBER *parameter-decl [, . . . parameter-decl]* **;**

STRING *parameter-decl [, . . . parameter-decl]* **;**

SET *[<scalar-type, . . . scalar-type >] parameter-decl [, . . . parameter-decl]* **;**

Parameters provide names for constants. Parameters are declared by specifying the parameter type followed by a list of parameter names. Declarations of parameters that have NUMBER or STRING types start with a *scalar-type* specification:

NUMBER | **NUM**

STRING | **STR**

The NUM and STR keywords are abbreviations for the NUMBER and STRING keywords, respectively.

The declaration of a parameter that has the set type begins with a *set-type* specification:

SET *[<scalar-type, . . . scalar-type >]*

In a *set-type* declaration, the SET keyword is followed by a list of *scalar-type* items that specify the member type. A set with scalar members is specified with a single *scalar-type* item. A set with tuple members has a *scalar-type* item for each tuple element. The *scalar-type* items specify the types of the elements at each tuple position.

If the SET keyword is not followed by a list of *scalar-type* items, then the set type is determined from the type of the initialization expression. The declared type defaults to SET<NUMBER> if no initialization expression is given or if the expression type cannot be determined.

For any parameter type, the type declaration is followed by a list of *parameter-decl* items that specify the names of the parameters to declare. In a *parameter-decl* item the parameter name can be followed by an optional index specification and any necessary options, as follows:

name [**{** *index-set* **}** *] [parameter-option(s)]*

The parameter *name* and *index-set* can be followed by a list of *parameter-options*. Dummy parameters declared in the *index-set* can be used in the *parameter-options*. The parameter options can be specified with the following forms:

= *expression*
: This option provides an explicit value for each parameter location. In this case the parameter acts like an alias for the *expression* value.

INIT *expression*
: This option specifies a default value that is used when a parameter value is required but no other value has been supplied. For example:

```
number n init 1;
set s init {'a', 'b', 'c'};
```

PROC OPTMODEL evaluates the expression for each parameter location the first time the parameter needs to be resolved. The expression is not used when the parameter already has a value.

= [*initializer(s)* **]**

This option provides a compact means to define the values for an array, in which each array location value can be individually specified by the *initializers*.

The =*expression* parameter option defines a parameter value by using a formula. The formula can refer to other parameters. The parameter value is updated when the referenced parameters change. The following example shows the effects of the update:

```
proc optmodel;
   number n;
   set<number> s = 1..n;
   number a{s};
   n = 3;
   a[1] = 2;    /* OK */
   a[7] = 19;   /* error, 7 is not in s */
   n = 10;
   a[7] = 19;   /* OK now */
```

In the preceding example the value of set s is resolved for each use of array a that has an index. For the first use of a[7], the value 7 is not a member of the set s. However, the value 7 is a member of s at the second use of a[7].

The **INIT** *expression* parameter option specifies a default value for a parameter. The following example shows the usage of this option:

```
proc optmodel;
   num a{i in 1..2} init i**2;
   a[1] = 2;
   put a[*]=;
```

When the value of a parameter is needed but no other value has been supplied, the default value specified by **INIT** *expression* is used, as shown in Figure 6.7.

```
a[1]=2 a[2]=4
```

Figure 6.7. INIT Option: Output

Note: Parameter values can also be read from files or specified with assignment statements. However, the value of a parameter that is assigned with the =*expression* or =[*initializer(s)*] forms can be changed only by modifying the parameters used in the defining expressions. Parameter values specified by the INIT option can be reassigned freely.

Initializing Arrays

Arrays can be initialized with the =[*initializer(s)*] form. This form is convenient when array location values need to be individually specified. This form of initialization is used in the following code:

```
proc optmodel;
   number a{1..3} = [5 4 7];
   put a[*]=;
```

Each array location receives a different value, as shown in Figure 6.8.

```
a[1]=5 a[2]=4 a[3]=7
```

Figure 6.8. Array Initialization

Each *initializer* takes the following form:

[**[** *index* **]** *] value*

The *value* specifies the value of an array location and can be a numeric or string constant, a set literal, or an expression enclosed in parentheses.

In array initializers, string constants can be specified using quoted strings. When the string text follows the rules for a SAS name, the text can also be specified without quotation marks. String constants that begin with a digit, contain blanks, or contain other special characters must be specified with a quoted string.

As an example, the following code defines an array parameter that could be used to map numeric days of the week to text strings:

```
proc optmodel;
   string dn{1..5} =
          [Monday Tuesday Wednesday Thursday Friday];
```

The optional *index* in square brackets specifies the index of the array location to initialize. The index specifies one or more numeric or string subscripts. The subscripts allow the same syntactic forms as the *value* items. Commas can be used to separate index subscripts. For example, location a[1,'abc'] of an array a could be specified with the index [1 abc]. The following example initializes just the diagonal locations in a square array:

```
proc optmodel;
   number m{1..3,1..3} = [[1 1] 0.1 [2 2] 0.2 [3 3] 0.3];
```

An index does not need to specify all the subscripts of an array location. If the index begins with a comma, then only the right-most subscripts of the index need to be specified. The preceding subscripts are supplied from the index that was used by the

preceding *initializer*. This can simplify the initialization of arrays that are indexed by multiple subscripts. For example, you can add new entries to the matrix of the previous example by using the following code:

```
proc optmodel;
   number m{1..3,1..3} = [[1 1] 0.1            [,3] 1
                                  [2 2] 0.2   [,3] 2
                                             [3 3] 0.3];
```

The spacing shows the layout of the example array. The previous example was updated by initializing two more values at m[1,3] and m[2,3].

If an index is omitted, then the next location in the order of the array's index set is initialized. If the index set has multiple *index-set-items*, then the rightmost indices are updated before indices to the left are updated. At the beginning of the initializer list, this is the first member of the index set. The index set must use a range expression to avoid unpredictable results when an index value is omitted.

The initializers can be followed by commas. The use of commas has no effect on the initialization. The comma can be used to clarify layout. For example, the comma could separate rows in a matrix.

Not every array location needs to be initialized. The locations without an explicit initializer are set to zero for numeric arrays, set to an empty string for string arrays, and set to an empty set for set arrays.

Note: An array location must not be initialized more than once during the processing of the initializer list.

VAR Declaration

VAR *var-decl [, . . . var-decl]* **;**

The VAR statement declares one or more optimization variables. Multiple VAR statements are permitted. A variable is not allowed to have the same name as a parameter or constraint.

Each *var-decl* specifies a variable name. The name can be followed by an array *index-set* specification and then variable options. Dummy parameters declared in the index set specification can be used in the following variable options.

Here is the syntax for a *var-decl*:

name [**{** *index-set* **}** *] [var-option(s)]*

For example, the following code declares a group of 100 variables, x[1]–x[100]:

```
proc optmodel;
   var x{1..100};
```

Here are the available variable options:

INIT *expression*
: sets an initial value for the variable. The expression is used only the first time the value is required. If no initial value is specified, then 0 is used by default.

>= *expression*
: sets a lower bound for the variable value. The default lower bound is $-\infty$.

<= *expression*
: sets an upper bound for the variable value. The default upper bound is ∞.

INTEGER
: requests that the solver assign the variable an integer value.

BINARY
: requests that the solver assign the variable a value of either 0 or 1.

For example, the following code declares a variable that has an initial value of 0.5. The variable is bounded between 0 and 1:

```
proc optmodel;
   var x init 0.5 >= 0 <= 1;
```

The values of the bounds can be determined later by using suffixed references to the variable. For example, the upper bound for variable x can be referred to as x.ub. In addition the bounds options can be overridden by explicit assignment to the suffixed variable name. Suffixes are described further in the section "Suffixes" on page 759.

When used in an expression, an unsuffixed variable name refers to the current value of the variable. Unsuffixed variables are not allowed in the expressions for options that define variable bounds or initial values. Such expressions have values that must be fixed during execution of the solver.

Programming Statements

PROC OPTMODEL supports several programming statements. You can perform various actions with these statements, such as reading or writing data sets, setting parameter values, generating text output, or invoking a solver.

Statements are read from the input and are executed immediately when complete. Note that certain statements can contain one or more substatements. The execution of substatements is held until the statements that contain them are submitted. Parameter values that are used by expressions in programming statements are resolved when the statement is executed. Note that this resolution might cause errors to be detected. For example, the use of undefined parameters is detected during resolution of the symbolic expressions from declarations.

A statement is terminated by a semicolon. The positions at which semicolons are placed are shown explicitly in the following statement syntax descriptions.

The programming statements can be grouped into these categories:

Control	Looping	General	Input/Output	Model
DO	CONTINUE	Assignment	CLOSEFILE	DROP
IF	FOR	CALL	CREATE DATA	EXPAND
Null (;)	DO Iterative	RESET OPTIONS	FILE	FIX
STOP	DO UNTIL		PRINT	RESTORE
	DO WHILE		PUT	SOLVE
	LEAVE		READ DATA	UNFIX
			SAVE MPS	
			SAVE QPS	

Assignment Statement

identifier-expression **=** *expression* **;**

The assignment statement assigns a variable or parameter value. The type of the target *identifier-expression* must match the type of the right-hand-side expression.

For example, the following code sets the current value for variable x to 3:

```
proc optmodel;
   var x;
   x = 3;
```

Note: Parameters that were declared with the equal sign (=) initialization forms must not be reassigned a value with an assignment statement. If this occurs, PROC OPTMODEL reports an error.

CALL Statement

CALL *name* **(** *argument-1 [, . . . argument-n]* **) ;**

The CALL statement invokes the named library subroutine. The values that are determined for each argument expression are passed to the subroutine when the subroutine is invoked. The subroutine can update the values of PROC OPTMODEL parameters and variables when an argument is an *identifier-expression* (see the section "Identifier Expressions" on page 679). For example, the following code sets the parameter array a to a random permutation of 1 to 4:

```
proc optmodel;
   number a{i in 1..4} init i;
   number seed init -1;
   call ranperm(seed, a[1], a[2], a[3], a[4]);
```

See Chapter 4, "Functions and CALL Routines," in *SAS Language Reference: Dictionary* for a list of CALL routines.

CLOSEFILE Statement

CLOSEFILE *file-specification(s)* **;**

The CLOSEFILE statement closes files that were opened by the FILE statement. Each file is specified by a logical name, a physical filename in quotation marks, or an expression enclosed in parentheses that evaluates to a physical filename. See the section "FILE Statement" on page 707 for more information about file specifications.

The following example shows how the CLOSEFILE statement is used with a logical filename:

```
filename greet 'hello.txt';
proc optmodel;
   file greet;
   put 'Hi!';
   closefile greet;
```

Generally you must close a file with a CLOSEFILE statement before external programs can access the file. However, any open files are automatically closed when PROC OPTMODEL terminates.

CONTINUE Statement

CONTINUE ;

The CONTINUE statement terminates the current iteration of the loop statement (iterative DO, DO UNTIL, DO WHILE, or FOR) that immediately contains the CONTINUE statement. Execution resumes at the start of the loop after checking WHILE or UNTIL tests. The FOR or iterative DO loops apply new iteration values.

CREATE DATA Statement

CREATE DATA *SAS-data-set* **FROM** *[* **[** *key-column(s)* **]** *[* **=***key-set]]*
column(s) **;**

The CREATE DATA statement creates a new SAS data set and copies data into it from PROC OPTMODEL parameters and variables. The CREATE DATA statement can create a data set with a single observation or a data set with observations for every location in one or more arrays. The data set is closed after the execution of the CREATE DATA statement.

The arguments to the CREATE DATA statement are as follows:

SAS-data-set
: specifies the output data set name and options.

key-column(s)
: declares index values and their corresponding data set variables. The values are used to index array locations in *column(s)*.

key-set
: specifies a set of index values for the *key-column(s)*.

column(s)

specifies data set variables as well as the PROC OPTMODEL source data for the variables.

Each *column* or *key-column* defines output data set variables and a data source for a column. For example, the following code generates the output SAS data set resdata from the PROC OPTMODEL array opt, which is indexed by the set indset:

```
create data resdata from [solns]=indset opt;
```

The output data set variable solns contains the index elements in indset.

Columns

Column(s) can have the following forms:

identifier-expression

transfers data from the PROC OPTMODEL parameter or variable specified by the *identifier-expression*. The output data set variable has the same name as the *name* part of the *identifier-expression* (see the section "Identifier Expressions" on page 679). If the *identifier-expression* refers to an array, then the index can be omitted when it matches the *key-column(s)*. The following example creates a data set with the variables m and n:

```
proc optmodel;
   number m = 7, n = 5;
   create data example from m n;
```

name **=** *expression*

transfers the value of a PROC OPTMODEL expression to the output data set variable *name*. The *expression* is reevaluated for each observation. If the *expression* contains any operators or function calls, then it must be enclosed in parentheses. If the *expression* is an *identifier-expression* that refers to an array, then the index can be omitted if it matches the *key-column(s)*. The following example creates a data set with the variable ratio:

```
proc optmodel;
   number m = 7, n = 5;
   create data example from ratio=(m/n);
```

COL(*name-expression***) =** *expression*

transfers the value of a PROC OPTMODEL expression to the output data set variable named by the string expression *name-expression*. The PROC OPTMODEL expression is reevaluated for each observation. If this expression contains any operators or function calls, then it must be enclosed in parentheses. If the PROC OPTMODEL *expression* is an *identifier-expression* that refers to an

array, then the index can be omitted if it matches the *key-column(s)*. The following example uses the COL expression to form the variable s5:

```
proc optmodel;
   number m = 7, n = 5;
   create data example from col("s"||n)=(m+n);
```

{ *index-set* **}** **<** *column(s)* **>**

performs the transfers by iterating each column specified by <*column(s)*> for each member of the *index set*. If there are n columns and m index set members, then $n \times m$ columns are generated. The dummy parameters from the index set can be used in the columns to generate distinct output data set variable names in the iterated columns, using COL expressions. The columns are expanded when the CREATE DATA statement is executed, before any output is performed. This form of *column(s)* cannot be nested. In other words, the following form of *column(s)* is NOT allowed:

{ *index-set* **}** **<** **{** *index-set* **}** **<** *column(s)* **>** **>**

The following example demonstrates the use of the iterated *column(s)* form:

```
proc optmodel;
   set<string> alph = {'a', 'b', 'c'};
   var x{1..3, alph} init 2;
   create data example from [i]=(1..3)
      {j in alph}<col("x"||j)=x[i,j]>;
```

The data set created by this code is shown in Figure 6.9.

Obs	i	xa	xb	xc
1	1	2	2	2
2	2	2	2	2
3	3	2	2	2

Figure 6.9. CREATE DATA with COL Expression

Note: When no *key-column(s)* are specified, the output data set has a single observation.

The following code incorporates several of the preceding examples to create and print a data set by using PROC OPTMODEL parameters:

```
proc optmodel;
   number m = 7, n = 5;
   create data example from m n ratio=(m/n) col("s"||n)=(m+n);
```

```
proc print;
   run;
```

The output from the PRINT procedure is shown in Figure 6.10.

Obs	m	n	ratio	s5
1	7	5	1.4	12

Figure 6.10. CREATE DATA for Single Observation

Key columns

Key-column(s) declare index values that enable multiple observations to be written from array *column(s)*. An observation is created for each unique index value combination. The index values supply the index for array *column(s)* that do not have an explicit index.

Key-column(s) define the data set variables where the index value elements are written. They can also declare local dummy parameters for use in expressions in the *column(s)*. *Key-column(s)* are syntactically similar to *column(s)*, but are more restricted in form. The following forms of *key-column(s)* are allowed:

name

transfers an index element value to the data set variable *name*. A local dummy parameter, *name*, is declared to hold the index element value.

COL(*name-expression***)** *[* **=** *index-name]*

transfers an index element value to the data set variable named by the string-valued *name-expression*. *Index-name* optionally declares a local dummy parameter to hold the index element value.

A *key-set* in the CREATE DATA statement explicitly specifies the set of index values. *Key-set* can be specified as a set expression, although it must be enclosed in parentheses if it contains any function calls or operators. *Key-set* can also be specified as an index set expression, in which case the *index-set* dummy parameters override any dummy parameters that are declared in the *key-column(s)* items. The following code creates a data set from the PROC OPTMODEL parameter m, a matrix whose only nonzero entries are located at (1, 1) and (4, 1):

```
proc optmodel;
   number m{1..5, 1..3} = [[1 1] 1 [4 1] 1];
   create data example
      from [i j] = {setof{i in 1..2}<i**2>, {1, 2}} m;
proc print data=example noobs;
   run;
```

The dummy parameter i in the SETOF expression takes precedence over the dummy parameter i declared in the *key-column(s)* item. The output from this code is shown in Figure 6.11.

i	j	m
1	1	1
1	2	0
4	1	1
4	2	0

Figure 6.11. CREATE: *Key-set* with SETOF Aggregation Expression

If no *key-set* is specified, then the set of index values is formed from the union of the index sets of the implicitly indexed *column(s)*. The number of index elements for each implicitly indexed array must match the number of *key-column(s)*. The type of each index element (string versus numeric) must match the element of the same position in other implicit indices.

The arrays for implicitly indexed columns in a CREATE DATA statement do not need to have identical index sets. A missing value is supplied for the value of an implicitly indexed array location when the implied index value is not in the array's index set.

In the following code, the *key-set* is unspecified. The set of index values is $\{1, 2, 3\}$, which is the union of the index sets of x and y. These index sets are not identical, so missing values are supplied when necessary. The results of this code are shown in Figure 6.12.

```
proc optmodel;
   number x{1..2} init 2;
   var y{2..3} init 3;
   create data exdata from [keycol] x y;
proc print;
   run;
```

Obs	keycol	x	y
1	1	2	.
2	2	2	3
3	3	.	3

Figure 6.12. CREATE: Unspecified *Key-set*

The types of the output data set variables match the types of the source values. The output variable type for a *key-column(s)* matches the corresponding element type in the index value tuple. A numeric element matches a NUMERIC data set variable, while a string element matches a CHAR variable. For regular *column(s)* the source

expression type determines the output data set variable type. A numeric expression produces a NUMERIC variable, while a string expression produces a CHAR variable.

Lengths of character variables in the output data set are determined automatically. The length is set to accommodate the longest string value output in that column.

You can use the iterated *column(s)* form to output selected rows of multiple arrays, assigning a different data set variable to each column. For example, the following code outputs the last two rows of the two-dimensional array, a, along with corresponding elements of the one-dimensional array, b:

```
proc optmodel;
   num m = 3;  /* number of rows/observations */
   num n = 4;  /* number of columns in a */
   num a{i in 1..m, j in 1..n} = i*j;  /* compute a */
   num b{i in 1..m} = i**2;  /* compute b */
   set<num> subset = 2..m;  /* used to omit first row */
   create data out
          from [i]=subset {j in 1..n}<col("a"||j)=a[i,j]> b;
```

To specify the data set to be created, the CREATE DATA statement uses the form *key-column(s)* {*index set*}<*column(s)*> *column(s)*. The preceding code creates a data set out, which has $m - 1$ observations and $n + 2$ variables. The variables are named i, a1 through an, and b, as shown in Figure 6.13.

Obs	i	a1	a2	a3	a4	b
1	2	2	4	6	8	4
2	3	3	6	9	12	9

Figure 6.13. CREATE DATA Set: The Iterated Column Form

See the section "Data Set Input/Output" on page 743 for more examples of using the CREATE DATA statement.

DO Statement

DO ; *statement(s)* **; END ;**

The DO statement groups a sequence of statements together as a single statement. Each statement within the list is executed sequentially. The DO statement can be used for grouping with the IF and FOR statements.

DO Statement, Iterative

DO *name* **=** *specification-1* **[,** *. . . specification-n]* **;** *statement(s)* **; END ;**

The iterative DO statement assigns the values from the sequence of *specification* items to a previously declared parameter or variable, *name*. The specified statement sequence is executed after each assignment. This statement corresponds to the iterative DO statement of the DATA step.

Each *specification* provides either a single number or a single string value, or a sequence of such values. Each *specification* takes the following form:

expression [**WHILE(** *logic-expression* **)** | **UNTIL(** *logic-expression* **)**]

The *expression* in the *specification* provides a single value or set of values to assign to the target *name*. Multiple values can be provided for the loop by giving multiple *specification* items that are separated by commas. For example, the following code outputs the values 1, 3, and 5:

```
proc optmodel;
   number i;
   do i=1,3,5;
      put i;
   end;
```

The same effect can be achieved with a single range expression in place of the explicit list of values, as in the following code:

```
proc optmodel;
   number i;
   do i=1 to 5 by 2;
      put 'value of i assigned by the DO loop = ' i;
      i=i**2;
      put 'value of i assigned in the body of the loop = ' i;
   end;
```

The output of this code is shown in Figure 6.14.

```
value of i assigned by the DO loop = 1
value of i assigned in the body of the loop = 1
value of i assigned by the DO loop = 3
value of i assigned in the body of the loop = 9
value of i assigned by the DO loop = 5
value of i assigned in the body of the loop = 25
```

Figure 6.14. DO Loop: Name Parameter Unaffected

Note that unlike the DATA step, a range expression requires the limit to be specified. Additionally the BY part, if any, must follow the limit expression. Moreover, while the *name* parameter can be reassigned in the body of the loop, the sequence of values that is assigned by the DO loop is unaffected.

Expression can also be an expression that returns a set of numbers or strings. For example, the following code produces the same output as the previous code but uses a set parameter value:

```
proc optmodel;
   set s = {1,3,5};
   number i;
```

```
do i = s;
   put i;
end;
```

Each *specification* can include a WHILE or UNTIL clause. A WHILE or UNTIL clause applies to the *expression* that immediately precedes the clause. The sequence that is specified by an *expression* can be terminated early by a WHILE or UNTIL clause. A WHILE *logic-expression* is evaluated for each sequence value before the nested *statements*. If the *logic-expression* returns a false (zero or missing) value, then the current sequence is terminated immediately. An UNTIL *logic-expression* is evaluated for each sequence value after the nested *statements*. The sequence from the current *specification* is terminated if the *logic-expression* returns a true value (non-zero and nonmissing). After early termination of a sequence due to a WHILE or UNTIL expression, the DO loop execution continues with the next *specification*, if any.

To demonstrate use of the WHILE clause, the following code outputs the values 1, 2, and 3. In this case the sequence of values from the set `s` is stopped when the value of `i` reaches 4.

```
proc optmodel;
   set s = {1,2,3,4,5};
   number i;
   do i = s while(i NE 4);
      put i;
   end;
```

DO UNTIL Statement

DO UNTIL (*logic-expression* **) ;** *statement(s)* **; END ;**

The DO UNTIL loop executes the specified sequence of statements repeatedly until the *logic-expression*, evaluated after the *statements*, returns true (a nonmissing non-zero value).

For example, the following code outputs the values 1 and 2:

```
proc optmodel;
   number i;
   i = 1;
   do until (i=3);
      put i;
      i=i+1;
   end;
```

Note that multiple criteria can be introduced using expression operators, as in the following example:

```
do until (i=3 and j=7);
```

For a list of expression operators, see Table 6.3 on page 677.

DO WHILE Statement

DO WHILE (*logic-expression* **) ;** *statement(s)* **; END ;**

The DO WHILE loop executes the specified sequence of statements repeatedly as long as the *logic-expression*, evaluated before the *statements*, returns true (a non-missing nonzero value).

For example, the following code outputs the values 1 and 2:

```
proc optmodel;
   number i;
   i = 1;
   do while (i<3);
      put i;
      i=i+1;
   end;
```

Note that multiple criteria can be introduced using expression operators, as in the following example:

```
do while (i<3 and j<7);
```

For a list of expression operators, see Table 6.3 on page 677.

DROP Statement

DROP *identifier-expression* **;**

The DROP statement causes the solver to ignore the specified constraint, constraint array, or constraint array location. The *identifier-expression* specifies the dropped constraint. An entire constraint array is dropped if the *identifier-expression* omits the index for an array name.

The following example code uses the DROP statement:

```
proc optmodel;
   var x{1..10};
   con c1: x[1] + x[2] <= 3;
   con disp{i in 1..9}: x[i+1] >= x[i] + 0.1;
   . . .
   drop c1;         /* drops the c1 constraint */
   drop disp[5];    /* drops just disp[5] */
   drop disp;       /* drops all disp constraints */
```

The constraint can be added back to the model with the RESTORE statement.

Multiple names can be specified in a single DROP statement. For example, the following line drops both the c1 and disp[5] constraints:

```
drop c1 disp[5];
```

EXPAND Statement

EXPAND *[identifier-expression] [/ options]* **;**

The EXPAND statement prints the specified constraint, variable, or objective declaration expressions after expanding aggregation operators, substituting the current value for parameters and indices, and resolving constant subexpressions. *Identifier-expression* is the name of a variable, objective, or constraint. If the name is omitted and no *options* are specified, then all variables, objectives, and undropped constraints are printed. The following code shows an example EXPAND statement:

```
proc optmodel;
   number n=2;
   var x{1..n};
   min z1=sum{i in 1..n}(x[i]-i)**2;
   max z2=sum{i in 1..n}(i-x[i])**3;
   con c{i in 1..n}: x[i]>=0;
   fix x[2]=3;
   expand;
```

This code produces the output in Figure 6.15.

```
Var x[1]
Fix x[2] = 3
Objective z1=(x[1] - 1)**2 + (x[2] - 2)**2
Maximize z2=(-x[1] + 1)**3 + (-x[2] + 2)**3
Constraint c[1]: x[1] >= 0
Constraint c[2]: x[2] >= 0
```

Figure 6.15. EXPAND Statement Output

Specifying an *identifier-expression* restricts output to the specified declaration. A nonarray name prints only the specified item. If an array name is used with a specific index, then information for the specified array location is output. Using an array name without an index restricts output to all locations in the array.

Use *options* to further control the EXPAND statement output. The supported options follow:

SOLVE

causes the EXPAND statement to print the variables, objectives, and constraints in the same form that would be seen by the solver if a SOLVE statement were executed. This includes any transformations by the OPTMODEL presolver (see the section "Presolver" on page 769). In this form any fixed variables are replaced by their values and any objectives are replaced by the expressions that define

them. Unless an *identifier-expression* specifies a particular nonarray item or array location, the EXPAND output is restricted to only the variables, the constraints, and the most recent objective seen in a MAX or MIN declaration or specified in a SOLVE statement.

The following options restrict the types of declarations output when no specific nonarray item or array location is requested. By default all types of declarations are output. Only the requested declaration types are output when one or more of the following options are used.

VAR

requests the output of unfixed variables. The VAR option can also be used in combination with the name of a variable array to display just the unfixed elements of the array.

FIX

requests the output of fixed variables. These variables might have been fixed by the FIX statement (or by the presolver if the SOLVE option is specified). The FIX option can also be used in combination with the name of a variable array to display just the fixed elements of the array.

OBJECTIVE / OBJ

requests the output of objectives. Only the most recent objective seen in a MAX or MIN declaration or specified in a SOLVE statement is considered when the SOLVE option is used.

CONSTRAINT / CON

requests the output of undropped constraints.

For example, you can see the effect of a FIX statement on the problem that is presented to the solver by using the SOLVE option. You can modify the previous example as follows:

```
proc optmodel;
   number n=2;
   var x{1..n};
   min z1=sum{i in 1..n}(x[i]-i)**2;
   max z2=sum{i in 1..n}(i-x[i])**3;
   con c{i in 1..n}: x[i]>=0;
   fix x[2]=3;
   expand / solve;
```

This code produces the output in Figure 6.16.

```
Var x[1] >= 0
Fix x[2] = 3
Maximize z2=(-x[1] + 1)**3 - 1
```

Figure 6.16. Expansion with Fixed Variable

Compare the results in Figure 6.16 to those in Figure 6.15. The constraint c[1] has been converted to a variable bound. The subexpression that uses the fixed variable has been resolved to a constant.

FILE Statement

FILE *file-specification* **[LRECL=***value***]** **;**

The FILE statement selects the current output file for the PUT statement. By default PUT output is sent to the SAS log. Use the FILE statement to manage a group of output files. The specified file is opened for output if it is not already open. The output file remains open until it is closed with the CLOSEFILE statement.

File-specification names the output file. It can use any of the following forms:

'external-file'
: specifies the physical name of an external file in quotation marks. The interpretation of the filename depends on the operating environment.

file-name
: specifies the logical name associated with a file by the FILENAME statement or by the operating environment. The names PRINT and LOG are reserved to refer to the SAS listing and log files, respectively.

  Note: Details about the FILENAME statement can be found in *SAS Language Reference: Dictionary*.

(expression)
: specifies an expression that evaluates to a string that contains the physical name of an external file.

The LRECL option sets the line length of the output file. If the option is omitted, then the line length defaults to 256 characters. The LRECL option is ignored if the file is already open or if the PRINT or LOG file is specified.

The LRECL *value* can be specified in these forms:

integer
: specifies the desired line length.

identifier-expression
: specifies the name of a numeric parameter that contains the length.

(expression)
: specifies a numeric expression in parentheses that returns the line length.

The following example shows how to use the FILE statement to handle multiple files:

```
proc optmodel;
   file 'file.txt' lrecl=80;   /* opens file.txt     */
   put 'This is line 1 of file.txt.';
   file print;                  /* selects the listing */
   put 'This goes to the listing.';
   file 'file.txt';            /* reselects file.txt */
   put 'This is line 2 of file.txt.';
   closefile 'file.txt';       /* closes file.txt    */
   file log;                    /* selects the SAS log */
   put 'This goes to the log.';

   /* using expression to open and write a collection of files */
   str ofile;
   num i;
   num l = 40;
   do i = 1 to 3;
      ofile = ('file' || i || '.txt');
      file (ofile) lrecl=(l*i);
      put ('This goes to ' || ofile);
      closefile (ofile);
   end;
```

The following code illustrates the usefulness of using a logical name associated with a file by FILENAME statement:

```
proc optmodel;
   /* assigns a logical name to file.txt */
   /* see FILENAME statement in */
   /* SAS Language Reference: Dictionary */
   filename myfile 'file.txt' mod;

   file myfile;
   put 'This is line 3 of file.txt.';
   closefile myfile;
   file myfile;
   put 'This is line 4 of file.txt.';
   closefile myfile;
```

Notice that the FILENAME statement opens the file referenced for append. Therefore, new data are appended to the end every time the logical name, myfile, is used in the FILE statement.

FIX Statement

FIX *identifier-list* [**=(** *expression* **)**] **;**

The FIX statement causes the solver to treat a list of variables, variable arrays, or variable array locations as fixed in value. The *identifier-list* consists of one or more variable names separated by spaces. Each member of the *identifier-list* is fixed to the same *expression*. For example, the following code fixes the variables x and y to 3:

```
proc optmodel;
   var x, y;
   num a = 2;
   fix x y=(a+1);
```

A variable is specified with an *identifier-expression* (see the section "Identifier Expressions" on page 679). An entire variable array is fixed if the *identifier-expression* names an array without providing an index. A new value can be specified with the *expression*. If the *expression* is a constant, then the parentheses can be omitted. For example, the following code fixes all locations in array x to 0 except x[10], which is fixed to 1:

```
proc optmodel;
   var x{1..10};
   fix x = 0;
   fix x[10] = 1;
```

If *expression* is omitted, the variable is fixed at its current value. For example, you can fix some variables to be their optimal values after the SOLVE statement is invoked.

The effect of FIX can be reversed using the UNFIX statement.

FOR Statement

FOR { *index-set* **}** *statement* **;**

The FOR statement executes its substatement for each member of the specified *index-set*. The index set can declare local dummy parameters. You can reference the value of these parameters in the substatement. For example, consider the following code:

```
proc optmodel;
    for {i in 1..2, j in {'a', 'b'}} put i= j=;
```

This code produces the output in Figure 6.17.

```
i=1 j=a
i=1 j=b
i=2 j=a
i=2 j=b
```

Figure 6.17. FOR Statement Output

As another example, the following code sets the current values for variable x to random values between 0 and 1:

```
proc optmodel;
   var x{1..10};
   for {i in 1..10}
       x[i] = ranuni(-1);
```

Multiple statements can be controlled by specifying a DO statement group for the substatement.

CAUTION: Avoid modifying the parameters that are used by the FOR statement index set from within the substatement. The set value that is used for the left-most index set item is not affected by such changes. However, the effect of parameter changes on later index set items cannot be predicted.

IF Statement

IF *logic-expression* **THEN** *statement* **;** *[* **ELSE** *statement* **;** *]*

The IF statement evaluates the logical expression and then conditionally executes the THEN or ELSE substatements. The substatement that follows the THEN keyword is executed when the logical expression result is nonmissing and nonzero. The ELSE substatement, if any, is executed when the logical expression result is a missing value or zero. The ELSE part is optional and must immediately follow the THEN substatement. When IF statements are nested, an ELSE is always matched to the nearest incomplete unmatched IF-THEN. Multiple statements can be controlled by using DO statements with the THEN or ELSE substatements.

Note: When an IF-THEN statement is used **without** an ELSE substatement, substatements of the IF statement are executed when possible as they are entered. Under certain circumstances, such as when an IF statement is nested in a FOR loop, the statement is not executed during interactive input until the next statement is seen. By following the IF-THEN statement with an extra semicolon, you can cause it to be executed upon submission, since the extra semicolon is handled as a null statement.

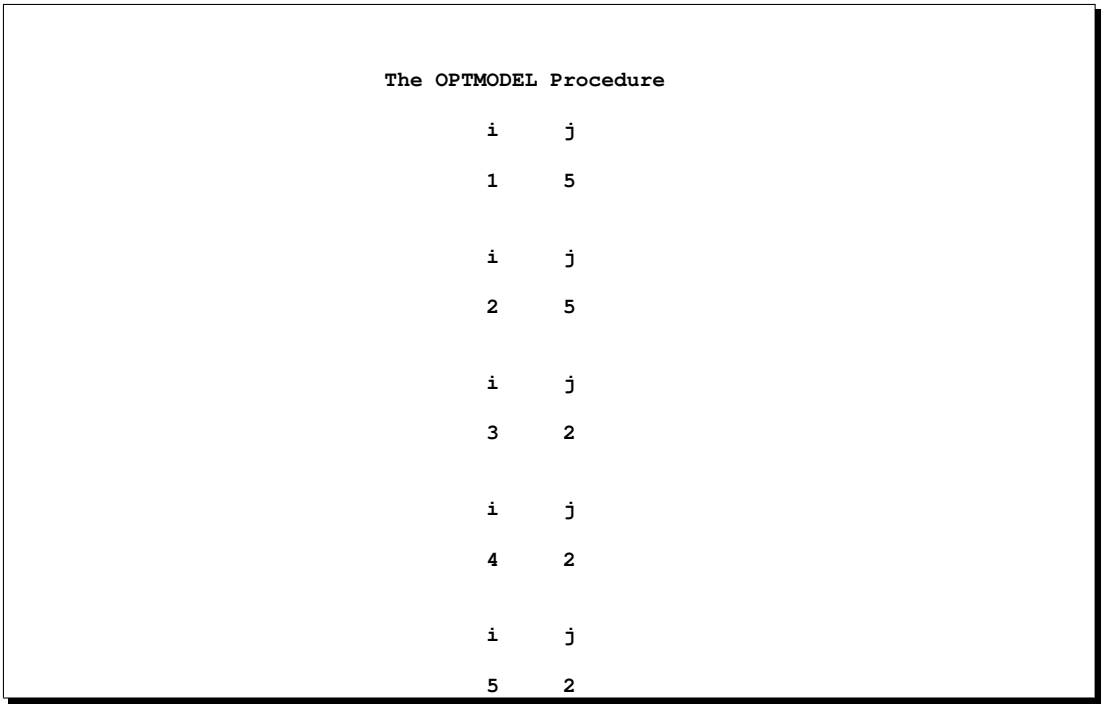
LEAVE Statement

LEAVE ;

The LEAVE statement terminates the execution of the entire loop body (iterative DO, DO UNTIL, DO WHILE, or FOR) that immediately contains the LEAVE statement. Execution resumes at the statement that follows the loop. The following example demonstrates a simple use of the LEAVE statement:

```
proc optmodel;
   number i, j;
   do i = 1..5;
      do j = 1..4;
            if i >= 3 and j = 2 then leave;
         end;
   print i j;
   end;
```

The results from this code are displayed in Figure 6.18.



```
The OPTMODEL Procedure

i     j

1     5


i     j

2     5


i     j

3     2


i     j

4     2


i     j

5     2
```

Figure 6.18. LEAVE Statement Output

For values of i equal to 1 or 2, the inner loop continues uninterrupted, leaving j with a value of 5. For values of i equal to 3, 4, or 5, the inner loop terminates early, leaving j with a value of 2.

Null Statement

;

The null statement is treated as a statement in the PROC OPTMODEL syntax, but its execution has no effect. It can be used as a placeholder statement.

PRINT Statement

PRINT *print-item(s)* **;**

The PRINT statement outputs string and numeric data in tabular form. The statement specifies a list of arrays or other data items to print. Multiple items can be output together as data columns in the same table.

If no format is specified, the PRINT statement handles the details of formatting automatically (see the section "Formatted Output" on page 748 for details). The default format for a numerical column is the fixed-point format (*w.d* format), which is chosen based on the values of the PDIGITS= and PWIDTH= options (see the section "PROC OPTMODEL Statement" on page 684) as well as the values in the column. The PRINT statement uses scientific notation (the E*w.* format) when a value is too large or too small to display in fixed format. The default format for a character col-

umn is the $*w.* format, where the width is set to be the length of the longest string (ignoring trailing blanks) in the column.

Print-item can be specified in the following forms:

identifier-expression [format]
: specifies a data item to output. *Identifier-expression* can name an array. In that case all defined array locations are output. *Format* specifies a SAS format that overrides the default format.

(*expression***)** *[format]*
: specifies a data value to output. *Format* specifies a SAS format that overrides the default format.

{*index-set***}** *identifier-expression [format]*
: specifies a data item to output under the control of an *index set*. The item is printed as if it were an array with the specified set of indices. This form can be used to print a subset of the locations in an array, such as a single column. If the *identifier-expression* names an array, then the indices of the array must match the indices of the *index-set*. *Format* specifies a SAS format that overrides the default format.

{*index-set***}** **(***expression***)** *[format]*
: specifies a data item to output under the control of an *index set*. The item is printed as if it were an array with the specified set of indices. In this form the *expression* is evaluated for each member of the *index-set* to create the array values for output. *Format* specifies a SAS format that overrides the default format.

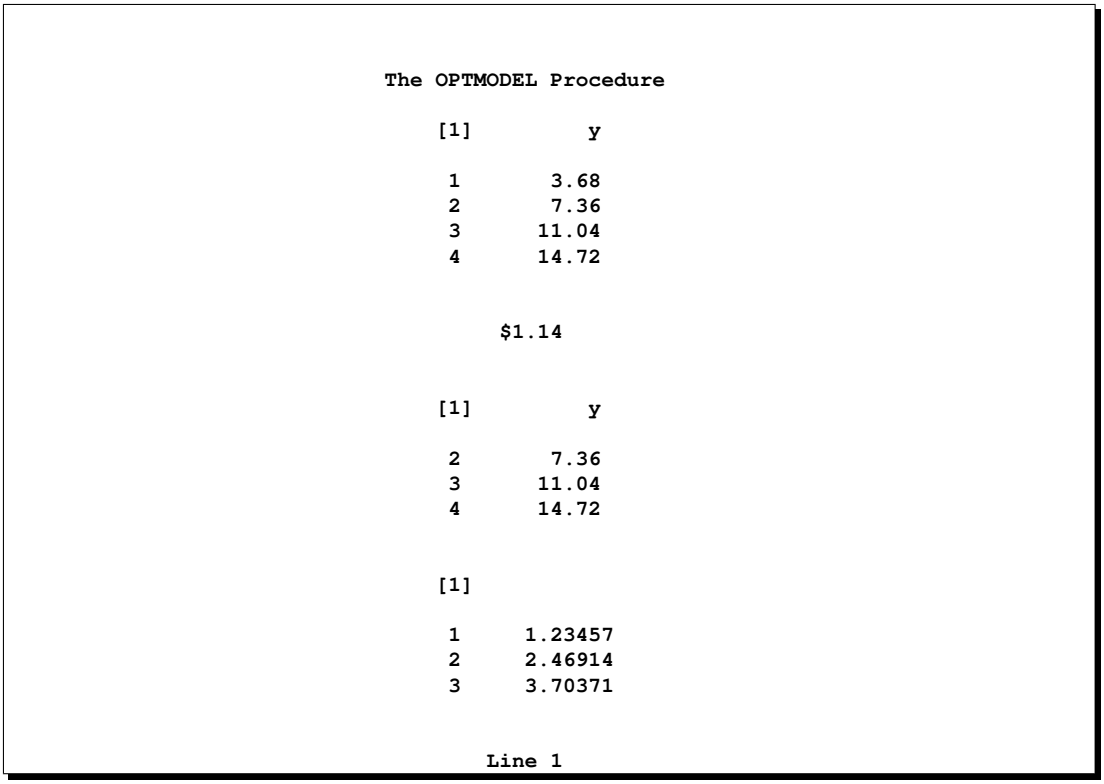
string
: specifies a string value to print.

PAGE
: specifies a page break.

The following example demonstrates the use of several *print-item* forms:

```
proc optmodel;
   num x = 4.3;
   var y{j in 1..4} init j*3.68;
   print y; /* identifier-expression */
   print (x * .265) dollar6.2; /* (expression) [format] */
   print {i in 2..4} y; /* {index-set} identifier-expression */
   print {i in 1..3}(i + i*.2345692) best7.;
                              /* {index-set} (expression) [format] */
   print "Line 1"; /* string */
```

The output is displayed in Figure 6.19.



```
The OPTMODEL Procedure

 [1]          y

  1        3.68
  2        7.36
  3       11.04
  4       14.72


     $1.14


 [1]          y

  2        7.36
  3       11.04
  4       14.72


 [1]

  1     1.23457
  2     2.46914
  3     3.70371


    Line 1
```

Figure 6.19. *Print-item* Forms

Adjacent print items that have similar indexing are grouped together and output in the same table. Items have similar indexing if they specify arrays that have the same number of indices and have matching index types (numeric versus string). Nonarray items are considered to have the same indexing as other nonarray items. The resulting table has a column for each array index followed by a column for each print item value. This format is called *list form*. For example, the following code produces a list form table:

```
proc optmodel;
   num a{i in 1..3} = i*i;
   num b{i in 3..5} = 4*i;
   print a b;
```

This code produces the listing output in Figure 6.20.

[1]	a	b
1	1	
2	4	
3	9	12
4		16
5		20

Figure 6.20. List Form PRINT Table

The array index columns show the set of valid index values for the print items in the table. The array index column for the ith index is labeled [i]. There is a row for each combination of index values that was used. The index values are displayed in sorted ascending order.

The data columns show the array values that correspond to the index values in each row. If a particular array index is invalid or the array location is undefined, then the corresponding table entry is displayed as blank for numeric arrays and as an empty string for string arrays. If the print items are scalar, then the table has a single row and no array index columns.

If a table contains a single array print item, the array is two-dimensional (has two indices), and the array is dense enough, then the array is shown in *matrix form*. In this format there is a single index column that contains the row index values. The label of this column is blank. This column is followed by a column for every unique column index value for the array. The latter columns are labeled by the column value. These columns contain the array values for that particular array column. Table entries that correspond to array locations that have invalid or undefined combinations of row and column indices are blank or (for strings) printed as an empty string.

The following code generates a simple example of matrix output:

```
proc optmodel;
   print {i in 1..6, j in i..6} (i*10+j);
```

The PRINT statement produces the output in Figure 6.21.

	1	2	3	4	5	6
1	11	12	13	14	15	16
2		22	23	24	25	26
3			33	34	35	36
4				44	45	46
5					55	56
6						66

Figure 6.21. Matrix Form PRINT Table

The PRINT statement prints single two-dimensional arrays in the form that uses fewer table cells (headings are ignored). Sparse arrays are normally printed in list form, and dense arrays are normally printed in matrix form. In a PROC OPTMODEL statement, the PMATRIX= option enables you to tune how the PRINT statement displays a two-dimensional array. The value of this option scales the total number of nonempty array elements, which is used to compute the tables cells needed for list form display. Specifying values for the PMATRIX= option less than 1 causes the list form to be used in more cases, while specifying values greater than 1 causes the matrix form to be used in more cases. If the value is 0, then list form is always used. The default value of the PMATRIX= option is 1. Changing the default can be done with the RESET OPTIONS statement.

The following code illustrates how the PMATRIX= option affects the display of the PRINT statement:

```
proc optmodel;
   num a{i in 1..6, i..i} = i;
   num b{i in 1..3, j in 1..3} = i*j;
   print a;
   print b;
   reset options pmatrix=3;
   print a;
   reset options pmatrix=0.5;
   print b;
```

The output is shown in Figure 6.22.

```
The OPTMODEL Procedure

   [1]    [2]     a

    1      1      1
    2      2      2
    3      3      3
    4      4      4
    5      5      5
    6      6      6


            b
        1      2      3

    1   1      2      3
    2   2      4      6
    3   3      6      9


                a
      1     2     3     4     5     6

1     1
2           2
3                 3
4                       4
5                             5
6                                   6


   [1]    [2]     b

    1      1      1
    1      2      2
    1      3      3
    2      1      2
    2      2      4
    2      3      6
    3      1      3
    3      2      6
    3      3      9
```

Figure 6.22. PRINT Statement: Effects of PMATRIX= Option

From Figure 6.22, it can be seen that by default, the PRINT statement tries to make the display compact. However, the default can be changed by using the PMATRIX= option.

PUT Statement

PUT *[put-item(s)] [@ | @@]* **;**

The PUT statement writes text data to the current output file. The syntax of the PUT statement in PROC OPTMODEL is similar to the syntax of the PROC IML and DATA step PUT statements. The PUT statement contains a list of items that specify data for output and provide instructions for formatting the data.

The current output file is initially the SAS log. This can be overridden with the FILE statement. An output file can be closed with the CLOSEFILE statement.

Normally the PUT statement outputs the current line after processing all items. Final @ or @@ operators suppress this automatic line output and cause the current column position to be retained for use in the next PUT statement.

Put-item can take any of the following forms.

identifier-expression [=] [format]
: outputs the value of the parameter or variable that is specified by the *identifier-expression*. The equal sign (=) causes a name for the location to be printed before each location value.

  Normally each item value is printed in a default format. Any leading and trailing blanks in the formatted value are removed and the value is followed by a blank space. When an explicit format is specified, the value is printed within the width determined by the format.

name[*] *[.suffix] [=] [format]*
: outputs each defined location value for an array parameter. The array name is specified as in the *identifier-expression* form except that the index list is replaced by an asterisk (*). The equal sign (=) causes a name for the location to be printed before each location value along with the actual index values to be substituted for the asterisk.

  Each item value normally prints in a default format. Any leading and trailing blanks in the formatted value are removed and the value is followed by a blank space. When an explicit format is specified, the value is printed within the width determined by the format.

(*expression*) *[=] [format]*
: outputs the value of the expression enclosed in parentheses. This produces similar results to the *identifier-expression* form except that the equal sign (=) uses the expression to form the name.

'*quoted-string*'
: copies the string to the output file.

@*integer*

@*identifier-expression*

@(*expression*)
: sets the absolute column position within the current line. The literal or expression value determines the new column position.

+*integer*

+*identifier-expression*

+(*expression*)
: sets the relative column position within the current line. The literal

or expression value determines the amount to update the column position.

/

outputs the current line and moves to the first column of the next line.

PAGE

outputs any pending line data and moves to the top of the next page.

READ DATA Statement

READ DATA *SAS-data-set* **[NOMISS] INTO** *[[set-name* **=]** *]* **[** *read-key-column(s)* **]** *] [read-column(s)]* **;**

The READ DATA statement reads data from a SAS data set into PROC OPTMODEL parameter and variable locations. The arguments to the READ DATA statement are as follows:

SAS-data-set

specifies the input data set name and options.

set-name

specifies a set parameter in which to save the set of observation key values read from the input data set.

read-key-column(s)

provide the index values for array destinations.

read-column(s)

specify the data values to read and the destination locations.

The following example uses the READ DATA statement to copy data set variables j and k from the SAS data set indata into parameters of the same name. The READ= data set option is used to specify a password.

```
proc optmodel;
   number j, k;
   read data indata(read=secret) into j k;
```

Key Columns

If any *read-key-column(s)* are specified, then the READ DATA statement reads all observations from the input data set. If no *read-key-column(s)* are specified, then only the first observation of the data set is read. The data set is closed after reading the requested information.

Each *read-key-column* declares a local dummy parameter and specifies a data set variable that supplies the column value. The values of the specified data set variables from each observation are combined into a key tuple. This combination is known as the *observation key*. The *observation key* is used to index array locations specified

by the *read-column(s)* items. The *observation key* is expected to be unique for each observation read from the data set.

The syntax for a *read-key-column(s)* is as follows:

name [= source-name] [/trim-option]

A *read-key-column* creates a local dummy parameter named *name* that holds an element of the *observation key* tuple. The dummy parameter can be used in subsequent *read-column(s)* items to reference the element value. If a *source-name* is given, then it specifies the data set variable that supplies the value. Otherwise the source data set variable has the same name as the dummy parameter, *name*. Use the special data set variable name _N_ to refer to the number identification of the observations.

You can specify a *set-name* to save the set of observation keys into a set parameter. If the *observation key* consists of a single scalar value, then the set member type must match the scalar type. Otherwise the set member type must be a tuple with element types that match the corresponding observation key element types.

The READ DATA statement initially assigns an empty set to the target *set-name* parameter. As observations are read, a tuple for each observation key is added to the set. A set used to index an array destination in the *read-column(s)* can be read at the same time as the array values. Consider a data set, invdata, created by the following code:

```
data invdata;
   input item $ invcount;
       datalines;
       table 100
       sofa  250
       chair 80
       ;
```

The following code reads the data set invdata, which has two variables, item and invcount. The READ DATA statement constructs a set of inventory items, Items. At the same time, the parameter location invcount[item] is assigned the value of the data set variable invcount in the corresponding observation.

```
proc optmodel;
   set<string> Items;
   number invcount{Items};
   read data invdata into Items=[item] invcount;
   print invcount;
```

The output of this code is shown in Figure 6.23.

```
The OPTMODEL Procedure

  [1]       invcount

  chair            80
  sofa            250
  table           100
```

Figure 6.23. READ DATA Statement: Key Column

When observations are read, the values of data set variables are copied to parameter locations. Numeric values are copied unchanged. For character values, *trim-option* controls how leading and trailing blanks are processed. *Trim-option* is ignored when the value type is numeric. Specify any of the following keywords for *trim-option*:

TRIM | **TR**
: removes leading and trailing blanks from the data set value. This is the default behavior.

LTRIM | **LT**
: removes only leading blanks from the data set value.

RTRIM | **RT**
: removes only trailing blanks from the data set value.

NOTRIM | **NT**
: copies the data set value with no changes.

Columns

Read-column(s) specify data set variables to read and PROC OPTMODEL parameter locations to which to assign the values. The types of the input data set variables must match the types of the parameters. Array parameters can be implicitly or explicitly indexed by the *observation key* values.

Normally, missing values from the data set are assigned to the parameters that are specified in the *read-column(s)*. The NOMISS keyword suppresses the assignment of missing values, leaving the corresponding parameter locations unchanged. Note that the parameter location does not need to have a valid index in this case. This permits a single statement to read data into multiple arrays that have different index sets.

Read-column(s) has the following forms:

identifier-expression [**=** *name* | **COL(** *name-expression* **)**] [/ *trim-option*]
: transfers an input data set variable to a target parameter or variable. *Identifier-expression* specifies the target. If the *identifier-expression* specifies an array without an explicit index, then the

observation key provides an implicit index. The name of the input data set variable can be specified with a *name* or a COL expression. Otherwise the data set variable name is given by the *name* part of the *identifier-expression*. For COL expressions, the string-valued *name-expression* is evaluated to determine the data set variable name. *Trim-option* controls removal of leading and trailing blanks in the incoming data. For example, the following code reads the data set variables column1 and column2 from the data set exdata into the PROC OPTMODEL parameters p and q, respectively. The observation numbers in exdata are read into the set indx, which indexes p and q.

```
data exdata;
   input column1 column2;
   datalines;
1 2
3 4
;

proc optmodel;
   number n init 2;
   set<num> indx;
   number p{indx}, q{indx};
   read data exdata into
      indx=[_N_] p=column1 q=col("column"||n);
   print p q;
```

The output is shown in Figure 6.24.

```
The OPTMODEL Procedure

   [1]     p     q

    1      1     2
    2      3     4
```

Figure 6.24. READ DATA Statement: Identifier Expressions

{ *index-set* **}** **<** *read-column(s)* **>**

performs the transfers by iterating each column specified by <*read-column(s)*> for each member of the *index-set*. If there are n columns and m index set members, then $n \times m$ columns are generated. The dummy parameters from the index set can be used in the columns to generate distinct input data set variable names in the iterated columns, using COL expressions. The columns are expanded when the READ DATA statement is executed, before any observations are read. This form of *read-column(s)* cannot be nested. In other words, the following form of *read-column(s)* is NOT allowed:

{ *index-set* **} < {** *index-set* **} <** *read-column(s)* **> >**

An example demonstrating the use of the iterated column *read-option* follows.

You can use an iterated column *read-option* to read multiple data set variables into the same array. For example, a data set might store an entire row of array data in a group of data set variables. The following code demonstrates how to read a data set containing demand data divided by day:

```
data dmnd;
   input loc $ day1 day2 day3 day4 day5;
   datalines;
East 1.1 2.3 1.3 3.6 4.7
West 7.0 2.1 6.1 5.8 3.2
;
proc optmodel;
   set DOW = 1..5;  /* days of week, 1=Monday, 5=Friday */
   set<string> LOC; /* locations */
   number demand{LOC, DOW};
   read data dmnd
        into LOC=[loc]
        {d in DOW} < demand[loc, d]=col("day"||d) >;
   print demand;
```

This reads a set of demand variables named DAY1–DAY5 from each observation, filling in the two-dimensional array demand. The output is shown in Figure 6.25.

	demand				
	1	2	3	4	5
East	1.1	2.3	1.3	3.6	4.7
West	7.0	2.1	6.1	5.8	3.2

Figure 6.25. Demand Data

RESET OPTIONS Statement

RESET OPTIONS *option(s)* **;**

RESET OPTION *option(s)* **;**

The RESET OPTIONS statement sets PROC OPTMODEL option values or restores them to their defaults. Options can be specified by using the same syntax as in the PROC OPTMODEL statement. The RESET OPTIONS statement provides two extensions to the option syntax. If an option normally requires a value (specified with an equal sign (=) operator), then specifying the option name alone resets it to its default value. You can also specify an expression enclosed in parentheses in place of a literal value. See the section "OPTMODEL Options" on page 773 for an example.

The RESET OPTIONS statement can be placed inside loops or conditional code. The statement is applied each time it is executed.

RESTORE Statement

RESTORE *identifier-expression* **;**

The RESTORE statement adds a constraint, constraint array, or constraint array location that was dropped by the DROP statement back into the solver model. *Identifier-expression* specifies the constraint. An entire constraint array is restored if the *identifier-expression* omits the index from an array name. For example, the following code declares a constraint array and then drops it:

```
con c{i in 1..4}: x[i] + y[i] <=1;
drop c;
```

The following statement restores the first constraint:

```
restore c[1];
```

Multiple names can be specified in a single RESTORE statement. The following statement restores the second and third constraints:

```
restore c[2] c[3];
```

If you want to restore all of them, you can submit the following statement:

```
restore c;
```

SAVE MPS Statement

SAVE MPS *SAS-data-set* **;**

The SAVE MPS statement saves the structure and coefficients for a linear programming model into a SAS data set. This data set can be used as input data for the OPTLP or OPTMILP procedure.

Note: The OPTMODEL presolver (see the section "Presolver" on page 769) is automatically bypassed so that the statement saves the original model *without* eliminating fixed variables, tightening bounds, etc.

The *SAS-data-set* argument specifies the output data set name and options. The output data set uses the MPS format described in Chapter 14. The generated data set contains observations that define different parts of the linear program.

Variables, constraints, and objectives are referenced in the data set by using label text based on the model name. For example, a model variable x[1] would be labeled "x[1]" in the data set. Labels are limited by default to 32 characters and are abbreviated to fit. You can change the maximum length for labels by using the MAXLABLEN= option. When needed, a programmatically generated number is added to labels to

avoid duplication. Only the most recent objective, which was specified in a MIN or MAX declaration or specified in a SOLVE statement, is included in the data set.

When an integer variable has been assigned a nondefault branching priority or direction, the MPS data set includes a BRANCH section. See Chapter 14 for more details.

The following code shows an example of the SAVE MPS statement. The model is specified using the OPTMODEL procedure. Then it is saved as the MPS data set MPSData, as shown in Figure 6.26. Next, PROC OPTLP is used to solve the resulting linear program.

```
proc optmodel;
   var x >= 0, y >= 0;
   con c: x >= y;
   con bx: x <= 2;
   con by: y <= 1;
   min obj=0.5*x-y;
   save mps MPSData;
quit;
proc optlp data=MPSData pout=PrimalOut dout=DualOut;
run;
```

Obs	FIELD1	FIELD2	FIELD3	FIELD4	FIELD5	FIELD6
1	NAME		MPSData	.		.
2	ROWS			.		.
3	N	obj		.		.
4	G	c		.		.
5	L	bx		.		.
6	L	by		.		.
7	COLUMNS			.		.
8		x	obj	0.5	c	1
9		x	bx	1.0		.
10		y	obj	-1.0	c	-1
11		y	by	1.0		.
12	RHS			.		.
13		.RHS.	bx	2.0		.
14		.RHS.	by	1.0		.
15	ENDATA			.		.

Figure 6.26. The MPS Data Set Generated by SAVE MPS Statement

SAVE QPS Statement

SAVE QPS *SAS-data-set* **;**

The SAVE QPS statement saves the structure and coefficients for a quadratic programming model into a SAS data set. This data set can be used as input data for the OPTQP procedure.

Note: The OPTMODEL presolver (see the section "Presolver" on page 769) is automatically bypassed so that the statement saves the original model *without* eliminating fixed variables, tightening bounds, etc.

The *SAS-data-set* argument specifies the output data set name and options. The output data set uses the QPS format described in Chapter 14. The generated data set contains observations that define different parts of the quadratic program.

Variables, constraints, and objectives are referenced in the data set by using label text based on the model name. For example, a model variable x[1] would be labeled "x[1]" in the data set. Labels are limited by default to 32 characters and are abbreviated to fit. You can change the maximum length for labels by using the MAXLABLEN= option. When needed, a programmatically generated number is added to labels to avoid duplication. Only the most recent objective, which was specified in a MIN or MAX declaration or specified in a SOLVE statement, is included in the output data set. The coefficients of the objective function appear in the QSECTION section.

The following code shows an example of the SAVE QPS statement. The model is specified using the OPTMODEL procedure. Then it is saved as the QPS data set QPSData, as shown in Figure 6.27. Next, PROC OPTQP is used to solve the resulting quadratic program.

```
proc optmodel;
   var x{1..2} >= 0;
      min z = 2*x[1] + 3 * x[2] + x[1]**2 + 10*x[2]**2
                  + 2.5*x[1]*x[2];
      con c1: x[1] - x[2] <= 1;
      con c2: x[1] + 2*x[2] >= 100;
      save qps QPSData;
quit;
proc optqp data=QPSData pout=PrimalOut dout=DualOut;
run;
```

Obs	FIELD1	FIELD2	FIELD3	FIELD4	FIELD5	FIELD6
1	NAME		QPSData	.		.
2	ROWS			.		.
3	N	z		.		.
4	L	c1		.		.
5	G	c2		.		.
6	COLUMNS			.		.
7		x[1]	z	2.0	c1	1
8		x[1]	c2	1.0		.
9		x[2]	z	3.0	c1	-1
10		x[2]	c2	2.0		.
11	RHS			.		.
12		.RHS.	c1	1.0		.
13		.RHS.	c2	100.0		.
14	QSECTION			.		.
15		x[1]	x[1]	2.0		.
16		x[1]	x[2]	2.5		.
17		x[2]	x[2]	20.0		.
18	ENDATA			.		.

Figure 6.27. The QPS Data Set Generated by SAVE QPS Statement

SOLVE Statement

SOLVE *[* **WITH** *solver] [(***OBJECTIVE**/**OBJ***) name] [/ options]* **;**

The SOLVE statement invokes a PROC OPTMODEL solver. The current model is first resolved to the numeric form that is required by the solver. The resolved model and possibly the current values of any optimization variables are passed to the solver. After the solver finishes executing, the SOLVE statement prints a short table giving a summary of results from the solver (see the section "ODS Table and Variable Names" on page 750) and updates the _OROPTMODEL_ macro variable.

Here are the arguments to the SOLVE statement:

solver
: selects the named solver, namely LP, MILP, QP, NLPC, NLPU, SQP, or IPNLP (see corresponding chapters in this book for details). If no WITH clause is specified, then a solver is chosen automatically, depending on the problem type.

name
: specifies the objective to use. You can abbreviate the OBJECTIVE keyword as OBJ. If no name is specified, then the solver uses the most recent objective seen in a MAX or MIN declaration or specified in a SOLVE statement.

options
: specifies solver options. Solver options can be specified only when the WITH clause is used. A list of the options available with the solver is provided in the individual chapters describing each solver.

Optimization techniques that use initial values obtain them from the current values of the optimization variables unless the NOINITVAR option is specified. When the solver finishes executing, the current value of each optimization variable is replaced by the optimal value found by the solver. These values can then be used as the initial values for subsequent solver invocations. The .init suffix location for each variable saves the initial value used for the most recent SOLVE statement.

Note: Should a solver fail, any currently pending statement is stopped and processing continues with the next complete statement read from the input. For example, if a SOLVE statement enclosed in a DO group (see the section "DO Statement" on page 701) fails, then the subsequent statements in the group are not executed and processing resumes at the point immediately following the DO group. Note that neither an infeasible result, an unbounded result, nor reaching an iteration limit is considered to be a solver failure.

Note: The information appearing in the macro variable _OROPTMODEL_ (see the section "Macro Variable _OROPTMODEL_" on page 728) varies by solver.

STOP Statement

STOP ;

The STOP statement halts the execution of all statements that contain it, including DO statements and other control or looping statements. Execution continues with the next top-level source statement. The following code demonstrates a simple use of the STOP statement:

```
proc optmodel;
   number i, j;
   do i = 1..5;
      do j = 1..4;
         if i = 3 and j = 2 then stop;
      end;
   end;
   print i j;
```

The output is shown in Figure 6.28.

```
The OPTMODEL Procedure

        i     j

        3     2
```

Figure 6.28. STOP Statement: Output

When the counters i and j reach 3 and 2, respectively, the STOP statement terminates both loops. Execution continues with the PRINT statement.

UNFIX Statement

UNFIX *identifier-list [* **=(** *expression* **)** *]* **;**

The UNFIX statement reverses the effect of FIX statements. The solver can vary the specified variables, variable arrays, or variable array locations specified by *identifier-list*. The *identifier-list* consists of one or more variable names separated by spaces.

Variable is an *identifier expression* (see the section "Identifier Expressions" on page 679). The UNFIX statement affects an entire variable array if the identifier expression omits the index from an array name. The *expression* specifies a new initial value that will be stored in each element of the *identifier-list*.

The following example demonstrates the UNFIX command:

```
proc optmodel;
   var x{1..3};
   fix x;        /* fixes entire array to 0 */
   unfix x[1];   /* x[1] can now be varied again */
   unfix x[2] = 2;  /* x[2] is given an initial value 2 */
```

```
                                                /* and can be varied now */
   unfix x;  /* all x indices can now be varied */
```

After the following code is executed, the variables x[1] and x[2] are not fixed. They each hold the value 4. The variable x[3] is fixed at a value of 2.

```
proc optmodel;
   var x{1..3} init 2;
   num a = 1;
   fix x;
   unfix x[1] x[2]=(a+3);
```

Macro Variable _OROPTMODEL_

The OPTMODEL procedure creates a macro variable named _OROPTMODEL_. You can inspect the execution of the most recently invoked solver from the value of the macro variable. The macro variable is defined at the start of the procedure and updated after each SOLVE statement is executed. The OPTMODEL procedure also updates the macro variable when an error is detected.

The _OROPTMODEL_ value is a string consisting of several "KEYWORD=*value*" items in sequence, separated by blanks; for example:

```
STATUS=OK SOLUTION_STATUS=OPTIMAL OBJECTIVE=9 ITERATIONS=1
PRESOLVE_TIME=0 SOLUTION_TIME=0
```

The information contained in _OROPTMODEL_ varies according to which solver was last called. For lists of keywords and possible values, see the individual solver chapters.

If a value has not been computed, then the corresponding element is not included in the value of the macro variable. When PROC OPTMODEL starts, for instance, the macro variable value is set to "STATUS=OK" because no SOLVE statement has been executed. If the STATUS= indicates an error, then the other values from the solver might not be available, depending on when the error occurred.

Note that PROC OPTMODEL reads a complete statement, such as a DO statement, before executing any code in it. But macro language statements are processed as the code is read. So you must be careful when using the _OROPTMODEL_ macro variable in code involving SOLVE statements nested in loops or DO statements. The following code demonstrates one example of this behavior:

```
proc optmodel;
   var x, y;
   min z=x**2 + (x*y-1)**2;
   for {n in 1..3} do;
      fix x=n;
      solve;
      %put Line 1 &_OROPTMODEL_;
```

```
      put 'Line 2 ' (symget("_OROPTMODEL_"));
   end;
quit;
```

In the preceding code the %PUT statement is executed once, before any SOLVE statements are executed. It displays PROC OPTMODEL's initial setting of the macro variable. But the PUT statement is executed after each SOLVE statement, and indicates the expected solution status.

OPTMODEL Expression Extensions

PROC OPTMODEL defines several new types of expressions for the manipulation of sets. Aggregation operators combine values of an expression that is evaluated over the members of an index set. Other operators create new sets by combining existing sets, or test relationships between sets. PROC OPTMODEL also supports an IF expression operator that can conditionally evaluate expressions. These and other such expressions are described in this section.

AND Aggregation Expression

AND{ *index-set* **}** *logic-expression*

The AND aggregation operator evaluates the logical expression *logic-expression* jointly for each member of the index set *index-set*. The index set enumeration finishes early if the *logic-expression* evaluation produces a false value (zero or missing). The expression returns 0 if a false value is found or returns 1 otherwise. The following code demonstrates both a true and a false result:

```
proc optmodel;
   put (and{i in 1..5} i < 10); /* returns 1 */
   put (and{i in 1..5} i NE 3); /* returns 0 */
```

CARD Function

CARD(*set-expression* **)**

The CARD function returns the number of members of its set operand. For example, the following code produces the output 3 since the set has 3 members:

```
proc optmodel;
   put (card(1..3));
```

CROSS Expression

set-expression **CROSS** *set-expression*

The CROSS expression returns the crossproduct of its set operands. The result is the set of tuples formed by concatenating the tuple value of each member of the

left operand with the tuple value of each member of the right operand. Scalar set members are treated as tuples of length 1. The following code demonstrates the CROSS operator:

```
proc optmodel;
   set s1 = 1..2;
   set<string> s2 = {'a', 'b'};
   set<number, string> s3=s1 cross s2;
   put 's3 is ' s3;
   set<number, string, number> s4 = s3 cross 4..5;
   put 's4 is ' s4;
```

This code produces the output in Figure 6.29.

```
s3 is {<1,'a'>,<1,'b'>,<2,'a'>,<2,'b'>}
s4 is {<1,'a',4>,<1,'a',5>,<1,'b',4>,<1,'b',5>,<2,'a',4>,<2,'a',5>,<2,'b',4>,<2,
'b',5>}
```

Figure 6.29. CROSS Expression Output

DIFF Expression

set-expression **DIFF** *set-expression*

The DIFF operator returns a set that contains the set difference of the left and right operands. The result set contains values that are members of the left operand but not members of the right operand. The operands must have compatible set types. The following code evaluates and prints a set difference:

```
proc optmodel;
   put ({1,3} diff {2,3}); /* outputs {1} */
```

IF-THEN/ELSE Expression

IF *logic-expression* **THEN** *expression-2 [* **ELSE** *expression-3]*

The IF-THEN/ELSE expression evaluates the logical expression *logic-expression* and returns the result of evaluating the second or third operand expression according to the logical test result. If the *logic-expression* is true (nonzero and nonmissing), then the result of evaluating *expression-2* is returned. If the *logic-expression* is false (zero or missing), then the result of evaluating *expression-3* is returned. The other subexpression that is not selected is not evaluated.

An ELSE clause is matched during parsing with the nearest IF-THEN clause that does not have a matching ELSE. The ELSE clause can be omitted for numeric expressions; the resulting IF-THEN is handled as if a default ELSE 0 clause were supplied.

Use the IF-THEN/ELSE expression to handle special cases in models. For example, an inventory model based on discrete time periods might require special handling for

the first or last period. In the following example the initial inventory for the first period is assumed to be fixed:

```
proc optmodel;
   number T;
   var inv{1..T}, order{1..T};
   number sell{1..T};
   number inv0;
   . . .
   /* balance inventory flow */
   con iflow{i in 1..T}:
       inv[i] = order[i] - sell[i] +
       if i=1 then inv0 else inv[i-1];
   . . .
```

The IF-THEN/ELSE expression in the example models the initial inventory for a time period i. Usually the inventory value is the inventory at the end of the previous period, but for the first time period the inventory value is given by the inv0 parameter. Note that the iflow constraints are linear since the IF-THEN/ELSE test subexpression does not depend on variables and the other subexpressions are linear.

IF-THEN/ELSE can be used as either a set expression or a scalar expression. The type of expression depends on the subexpression between the THEN and ELSE keywords. The type used affects the parsing of the subexpression that follows the ELSE keyword because the set form has a lower operator precedence. For example, the following two expressions are equivalent because the numeric IF-THEN/ELSE has a higher precedence than the range operator (..):

```
IF logic THEN 1 ELSE 2 .. 3

(IF logic THEN 1 ELSE 2) .. 3
```

But the set form of IF-THEN/ELSE has lower precedence than the range expression operator. So the following two expressions are equivalent:

```
IF logic THEN 1 .. 2 ELSE 3 .. 4

IF logic THEN (1 .. 2) ELSE (3 .. 4)
```

The IF-THEN and IF-THEN/ELSE operators always have higher precedence than the logic operators. So, for example, the following two expressions are equivalent:

```
IF logic THEN numeric1 < numeric2

(IF logic THEN numeric1) < numeric2
```

It is best to use parentheses when in doubt about precedence.

IN Expression

expression **IN** *set-expression*

expression **NOT IN** *set-expression*

The IN expression returns 1 if the value of the left operand is a member of the right operand set. Otherwise the IN expression returns 0. The NOT IN operator logically negates the returned value. Unlike the DATA step, the right operand is an arbitrary set expression. The left operand can be a tuple expression. The following example demonstrates the IN and NOT IN operators:

```
proc optmodel;
   set s = 1..10;
   put (5 in s);          /* outputs 1 */
   put (-1 not in s);     /* outputs 1 */
   set<num, str> t = {<1,'a'>, <2,'b'>, <2,'c'>};
   put (<2, 'b'> in t);    /* outputs 1 */
   put (<1, 'b'> in t);   /* outputs 0 */
```

Index Set Expression

{ *index-set* **}**

The index set expression returns the members of an index set. This expression is distinguished from a set constructor (see the section "Set Constructor Expression" on page 735) because it contains a list of set expressions.

The following code uses an index set with a selection expression that excludes the value 3:

```
proc optmodel;
   put ({i in 1..5 : i NE 3}); /* outputs {1,2,4,5} */
```

INTER Expression

set-expression **INTER** *set-expression*

The INTER operator returns a set that contains the intersection of the left and right operands. This is the set that contains values that are members of both operand sets. The operands must have compatible set types.

The following code evaluates and prints a set intersection:

```
proc optmodel;
   put ({1,3} inter {2,3}); /* outputs {3} */
```

INTER Aggregation Expression

INTER{ *index-set* **}** *set-expression*

The INTER aggregation operator evaluates the *set-expression* for each member of the index set *index-set*. The result is the set containing the intersection of the set of values that were returned by the *set-expression* for each member of the index set. An empty index set causes an expression evaluation error.

The following code uses the INTER aggregation operator to compute the value of $\{1,2,3,4\} \cap \{2,3,4,5\} \cap \{3,4,5,6\}$:

```
proc optmodel;
   put (inter{i in 1..3} i..i+3); /* outputs {3,4} */
```

MAX Aggregation Expression

MAX{ *index-set* **}** *expression*

The MAX aggregation operator evaluates the numeric expression *expression* for each member of the index set *index-set*. The result is the maximum of the values that are returned by the *expression*. Missing values are handled with the SAS numeric sort order, so a missing value is treated as smaller than any nonmissing value. If the index set is empty, then the result is the negative number that has the largest absolute value representable on the machine.

The following example produces the output `0.5`:

```
proc optmodel;
   put (max{i in 2..5} 1/i);
```

MIN Aggregation Expression

MIN{ *index-set* **}** *expression*

The MIN aggregation operator evaluates the numeric expression *expression* for each member of the index set *index-set*. The result is the minimum of the values that are returned by the *expression*. Missing values are handled with the SAS numeric sort order, so a missing value is treated as smaller than any nonmissing value. If the index set is empty, then the result is the largest positive number representable on the machine.

The following example produces the output `0.2`:

```
proc optmodel;
   put (min{i in 2..5} 1/i);
```

OR Aggregation Expression

OR{ *index-set* **}** *logic-expression*

The OR aggregation operator evaluates the logical expression *logic-expression* for each member of the index set *index-set*. The index set enumeration finishes early if the *logic-expression* evaluation produces a true value (nonzero and nonmissing). The result is 1 if a true value is found, or 0 otherwise. The following code demonstrates both a true and a false result:

```
proc optmodel;
   put (or{i in 1..5} i = 2); /* returns 1 */
   put (or{i in 1..5} i = 7); /* returns 0 */
```

PROD Aggregation Expression

PROD{ *index-set* **}** *expression*

The PROD aggregation operator evaluates the numeric expression *expression* for each member of the index set *index-set*. The result is the product of the values that are returned by the *expression*. This operator is analogous to the $\prod$ operator used in mathematical notation. If the index set is empty, then the result is 1.

The following example uses the PROD operator to evaluate a factorial:

```
proc optmodel;
   number n = 5;
   put (prod{i in 1..n} i); /* outputs 120 */
```

Range Expression

expression **..** *expression* *[* **BY** *expression* *]*

The range expression returns the set of numbers from the specified arithmetic progression. The sequence proceeds from the left operand value up to the right operand limit. The increment between numbers is 1 unless a different value is specified with a BY clause. If the increment is negative, then the progression is from the left operand down to the right operand limit. The result can be an empty set.

For compatibility with the DATA step iterative DO loop construct, the keyword TO can substitute for the range (..) operator.

The limit value is not included in the resulting set unless it belongs in the arithmetic progression. For example, the following range expression does not include 30:

```
proc optmodel;
   put (10..30 by 7); /* outputs {10,17,24} */
```

The actual numbers that the range expression "f..l by i" produces are in the arithmetic sequence

$$f, f+i, f+2i, \ldots, f+ni$$

where

$$n = \left\lfloor \frac{l-f}{i} + \sqrt{\epsilon} \right\rfloor$$

and ϵ represents the relative machine precision. The limit is adjusted to avoid arithmetic roundoff errors.

PROC OPTMODEL represents the set specified by a range expression compactly when the value is stored in a parameter location, used as a set operand of an IN/NOTIN expression, used by an iterative DO loop, or used in an index set. For example, the following expression is evaluated efficiently:

```
999998.5 IN 1..1000000000
```

Set Constructor Expression

{ *[expression-1 [, . . . expression-n]]* **}**

The set constructor expression returns the set of the expressions in the member list. Duplicated values are added to the set only once. A warning message is produced when duplicates are detected. The constructor expression consists of 0 or more subexpressions of the same scalar type or of tuple expressions that match in length and in element types.

The following code outputs a three-member set and warns about the duplicated value 2:

```
proc optmodel;
   put ({1,2,3,2}); /* outputs {1,2,3} */
```

The following example produces a three-member set of tuples, using PROC OPTMODEL parameters and variables. The output is displayed in Figure 6.30.

```
proc optmodel;
   number m = 3, n = 4;
   var x{1..4} init 1;
   string y = 'c';
   put ({<'a', x[3]>, <'b', m>, <y, m/n>});
```

```
{<'a',1>,<'b',3>,<'c',0.75>}
```

Figure 6.30. Set Constructor Expression Output

Set Literal Expression

/ member(s) /

The set literal expression provides compact specification of simple set values. It is equivalent in function to the set constructor expression but minimizes typing for sets that contain numeric and string constant values. The set members are specified by *members*, which are literal values. As with the set constructor expression, each member must have the same type.

The following code specifies a simple numeric set:

```
/1 2.5 4/
```

The set contains the members `1`, `2.5`, and `4`. A string set could be specified as follows:

```
/Miami 'San Francisco' Seattle 'Washington, D.C.'/
```

This set contains the strings `'Miami'`, `'San Francisco'`, `'Seattle'`, and `'Washington, D.C.'`. You can specify string values in set literals without quotation marks when the text follows the rules for a SAS name. Strings that begin with a digit or contain blanks or other special characters must be specified with quotation marks.

Specify tuple members of a set by enclosing the tuple elements within angle brackets (<*element(s)*>). The tuple elements can be specified with numeric and string literals. The following example includes the tuple elements `<'New York', 4.5>` and `<'Chicago', -5.7>`:

```
/<'New York' 4.5> <Chicago -5.7>/
```

SETOF Aggregation Expression

SETOF{ *index-set* **}** *expression*

The SETOF aggregation operator evaluates the expression *expression* for each member of the index set *index-set*. The result is the set that is formed by collecting the values returned by the operand expression. The operand can be a tuple expression. For example, the following code produces a set of tuples of numbers with their squared and cubed values:

```
proc optmodel;
   put (setof{i in 1..3}<i, i*i, i**3>);
```

Figure 6.31 shows the displayed output.

```
{<1,1,1>,<2,4,8>,<3,9,27>}
```

Figure 6.31. SETOF Aggregation Expression Output

SLICE Expression

SLICE(< *element-1*, . . . *element-n* >, *set-expression*)

The SLICE expression produces a new set by selecting members in the operand set that match a pattern tuple. The pattern tuple is specified by the element list in angle brackets. Each *element* in the pattern tuple must specify a numeric or string expression. The expressions are used to match the values of the corresponding elements in the operand set member tuples. You can also specify an *element* by using an asterisk (*). The sequence of element values that correspond to asterisk positions in each matching tuple is combined into a tuple of the result set. At least one asterisk *element* must be specified.

The following code demonstrates the SLICE expression:

```
proc optmodel;
   put (slice(<1,*>, {<1,3>, <1,0>, <3,1>}));
   put (slice(<*,2,*>, {<1,2,3>, <2,4,3>, <2,2,5>}));
```

This code produces the output in Figure 6.32.

```
{3,0}
{<1,3>,<2,5>}
```

Figure 6.32. SLICE Expression Output

For the first PUT statement, <1,*> matches set members <1,3> and <1,0> but not <3,1>. The second element of each matching set tuple, corresponding to the asterisk element, becomes the value of the resulting set member. In the second PUT statement, the values of the first and third elements of the operand set member tuple are combined into a two-position tuple in the result set.

The following code uses the SLICE expression to help compute the transitive closure of a set of tuples representing a relation by using Warshall's algorithm. In this code the set parameter dep represents a direct dependency relation.

```
proc optmodel;
   set<str,str> dep = {<'B','A'>, <'C','B'>, <'D','C'>};
   set<str,str> cl;
   set<str> cn;
   cl = dep;
   cn =  (setof{<i,j> in dep} i) inter (setof{<i,j> in dep} j);
```

```
for {node in cn}
    cl = cl union (slice(<*,node>,cl) cross slice(<node,*>,cl));
put cl;
```

The local dummy parameter node in the FOR statement iterates over the set cn of possible intermediate nodes that can connect relations transitively. At the end of each FOR iteration, the set parameter cl contains all tuples from the original set as well as all transitive tuples found in the current or previous iterations.

The output in Figure 6.33 includes the indirect as well as direct transitive dependencies from the set dep.

```
{<'B','A'>,<'C','B'>,<'D','C'>,<'C','A'>,<'D','B'>,<'D','A'>}
```

Figure 6.33. Warshall's Algorithm Output

A special form of *index-set-item* uses the SLICE expression implicitly. See the section "More on Index Sets" on page 776 for details.

SUM Aggregation Expression

SUM{ *index-set* **}** *expression*

The SUM aggregation operator evaluates the numeric expression *expression* for each member in the index set *index-set*. The result is the sum of the values that are returned by the *expression*. If the index set is empty, then the result is 0. This operator is analogous to the $\sum$ operator that is used in mathematical notation. The following code demonstrates the use of the SUM aggregation operator:

```
proc optmodel;
   put (sum {i in 1..10} i);  /* outputs 55 */
```

SYMDIFF Expression

set-expression **SYMDIFF** *set-expression*

The SYMDIFF expression returns the symmetric set difference of the left and right operands. The result set contains values that are members of either the left or right operand but are not members of both operands. The operands must have compatible set types.

The following example demonstrates a symmetric difference:

```
proc optmodel;
   put ({1,3} symdiff {2,3}); /* outputs {1,2} */
```

Tuple Expression

< *expression-1*, . . . *expression-n* **>**

A tuple expression represents the value of a member in a set of tuples. Each scalar subexpression inside the angle brackets represents the value of a tuple element. This form is used only with IN, SETOF, and set constructor expressions.

The following code demonstrates the tuple expression:

```
proc optmodel;
   put (<1,2,3> in setof{i in 1..2}<i,i+1,i+2>);
   put ({<1,'a'>, <2,'b'>} cross {<3,'c'>, <4,'d'>});
```

The first PUT statement checks whether the tuple <1, 2, 3> is a member of a set of tuples. The second PUT statement outputs the cross product of two sets of tuples constructed by the set constructor.

This code produces the output in Figure 6.34.

```
1
{<1,'a',3,'c'>,<1,'a',4,'d'>,<2,'b',3,'c'>,<2,'b',4,'d'>}
```

Figure 6.34. Tuple Expression Output

UNION Expression

set-expression **UNION** *set-expression*

The UNION expression returns the set union of the left and right operands. The result set contains values that are members of either the left or right operand. The operands must have compatible set types. The following example performs a set union:

```
proc optmodel;
   put ({1,3} union {2,3}); /* outputs {1,3,2} */
```

UNION Aggregation Expression

UNION{ *index-set* **}** *set-expression*

The UNION aggregation expression evaluates the *set-expression* for each member of the index set *index-set*. The result is the set union of the values that are returned by the *set-expression*. If the index set is empty, then the result is an empty set.

The following code demonstrates a UNION aggregation. The output is the value of $\{1,2,3,4\} \cup \{2,3,4,5\} \cup \{3,4,5,6\}$.

```
proc optmodel;
   put (union{i in 1..3} i..i+3); /* outputs {1,2,3,4,5,6} */
```

WITHIN Expression

set-expression **WITHIN** *set-expression*

set-expression **NOT WITHIN** *set-expression*

The WITHIN expression returns 1 if the left operand set is a subset of the right operand set and returns 0 otherwise. (That is, the operator returns true if every member of the left operand set is a member of the right operand set.) The NOT WITHIN form logically negates the result value. The following code demonstrates the WITHIN and NOT WITHIN operators:

```
proc optmodel;
   put ({1,3} within {2,3});     /* outputs 0 */
   put ({1,3} not within {2,3}); /* outputs 1 */
   put ({1,3} within {1,2,3});   /* outputs 1 */
```

Details: OPTMODEL Procedure

Conditions of Optimality

Linear Programming

A standard linear program has the following formulation:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where

$\mathbf{x}$	$\in$	$\mathbb{R}^n$	is the vector of decision variables
$\mathbf{A}$	$\in$	$\mathbb{R}^{m\times n}$	is the matrix of constraints
$\mathbf{c}$	$\in$	$\mathbb{R}^n$	is the vector of objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{R}^m$	is the vector of constraints right-hand sides (RHS)

This formulation is called the primal problem. The corresponding dual problem (see the section "Dual Values" on page 763) is

$$\begin{aligned} \text{maximize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the vector of dual variables.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are optimal to the primal and dual problems, respectively, only if there exist primal slack variables $\mathbf{s} = \mathbf{A}\mathbf{x} - \mathbf{b}$ and dual slack variables $\mathbf{w} = \mathbf{A}^T\mathbf{y} - \mathbf{c}$ such that the following *Karush-Kuhn-Tucker (KKT) conditions* are satisfied:

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{s} &= \mathbf{b}, \quad \mathbf{x} \geq 0, \quad \mathbf{s} \geq 0 \\ \mathbf{A}^T\mathbf{y} + \mathbf{w} &= \mathbf{c}, \quad \mathbf{y} \geq 0, \quad \mathbf{w} \geq 0 \\ \mathbf{s}^T\mathbf{y} &= 0 \\ \mathbf{w}^T\mathbf{x} &= 0 \end{aligned}$$

The first line of equations defines primal feasibility, the second line of equations defines dual feasibility, and the last two equations are called the complementary slackness conditions.

Nonlinear Programming

To facilitate discussion of optimality conditions in nonlinear programming, we write the general form of nonlinear optimization problems by grouping the equality constraints and inequality constraints. We also write all the general nonlinear inequality constraints and bound constraints in one form as "$\geq$" inequality constraints. Thus we have the following formulation:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \\ & c_i(x) \geq 0, \quad i \in \mathcal{I} \end{aligned}$$

where $\mathcal{E}$ is the set of indices of the equality constraints, $\mathcal{I}$ is the set of indices of the inequality constraints, and $m = |\mathcal{E}| + |\mathcal{I}|$.

A point x is *feasible* if it satisfies all the constraints $c_i(x) = 0, i \in \mathcal{E}$ and $c_i(x) \geq 0, i \in \mathcal{I}$. The feasible region $\mathcal{F}$ consists of all the feasible points. In unconstrained cases, the feasible region $\mathcal{F}$ is the entire $\mathbb{R}^n$ space.

A feasible point x^* is a *local solution* of the problem if there exists a neighborhood $\mathcal{N}$ of x^* such that

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

Further, a feasible point x^* is a *strict local solution* if strict inequality holds in the preceding case; i.e.,

$$f(x) > f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

A feasible point x^* is a *global solution* of the problem if no point in $\mathcal{F}$ has a smaller function value than $f(x^*)$; i.e.,

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{F}$$

Unconstrained Optimization

The following conditions hold true for unconstrained optimization problems:

- **First-order necessary conditions**: If x^* is a local solution and $f(x)$ is continuously differentiable in some neighborhood of x^*, then

$$\nabla f(x^*) = 0$$

- **Second-order necessary conditions**: If x^* is a local solution and $f(x)$ is twice continuously differentiable in some neighborhood of x^*, then $\nabla^2 f(x^*)$ is positive semidefinite.
- **Second-order sufficient conditions**: If $f(x)$ is twice continuously differentiable in some neighborhood of x^*, $\nabla f(x^*) = 0$, and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local solution.

Constrained Optimization

For constrained optimization problems, the *Lagrangian function* is defined as follows:

$$L(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

where $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$, are called *Lagrange multipliers*. $\nabla_x L(x, \lambda)$ is used to denote the gradient of the Lagrangian function with respect to x, and $\nabla_x^2 L(x, \lambda)$ is used to denote the Hessian of the Lagrangian function with respect to x. The active set at a feasible point x is defined as

$$\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : c_i(x) = 0\}$$

We also need the following definition before we can state the first-order and second-order necessary conditions:

- **Linear independence constraint qualification and regular point**: A point x is said to satisfy the *linear independence constraint qualification* if the gradients of active constraints

$$\nabla c_i(x), \quad i \in \mathcal{A}(x)$$

  are linearly independent. Further, we refer to such a point x as a *regular point*.

We now state the theorems that are essential in the analysis and design of algorithms for constrained optimization:

- **First-order necessary conditions**: Suppose that x^* is a local minimum and also a regular point. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are continuously differentiable, there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ such that the following conditions hold:

$$\nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*) = 0$$

$$\begin{aligned}
c_i(x^*) &= 0, \quad i \in \mathcal{E} \\
c_i(x^*) &\geq 0, \quad i \in \mathcal{I} \\
\lambda_i^* &\geq 0, \quad i \in \mathcal{I} \\
\lambda_i^* c_i(x^*) &= 0, \quad i \in \mathcal{I}
\end{aligned}$$

The preceding conditions are often known as the *Karush-Kuhn-Tucker conditions*, or *KKT conditions* for short.

- **Second-order necessary conditions**: Suppose that x^* is a local minimum and also a regular point. Let λ^* be the Lagrange multipliers that satisfy the KKT conditions. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are twice continuously differentiable, the following conditions hold:

$$z^T \nabla_x^2 L(x^*, \lambda^*) z \geq 0$$

for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^T z = 0, \quad i \in \mathcal{A}(x^*)$$

- **Second-order sufficient conditions**: Suppose there exist a point x^* and some Lagrange multipliers λ^* such that the KKT conditions are satisfied. If

$$z^T \nabla_x^2 L(x^*, \lambda^*) z > 0$$

for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^T z = 0, \quad i \in \mathcal{A}(x^*)$$

then x^* is a strict local solution.

Note that the set of all such z's forms the null space of the matrix $\left[\nabla c_i(x^*)^T\right]_{i \in \mathcal{A}(x^*)}$. Thus we can search for strict local solutions by numerically checking the Hessian of the Lagrangian function projected onto the null space. For a rigorous treatment of the optimality conditions, see Fletcher (1987) and Nocedal and Wright (1999).

Data Set Input/Output

You can use the CREATE DATA and READ DATA statements to exchange PROC OPTMODEL data with SAS data sets. The statements can move data into and out of PROC OPTMODEL parameters and variables. For example, the following code uses a CREATE DATA statement to save the results from an optimization into a data set:

```
proc optmodel;
   var x;
   min z = (x-5)**2;
   solve;
   create data optdata from xopt=x z;
```

This code writes a single observation into the data set OPTDATA. The data set contains two variables, xopt and z, and the values contain the optimized values of the PROC OPTMODEL variable x and objective z, respectively. The code "xopt=x" renames the variable x to xopt.

The group of values held by a data set variable in different observations of a data set is referred to as a *column*. The READ DATA and CREATE DATA statements specify

a set of columns for a data set and define how data are to be transferred between the columns and PROC OPTMODEL parameters.

Columns in square brackets ([ ]) are handled specially. Such columns are called *key columns*. Key columns specify element values that provide an implicit index for subsequent array columns. The following example uses key columns with the CREATE DATA statement to write out variable values from an array:

```
proc optmodel;
   set LOCS = {'New York', 'Washington', 'Boston'};  /* locations */
   set DOW = 1..7;  /* day of week */
   var s{LOCS, DOW} init 1;
   create data soldata from [location day_of_week]={LOCS, DOW} sale=s;
```

In this case the optimization variable s is initialized to a value of 1 and is indexed by values from the set parameters LOCS and DOW. The output data set contains an observation for each combination of values in these sets. The output data set contains three variables, location, day_of_week, and sale. The data set variables location and day_of_week save the index element values for the optimization variable s that is written in each observation. The data set created is shown in Figure 6.35.

Data Set: SOLDATA

Obs	location	day_of_ week	sale
1	New York	1	1
2	New York	2	1
3	New York	3	1
4	New York	4	1
5	New York	5	1
6	New York	6	1
7	New York	7	1
8	Washington	1	1
9	Washington	2	1
10	Washington	3	1
11	Washington	4	1
12	Washington	5	1
13	Washington	6	1
14	Washington	7	1
15	Boston	1	1
16	Boston	2	1
17	Boston	3	1
18	Boston	4	1
19	Boston	5	1
20	Boston	6	1
21	Boston	7	1

Figure 6.35. Data Sets Created

Note that the key columns in the preceding example do not name existing PROC OPTMODEL variables. They create new local dummy parameters, location and day_of_week, in the same manner as dummy parameters in index sets. These local parameters can be used in subsequent columns. For example, the following code

demonstrates how to use a key column value in an expression for a later column value:

```
proc optmodel;
   create data tab
          from [i]=(1..10)
          Square=(i*i) Cube=(i*i*i);
```

This creates a data set that has 10 observations that hold squares and cubes of the numbers from 1 to 10. The key column variable here is named i and is explicitly assigned the values from 1 to 10, while the data set variables Square and Cube hold the square and cube, respectively, of the corresponding value of i.

In the preceding example the key column values are simply the numbers from 1 to 10. The value is the same as the observation number, so the variable i is redundant. You can remove the data set variable for a key column via the DROP data set option, as follows:

```
proc optmodel;
   create data tab2 (drop=i)
          from [i] =(1..10)
          Square=(i*i) Cube=(i*i*i);
```

The local parameters declared by key columns receive their values in various ways. For a READ DATA statement the key column values come from the data set variables for the column. In a CREATE DATA statement the values can be defined explicitly, as shown in the previous example. Otherwise the CREATE DATA statement generates a set of values that combines the index sets of array columns that need implicit indexing. The code producing the output in Figure 6.35 demonstrates implicit indexing.

Use a suffix ("Suffixes" on page 759) to read or write auxiliary values, such as variable bounds or constraint duals. For example, consider the following code:

```
data pdat;
   input p $ maxprod cost;
   datalines;
ABQ    12  0.7
MIA     9  0.6
CHI    14  0.5
run;
proc optmodel;
   set<string> plants;
   var prod{plants} >= 0;
   number cost{plants};
   read data pdat into plants=[p] prod.ub=maxprod cost;
```

The code "plants=[p]" in the READ DATA statement declares p as a key column and instructs PROC OPTMODEL to store the set of plant names from the data set

variable p into the set parameter plants. The statement assigns the upper bound for the variable prod indexed by p to be the value of the data set variable maxprod. The cost parameter location indexed by p is also assigned to be the value of the data set variable cost.

The target variables prod and cost in the preceding example use implicit indexing. Indexing can also be performed explicitly. The following version of the READ DATA statement makes the indices explicit:

```
read data pdat into plants=[p] prod[p].ub=maxprod cost[p];
```

Explicit indexing is useful when array indices need to be transformed from the key column values in the data set. For example, the following code reverses the order in which elements from the data set are stored in an array:

```
data abcd;
input letter $ @@;
datalines;
a b c d
;


proc optmodel;
   set<num> subscripts=1..4;
   string  letter{subscripts};
   read data abcd into [_N_] letter[5-_N_];
   print letter;
```

The output from this example appears in Figure 6.36.

```
The OPTMODEL Procedure

    [1]    letter

     1     d
     2     c
     3     b
     4     a
```

Figure 6.36. READ DATA Statement: Explicit Indexing

The following example demonstrates the use of explicit indexing to save sequential subsets of an array in individual data sets:

```
data revdata;
   input month rev @@;
   datalines;
1 200 2 345 3 362 4 958
5 659 6 804 7 487 8 146
9 683 10 732 11 652 12 469
;
```

```
proc optmodel;
   set m = 1..3;
   var revenue{1..12};
   read data revdata into [_N_] revenue=rev;
   create data qtr1 from [month]=m revenue[month];
   create data qtr2 from [month]=m revenue[month+3];
   create data qtr3 from [month]=m revenue[month+6];
   create data qtr4 from [month]=m revenue[month+9];
```

Each CREATE DATA statement generates a data set representing one quarter of the year. Each data set contains the variables month and revenue. The data set qtr2 is shown in Figure 6.37.

Obs	month	revenue
1	1	958
2	2	659
3	3	804

Figure 6.37. CREATE DATA Statement: Explicit Indexing

Control Flow

Most of the control flow statements in PROC OPTMODEL are familiar to users of the DATA step or PROC IML. PROC OPTMODEL supports the IF statement, DO blocks, the iterative DO statement, the DO WHILE statement, and the DO UNTIL statement. You can also use the CONTINUE, LEAVE, and STOP statements to modify control flow.

PROC OPTMODEL adds the FOR statement. This statement is similar in operation to an iterative DO loop. However, the iteration is performed over the members of an index set. This form is convenient for iteration over all the locations in an array, since the valid array indices are also defined using an index set. For example, the following code initializes the array parameter A, indexed by i and j, to random values sampled from a normal distribution with mean 0 and variance 1:

```
proc optmodel;
   set R=1..10;
   set C=1..5;
   number A{R, C};
   for {i in R, j in C}
       A[i, j]=rannor(-1);
```

The FOR statement provides a convenient way to perform a statement such as the preceding assignment statement for each member of a set.

Formatted Output

PROC OPTMODEL provides two primary means of producing formatted output. The PUT statement provides output of data values with detailed format control. The PRINT statement handles arrays and produces formatted output in tabular form.

The PUT statement is similar in syntax to the PUT statement in the DATA step and in PROC IML. The PUT statement can output data to the SAS log, the SAS listing, or an external file. Arguments to the PUT statement specify the data to output and provide instructions for formatting. The PUT statement provides enough control to create reports within PROC OPTMODEL. However, typically the PUT statement is used to produce output for debugging or to quickly check data values.

The following example demonstrates some features of the PUT statement:

```
proc optmodel;
   number a=1.7, b=2.8;
   set s={a,b};
   put a b;         /* list output */
   put a= b=;       /* named output */
   put 'Value A: ' a 8.1 @30 'Value B: ' b 8.; /* formatted */
   string str='Ratio (A/B) is:';
   put str (a/b);  /* strings and expressions */
   put s=;          /* named set output */
```

This code produces the output in Figure 6.38.

```
1.7 2.8
a=1.7 b=2.8
Value A:      1.7            Value B:        3
Ratio (A/B) is: 0.6071428571
s={1.7,2.8}
```

Figure 6.38. PUT Statement Output

The first PUT statement demonstrates list output. The numeric data values are output in a default format, BEST12., with leading and trailing blanks removed. A blank space is inserted after each data value is output. The second PUT statement uses the equal sign (=) to request that the variable name be output along with the regular list output.

The third PUT statement demonstrates formatted output. It uses the @ operator to position the output in a specific column. This style of output can be used in report generation. Note that the format specification "8." causes the displayed value of parameter b to be rounded.

The fourth PUT statement shows the output of a string value, str. It also outputs the value of an expression enclosed in parentheses. The final PUT statement outputs a set along with its name.

The default destination for PUT statement output is the SAS log. The FILE and CLOSEFILE statements can be used to send output to the SAS listing or to an external

data file. Multiple files can be open at the same time. The FILE statement selects the current destination for PUT statement output, and the CLOSEFILE statement closes the corresponding file. See the section "FILE Statement" on page 707 for more details.

The PRINT statement is designed to output numeric and string data in the form of tables. The PRINT statement handles the details of formatting automatically. However, the output format can be overridden by PROC OPTMODEL options and through Output Delivery System (ODS) facilities.

The PRINT statement can output array data in a table form that contains a row for each combination of array index values. This form uses columns to display the array index values for each row and uses other columns to display the value of each requested data item. The following code demonstrates the table form:

```
proc optmodel;
   number square{i in 0..5} = i*i;
   number recip{i in 1..5} = 1/i;
   print square recip;
```

The PRINT statement produces the output in Figure 6.39.

[1]	square	recip
0	0	
1	1	1.00000
2	4	0.50000
3	9	0.33333
4	16	0.25000
5	25	0.20000

Figure 6.39. PRINT Statement Output (List Form)

The first table column, labeled "[1]," contains the index values for the parameters square and recip. The columns that are labeled "square" and "recip" contain the parameter values for each array index. For example, the last row corresponds to the index 5 and the value in the last column is 0.2, which is the value of recip[5].

Note that the first row of the table contains no value in the recip column. Parameter location recip[0] does not have a valid index, so no value is printed. The PRINT statement does not display variables that are undefined or have invalid indices. This permits arrays that have similar indexing to be printed together. The sets of defined indices in the arrays are combined to generate the set of indices shown in the table.

Also note that the PRINT statement has assigned formats and widths that differ between the square and recip columns. The PRINT statement assigns a default fixed-point format to produce the best overall output for each data column. The format that is selected depends on the PDIGITS= and PWIDTH= options.

The PDIGITS= and PWIDTH= options specify the desired significant digits and formatted width, respectively. If the range of magnitudes is large enough that no suitable

format can be found, then the data item is displayed in scientific format. The table in the preceding example displays the last column with 5 decimal places in order to display the 5 significant digits that were requested by the default PDIGITS value. The square column, on the other hand, does not need any decimal places.

The PRINT statement can also display two-dimensional arrays in matrix form. If the list following the PRINT statement contains only a single array that has two index elements, then the array is displayed in matrix form when it is sufficiently dense (otherwise the display is in table form). In this form the left-most column contains the values of the first index element. The remaining columns correspond to and are labeled by the values of the second index element. The following code prints an example of matrix form:

```
proc optmodel;
   set R=1..6;
   set C=1..4;
   number a{i in R, j in C} = 10*i+j;
   print a;
```

The PRINT statement produces the output in Figure 6.40.

	a			
	1	2	3	4
1	11	12	13	14
2	21	22	23	24
3	31	32	33	34
4	41	42	43	44
5	51	52	53	54
6	61	62	63	64

Figure 6.40. PRINT Statement Output (Matrix Form)

In the example the first index element ranges from 1 to 6 and corresponds to the table rows. The second index element ranges from 1 to 4 and corresponds to the table columns. Array values can be found based on the row and column values. For example, the value of parameter a[3,2] is 32. This location is found in the table in the row labeled "3" and the column labeled "2."

ODS Table and Variable Names

PROC OPTMODEL assigns a name to each table it creates. You can use these names to reference the table when using the Output Delivery System (ODS) to select tables and create output data sets. These names are listed in Table 6.8. For more information about ODS, see *SAS Output Delivery System: User's Guide*.

Table 6.8. ODS Tables Produced in PROC OPTMODEL

ODS Table Name	Description	Statement/Option
PrintTable	Specified parameter and/or variable values	PRINT
ProblemSummary	Description of objective, variables, and constraints	SOLVE
SolverOptions	List of solver options and their values	SOLVE
DerivMethods	List of derivatives used by the solver, including the method of computation	SOLVE
SolutionSummary	Overview of solution, including solver-dependent solution quality values	SOLVE
OptStatistics	Solver-dependent description of the resources required for solution, including function evaluations and solver time	SOLVE

Table 6.9 lists the variable names of the preceding tables used in the ODS template of the OPTMODEL procedure.

Table 6.9. Variable Names for the ODS Tables Produced in PROC OPTMODEL

Table Name	Variables
PrintTable (table form)	COL1 - COLn, *identifier-expression(_suffix)*
PrintTable (matrix form)	ROW, COL1 - COLn, *identifier-expression(_suffix)*
ProblemSummary	Label1, cValue1, and nValue1
DerivMethods	Label1, cValue1, and nValue1
SolverOptions	Label1, cValue1, nValue1, cValue2, and nValue2
SolutionSummary	Label1, cValue1, and nValue1
OptStatistics	Label1, cValue1, and nValue1

The PRINT statement produces an ODS table named "PrintTable." The variable names used depend on the display format used. See the section "Formatted Output" on page 748 for details on choosing the display format.

For the PRINT statement with table format, the columns that display array indices are named COL1–COLn, where n is the number of index elements. Columns that display values from identifier expressions are named based on the expression's name and suffix. The identifier name becomes the output variable name if no suffix is used. Otherwise the variable name is formed by appending an underscore (_) and the suffix to the identifier name. Columns that display the value of expressions are named COLn, where n is the column number in the table.

For the PRINT statement with matrix format, the first column has the variable name ROW. The remaining columns are named COL1–COLn where n is the number of distinct column indices. Columns that display values from identifier expressions are named based on the expression's name and suffix, as described in the case of table format.

The PRINTLEVEL= option controls the tables produced by the SOLVE statement. When PRINTLEVEL=0, the SOLVE statement produces no ODS tables. When PRINTLEVEL=1, the SOLVE statement produces the "ProblemSummary" and "SolutionSummary" tables. When PRINTLEVEL=2, the SOLVE statement produces the "ProblemSummary," "SolverOptions," "DerivMethods," "SolutionSummary," and "OptStatistics" tables.

The following code generates several ODS tables and writes each table to a SAS data set:

```
proc optmodel printlevel=2;
   ods output PrintTable=expt ProblemSummary=exps DerivMethods=exdm
              SolverOptions=exso SolutionSummary=exss OptStatistics=exos;
   var x{1..2} >= 0;
   min z = 2*x[1] + 3 * x[2] + x[1]**2 + 10*x[2]**2
           + 2.5*x[1]*x[2] + x[1]**3;
   con c1: x[1] - x[2] <= 1;
   con c2: x[1] + 2*x[2] >= 100;
   solve;
   print x;
```

The data set expt contains the Print table ("PrintTable") and is shown in Figure 6.41. The variable names are COL1 and x_INIT.

Obs	COL1	x
1	1	10.448
2	2	44.776

Figure 6.41. ODS Table "PrintTable"

The data set exps contains the Problem Summary table ("ProblemSummary") and is shown in Figure 6.42. The variable names are Label1, cValue11, and nValue1. The rows describe the objective function, variables, and constraints. The rows depend on the form of the problem.

Obs	Label1	cValue1	nValue1
1	Objective Sense	Minimization	.
2	Objective Function	z	.
3	Objective Type	Nonlinear	.
4			.
5	Number of Variables	2	2.000000
6	Bounded Above	0	0
7	Bounded Below	2	2.000000
8	Bounded Below and Above	0	0
9	Free	0	0
10	Fixed	0	0
11			.
12	Number of Constraints	2	2.000000
13	Linear LE (<=)	1	1.000000
14	Linear EQ (=)	0	0
15	Linear GE (>=)	1	1.000000
16	Linear Range	0	0

Figure 6.42. ODS Table "ProblemSummary"

The data set exso contains the Solver Options table ("SolverOptions") and is shown in Figure 6.43. The variable names are Label1, cValue11, nValue1, cValue1, and nValue1. The rows, which depend on the solver called by PROC OPTMODEL, list the values taken by each of the solver options. The presence of an asterisk (*) next to an option indicates that a nondefault value has been specified for that option.

Obs	Label1	cValue1	nValue1	c Value2	nValue2
1	TECH	TRUREG	.		.
2	ABSOPTTOL	0.001	0.001000		.
3	MAXFUNC	3000	3000.000000		.
4	MAXITER	500	500.000000		.
5	MAXTIME	I	I		.
6	OBJLIMIT	1E20	1E20		.
7	PRINTFREQ	0	0		.
8	RELOPTTOL	1E-6	0.000001000		.

Figure 6.43. ODS Table "SolverOptions"

The data set exdm contains the Methods of Derivative Computation table ("DerivMethods") and is shown in Figure 6.44. The variable names are Label1, cValue11, and nValue1. The rows, which depend on the derivatives used by the solver, specify the method used to calculate each derivative.

Obs	Label1	cValue1	nValue1
1	Objective Gradient	Analytic Formulas	.
2	Objective Hessian	Analytic Formulas	.

Figure 6.44. ODS Table "DerivMethods"

The data set exss contains the Solution Summary table ("SolutionSummary") and is shown in Figure 6.45. The variable names are Label1, cValue11, and nValue1. The rows give an overview of the solution, including the solver chosen, the objective value, and the solution status. Depending on the values returned by the solver, the Solution Summary table might also include some solution quality values such as optimality error and infeasibility. The values in the Solution Summary table appear in the _OROPTMODEL_ macro variable; each solver chapter has a section describing the solver's contribution to this macro variable.

Obs	Label1	cValue1	nValue1
1	Solver	NLPC/Trust Region	.
2	Objective Function	z	.
3	Solution Status	Optimal	.
4	Objective Value	22623	22623
5	Iterations	4	4.000000
6			.
7	Absolute Optimality Error	8.6694316E-6	0.000008669
8	Relative Optimality Error	9.3760097E-9	9.3760097E-9
9	Absolute Infeasibility	1.065814E-14	1.065814E-14
10	Relative Infeasibility	1.055261E-16	1.055261E-16

Figure 6.45. ODS Table "SolutionSummary"

The data set exos contains the Optimization Statistics table ("OptStatistics") and is shown in Figure 6.46. The variable names are Label1, cValue11, and nValue1. The rows, which depend on the solver called by PROC OPTMODEL, describe the amount of time and function evaluations used by the solver.

Obs	Label1	cValue1	nValue1
1	Function Evaluations	9	9.000000
2	Gradient Evaluations	2	2.000000
3	Hessian Evaluations	5	5.000000
4	Problem Generation Time	0.015	0.015000
5	Code Generation Time	0.015	0.015000
6	Presolver Time	0.000	0
7	Solver Time	0.015	0.015000

Figure 6.46. ODS Table "OptStatistics"

Constraints

You can add constraints to a PROC OPTMODEL model. The solver tries to satisfy the specified constraints while minimizing or maximizing the objective.

Constraints in PROC OPTMODEL have names. By using the name, you can examine various attributes of the constraint, such as the dual value that is returned by the solver (see the section "Suffixes" on page 759 for details). A constraint is not allowed to have the same name as any other model component.

PROC OPTMODEL provides a default name if none is supplied by the constraint declaration. The predefined array _ACON_ provides names for otherwise anonymous constraints. The predefined numeric parameter _NACON_ contains the number of such constraints. The constraints are assigned integer indices in sequence, so _ACON_[1] refers to the first unnamed constraint declared, while _ACON_[_NACON_] refers to the newest.

Consider the following example of a simple model that has a constraint:

```
proc optmodel;
   var x, y;
   min r = x**2 + y**2;
   con c: x+y >= 1;
   solve;
   print x y;
```

Without the constraint named c, the solver would find the point $x = y = 0$ that has an objective value of 0. However, the constraint makes this point infeasible. The resulting output is shown in Figure 6.47.

```
                    The OPTMODEL Procedure

                       Problem Summary

          Objective Sense            Minimization
          Objective Function                    r
          Objective Type                Quadratic

          Number of Variables                   2
          Bounded Above                         0
          Bounded Below                         0
          Bounded Below and Above               0
          Free                                  2
          Fixed                                 0

          Number of Constraints                 1
          Linear LE (<=)                        0
          Linear EQ (=)                         0
          Linear GE (>=)                        1
          Linear Range                          0


                    The OPTMODEL Procedure

                       Solution Summary

        Solver                        NLPC/Trust Region
        Objective Function                            r
        Solution Status                         Optimal
        Objective Value                             0.5
        Iterations                                    0

        Absolute Optimality Error                     0
        Relative Optimality Error                     0
        Absolute Infeasibility                        0
        Relative Infeasibility                        0

                           x       y

                         0.5     0.5
```

Figure 6.47. Constrained Model Solution

The solver has found the point where the objective function is minimized in the region $x + y \geq 1$. This is actually on the border of the region, or we say the constraint `c` is active (see the section "Dual Values" on page 763 for details).

In the preceding example the constraint `c` had only a lower bound. You can specify constraints that have both upper and lower bounds. For example, replacing the constraint `c` in the previous example would further restrict the feasible region:

```
con c: 3 >= x+y >= 1;
```

PROC OPTMODEL standardizes constraints to collect the expression terms that depend on variables and to separate the expression terms that are constant. When there

is a single equality or inequality operator, the separable constant terms are moved to the right operand while the variable terms are moved to the left operand. For range constraints the separable constant terms from the middle expression are subtracted from the lower and upper bounds. You can see the standardized constraints with the use of the EXPAND statement in the following example. Consider the following PROC OPTMODEL code:

```
proc optmodel;
   var x{1..3};
   con b: sum{i in 1..3}(x[i] - i) = 0;
   expand b;
```

This produces an optimization problem with the following constraint:

```
(x[1] - 1) + (x[2] - 2) + (x[3] - 3) = 0
```

The EXPAND statement produces the output in Figure 6.48.

```
Constraint b: x[1] + x[2] + x[3] = 6
```

Figure 6.48. Expansion of a Standardized Constraint

Here the i separable constant terms in the operand of the SUM operation were moved to the right-hand side of the constraint. The sum of these i values is 6.

After standardization the constraint expression that contains all the variables is called the *body* of the constraint. You can reference the current value of the body expression by attaching the .body suffix to the constraint name. Similarly, the upper and lower bound expressions can be referenced by using the .ub and .lb suffixes, respectively. (See the section "Suffixes" on page 759 for more information.)

As a result of standardization, the value of a body expression depends on how the corresponding constraint is entered. The following example demonstrates how using equivalent relational syntax can result in different .body values:

```
proc optmodel;
    var x init 1;
    con c1: x**2 <= 5;
    con c2: 5 >= x**2;
    con c3: -x**2 >= -5;
    con c4: -5 <= -x**2;
    expand;
    print c1.body c2.body c3.body c4.body;
```

The EXPAND and PRINT statements produce the output in Figure 6.49.

```
                          The OPTMODEL Procedure

Var x
Constraint c1: x**2 <= 5
Constraint c2: -x**2 >= -5
Constraint c3: -x**2 >= -5
Constraint c4: --x**2 <= 5


                   c1.BODY    c2.BODY    c3.BODY    c4.BODY

                         1         -1         -1          1
```

Figure 6.49. Expansion and Body Values of Standardized Constraints

CAUTION: Each constraint has an associated *dual value* (see "Dual Values" on page 763). As a result of standardization, the sign of a dual value depends in some instances on the way in which the corresponding constraint is entered into PROC OPTMODEL. In the case of a minimization objective with one-sided constraint $g(x) \geq L$, avoid entering the constraint as $L \leq g(x)$. For example, the following code produces a value of 2:

```
proc optmodel;
   var x;
   min o1 = x**2;
   con c1: x >= 1;
   solve with nlpc;
   print (c1.dual);
```

Replacing the constraint as follows results in a value of –2:

```
con c1: 1 <= x;
```

results in a value of -2.

In the case of a maximization objective with the one-sided constraint $g(x) \leq U$, avoid entering the constraint as $U \geq g(x)$.

When a constraint has variables on both sides, the sign of the dual value depends on the direction of the inequality. For example, you can enter the following constraint:

```
con c1: x**5 - y + 8 <= 5*x + y**2;
```

This is a $\leq$ constraint, so c1.dual is nonpositive. If you enter the same constraint as follows, then c1.dual is nonnegative:

```
con c1: 5*x + y**2 >= x**5 - y + 8;
```

It is also important to note that the signs of the dual values are negated in the case of maximization. The following code outputs a value of 2:

```
proc optmodel;
   var x;
   min o1 = x**2;
   con c1: 1 <= x <= 2;
   solve with nlpc;
   print (c1.dual);
```

Changing the objective function as follows yields the same value of x, but c1.dual now holds the value –2:

```
   max o1 = -x**2;
```

Note: A simple bound constraint on a decision variable x can be entered either by using a CONSTRAINT declaration or by assigning values to x.lb and x.ub. If you require dual values for simple bound constraints, use the CONSTRAINT declaration.

Constraints can be linear or nonlinear. PROC OPTMODEL determines the type of constraint automatically by examining the form of the body expression. Subexpressions that do not involve variables are treated as constants. Constant subexpressions that are multiplied by or added to linear subexpressions produce new linear subexpressions. For example, constraint A in the following code is linear:

```
proc optmodel;
   var x{1..3};
   con A: 0.5*(x[1]-x[2]) + x[3] >= 0;
```

Suffixes

Use suffixes with *identifier-expressions* to retrieve and modify various auxiliary values maintained by the solver. The values of the suffixes can come from expressions in the declaration of the name that is suffixed. For example, the following declaration of variable v provides the values of several suffixes of v at the same time:

```
var v >= 0 <= 2 init 1;
```

The values of the suffixes also come from the solver or from values assigned by assignment or READ DATA statements (see an example in the section "Data Set Input/Output" on page 743).

Table 6.10 shows the names of the available suffixes.

Table 6.10. Suffix Names

Name Kind	Suffix	Modifiable	Description
Variable	.init	No	initial value for the solver
Variable	.lb	Yes	lower bound
Variable	.ub	Yes	upper bound
Variable	.sol	No	current solution value
Variable	.rc	No	reduced cost (LP) / gradient of Lagrangian function
Variable	.dual	No	reduced cost (LP) / gradient of Lagrangian function
Variable	.relax	Yes	relaxation of integrality restriction
Variable	.priority	Yes	branching priority
Variable	.direction	Yes	branching direction
Objective	.sol	No	current objective value
Constraint	.body	No	current constraint body value
Constraint	.dual	No	dual value from the solver
Constraint	.lb	Yes	current lower bound
Constraint	.ub	Yes	current upper bound

Note: The .init value of a variable represents the value it had before the most recent SOLVE statement that used the variable. The value is zero before a successful completion of a SOLVE statement that uses the variable.

The .sol suffix for a variable or objective can be used within a declaration to reference the current value of the symbol. It is treated as a constant in such cases. When processing a SOLVE statement, the value is fixed at the start of the SOLVE. Outside of declarations, a variable or objective name with the .sol suffix is equivalent to the unsuffixed name.

You must use suffixes with names of the appropriate kind. For example, the .init suffix cannot be used with the name of an objective. In particular, parameter names cannot have suffixes.

Suffixed names can be used wherever a parameter name is accepted, provided only the value is required. However, you are not allowed to change the value of certain suffixes. Table 6.10 marks these suffixes as not modifiable. Suffixed names that are used as a target in an assignment or READ DATA statement must be modifiable.

The following code formulates a trivial linear programming problem. The objective value is unbounded, which is reported after the execution of the SOLVE statement. The PRINT statements illustrate the corresponding default auxiliary values. This is shown in Figure 6.50.

```
proc optmodel;
     var x, y;
     min z = x + y;
```

```
con c: x + 2*y <= 3;
solve;
print x.lb x.ub x.init x.sol;
print y.lb y.ub y.init y.sol;
print c.lb c.ub c.body c.dual;
```

```
                The OPTMODEL Procedure

        x.LB           x.UB    x.INIT    x.SOL

-1.7977E308    1.79769E308         0        0


        y.LB           y.UB    y.INIT    y.SOL

-1.7977E308    1.79769E308         0        0


          c.LB    c.UB    c.BODY    c.DUAL

 -1.7977E308       3         0         0
```

Figure 6.50. Using a Suffix: Retrieving Auxiliary Values

Next, continue to submit the following statements to change the default bounds and solve again. The output is shown in Figure 6.51.

```
x.lb=0;
y.lb=0;
c.lb=1;
solve;
print x.lb x.ub x.init x.sol;
print y.lb y.ub y.init y.sol;
print c.lb c.ub c.body c.dual;
```

```
                The OPTMODEL Procedure

      x.LB           x.UB    x.INIT    x.SOL

         0    1.79769E308         0        0


      y.LB           y.UB    y.INIT    y.SOL

         0    1.79769E308         0      0.5


        c.LB    c.UB    c.BODY    c.DUAL

           1       3         1       0.5
```

Figure 6.51. Using a Suffix: Modifying Auxiliary Values

CAUTION: Spaces are significant. The form NAME.␣TAG is treated as a SAS format name followed by the tag name, not as a suffixed identifier. The forms NAME.TAG, NAME␣.␣TAG, and NAME␣.TAG (note the location of spaces) are interpreted as suffixed references.

Integer Variable Suffixes

The suffixes .relax, .priority, and .direction are applicable to integer variables.

For an integer variable x, setting x.relax to a nonzero, nonmissing value relaxes the integrality restriction. The value of x.relax is read as either 1 or 0, depending on whether or not integrality is relaxed. This suffix is ignored for noninteger variables.

The value contained in x.priority sets the branching priority of an integer variable x for use with the MILP solver. This value can be any nonnegative, nonmissing number. The default value is 0, which indicates default branching priority. Variables with positive .priority values are assigned greater priority than the default. Variables with the highest .priority values are assigned the highest priority. Variables with the same .priority value are assigned the same branching priority.

The value of x.direction assigns a branching direction to an integer variable x. This value should be an integer in the range –1 to 3. A noninteger value in this range is rounded on assignment. The default value is 0. The significance of each integer is found in Table 6.11.

Table 6.11. Branching Directions

Value	Direction
–1	Round down to nearest integer
0	Default
1	Round up to nearest integer
2	Round to nearest integer
3	Round to closest presolved bound

Suppose the solver will branch next on an integer variable x whose last LP relaxation solution is 3.3. Suppose also that after passing through the presolver, the lower bound of x is 0 and the upper bound of x is 10. If the value in x.direction is –1 or 2, then the solver sets x to 3 for the next iteration. If the value in x.direction is 1, then the solver sets x to 4. If the value in x.direction is 3, then the solver sets x to 0.

The MPS data set created by the SAVE MPS statement ("SAVE MPS Statement" on page 723) will include a BRANCH section if any nondefault .priority or .direction values have been specified for integer variables.

Dual Values

A dual value is associated with each constraint. To access the dual value of a constraint, use the constraint name followed by the suffix .dual.

For linear programming problems, the dual value associated with a constraint is also known as the dual price (or the shadow price). The latter is usually interpreted economically as the rate at which the optimal value changes with respect to a change in some right-hand side that represents a resource supply or demand requirement.

For nonlinear programming problems, the dual values correspond to the values of the optimal Lagrange multipliers. For more details about duality in nonlinear programming, see Bazaraa, Sherali, and Shetty (1993).

From the dual value associated with the constraint, you can also tell whether the constraint is active or not. A constraint is said to be active, or tight at a point, if it holds with equality at that point. It can be informative to identify active constraints at the optimal point and check their corresponding dual values. Relaxing the active constraints might improve the objective value.

Background on Duality in Mathematical Programming

For a minimization problem, there exists an associated problem with the following property: any feasible solution to the associated problem provides a lower bound for the original problem, and conversely any feasible solution to the original problem provides an upper bound for the associated problem. The original and the associated problems are referred to as the primal and the dual problem, respectively. More specifically, consider the following primal problem:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) & \\ \text{subject to} & c_i(x) = 0, & i \in \mathcal{E} \\ & c_i(x) \leq 0, & i \in \mathcal{L} \\ & c_i(x) \geq 0, & i \in \mathcal{G} \end{array}$$

where $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{G}$ denote the sets of equality, $\leq$ inequality, and $\geq$ inequality constraints, respectively. Variables $x \in \mathbb{R}^n$ are called the primal variables. The Lagrangian function of the primal problem is defined as

$$L(x, \lambda, \mu, \nu) = f(x) - \sum_{i \in \mathcal{E}} \lambda_i c_i(x) - \sum_{i \in \mathcal{L}} \mu_i c_i(x) - \sum_{i \in \mathcal{G}} \nu_i c_i(x)$$

where $\lambda_i \in \mathbb{R}$, $\mu_i \leq 0$, and $\nu_i \geq 0$. By convention, the Lagrange multipliers for inequality constraints have to be nonnegative. Hence λ, $-\mu$ and ν correspond to the Lagrange multipliers in the preceding Lagrangian function. It can be seen that the Lagrangian function is a linear combination of the objective function and constraints of the primal problem.

The Lagrangian function plays a fundamental role in nonlinear programming. It is used to define the optimality conditions that characterize a local minimum of the

primal problem. It is also used to formulate the dual problem of the preceding primal problem. To this end, consider the following *dual* function:

$$d(\lambda, \mu, \nu) = \inf_{x} \; L(x, \lambda, \mu, \nu)$$

The dual problem is defined as

$$\begin{aligned} \underset{\lambda,\mu,\nu}{\text{maximize}} \quad & d(\lambda, \mu, \nu) \\ \text{subject to} \quad & \mu \leq 0 \\ & \nu \geq 0. \end{aligned}$$

The variables λ, μ, and ν are called the dual variables. Note that the dual variables associated with the equality constraints (λ) are free, whereas those associated with $\leq$ inequality constraints (μ) have to be nonpositive and those associated with $\geq$ inequality constraints (ν) have to be nonnegative.

The relation between the primal and the dual problems provides a nice connection between the optimal solutions of the problems. Suppose x^* is an optimal solution of the primal problem and $(\lambda^*, \mu^*, \nu^*)$ is an optimal solution of the dual problem. The difference between the objective values of the primal and dual problems, $\delta = f(x^*) - d(\lambda^*, \mu^*, \nu^*) \geq 0$, is called the duality gap. For some restricted class of convex nonlinear programming problems, both the primal and the dual problems have an optimal solution and the optimal objective values are equal—i.e., the duality gap $\delta = 0$. In such cases, the optimal values of the dual variables correspond to the optimal Lagrange multipliers of the primal problem with the correct signs.

A maximization problem is treated analogously to a minimization problem. For the maximization problem

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & f(x) \\ \text{subject to} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \\ & c_i(x) \leq 0, \quad i \in \mathcal{L} \\ & c_i(x) \geq 0, \quad i \in \mathcal{G}, \end{aligned}$$

the dual problem is

$$\begin{aligned} \underset{\lambda,\mu,\nu}{\text{minimize}} \quad & d(\lambda, \mu, \nu) \\ \text{subject to} \quad & \mu \geq 0 \\ & \nu \leq 0. \end{aligned}$$

where the dual function is defined as $d(\lambda, \mu, \nu) = \sup_{x} \; L(x, \lambda, \mu, \nu)$ and the Lagrangian function $L(x, \lambda, \mu, \nu)$ is defined the same as earlier. In this case, λ, μ, and $-\nu$ correspond to the Lagrange multipliers in $L(x, \lambda, \mu, \nu)$.

Minimization Problems

For inequality constraints in minimization problems, a positive optimal dual value indicates that the associated $\geq$ inequality constraint is active at the solution, and a negative optimal dual value indicates that the associated $\leq$ inequality constraint is active at the solution. In PROC OPTMODEL, the optimal dual value for a *range constraint* (a constraint with both upper and lower bounds) is the sum of the dual values associated with the upper and lower inequalities. Since only one of the two inequalities can be active, the sign of the optimal dual value, if nonzero, identifies which one is active.

For equality constraints in minimization problems, the optimal dual values are unrestricted in sign. A positive optimal dual value for an equality constraint implies that, starting close enough to the primal solution, the same optimum could be found if the equality constraint were replaced with a $\geq$ inequality constraint. A negative optimal dual value for an equality constraint implies that the same optimum could be found if the equality constraint were replaced with a $\leq$ inequality constraint.

The following is an example where simple linear programming is considered:

```
proc optmodel;
   var x, y;
   min z = 6*x + 7*y;
   con
      4*x +   y >=  5,
       -x - 3*y <= -4,
        x +   y <=  4;
   solve;
   print x y;
   expand _ACON_ ;
   print _ACON_.dual _ACON_.body;
```

The PRINT statements generate the output shown in Figure 6.52.

```
                          The OPTMODEL Procedure

                             Problem Summary

                    Objective Sense            Minimization
                    Objective Function                    z
                    Objective Type                   Linear

                    Number of Variables                   2
                    Bounded Above                         0
                    Bounded Below                         0
                    Bounded Below and Above               0
                    Free                                  2
                    Fixed                                 0

                    Number of Constraints                 3
                    Linear LE (<=)                        2
                    Linear EQ (=)                         0
                    Linear GE (>=)                        1
                    Linear Range                          0


                          The OPTMODEL Procedure

                             Solution Summary

                      Solver                  Dual Simplex
                      Objective Function                 z
                      Solution Status             Optimal
                      Objective Value                  13
                      Iterations                        2

                      Primal Infeasibility              0
                      Dual Infeasibility                0
                      Bound Infeasibility               0


                                      x     y

                                      1     1
Constraint _ACON_[1]: 4*x + y >= 5
Constraint _ACON_[2]: - x - 3*y <= -4
Constraint _ACON_[3]: x + y <= 4


                                  _ACON_.     _ACON_.
                            [1]       DUAL        BODY

                             1           1           5
                             2          -2          -4
                             3           0           2
```

Figure 6.52. Dual Values in Minimization Problem: Display

It can be seen that the first and second constraints are active, with dual values 1 and −2. Continue to submit the following statements. Notice how the objective value is changed in Figure 6.53.

```
_ACON_[1].lb = _ACON_[1].lb - 1;
solve;
_ACON_[2].ub = _ACON_[2].ub + 1;
solve;
```

```
                The OPTMODEL Procedure

                   Solution Summary

Solver                  Dual Simplex
Objective Function                 z
Solution Status              Optimal
Objective Value                   12
Iterations                         2

Primal Infeasibility               0
Dual Infeasibility                 0
Bound Infeasibility                0


                The OPTMODEL Procedure

                   Solution Summary

Solver                  Dual Simplex
Objective Function                 z
Solution Status              Optimal
Objective Value                   10
Iterations                         2

Primal Infeasibility    2.775558E-17
Dual Infeasibility                 0
Bound Infeasibility                0
```

Figure 6.53. Dual Values in Minimization Problem: Interpretation

The change is just as the dual values imply. After the first constraint is relaxed by 1 unit, the objective value is improved by 1 unit. For the second constraint, the relaxation and improvement are 1 unit and 2 units, respectively.

CAUTION: The signs of dual values produced by PROC OPTMODEL depend, in some instances, on the way in which the corresponding constraints are entered. See the section "Constraints" on page 755 for details.

Maximization Problems

For inequality constraints in maximization problems, a positive optimal dual value indicates that the associated $\leq$ inequality constraint is active at the solution, and a negative optimal dual value indicates that the associated $\geq$ inequality constraint is active at the solution. The optimal dual value for a range constraint is the sum of the dual values associated with the upper and lower inequalities. The sign of the optimal dual value identifies which inequality is active.

For equality constraints in maximization problems, the optimal dual values are unrestricted in sign. A positive optimal dual value for an equality constraint implies that,

starting close enough to the primal solution, the same optimum could be found if the equality constraint were replaced with a $\leq$ inequality constraint. A negative optimal dual value for an equality constraint implies that the same optimum could be found if the equality constraint were replaced with a $\geq$ inequality constraint.

CAUTION: The signs of dual values produced by PROC OPTMODEL depend, in some instances, on the way in which the corresponding constraints are entered. See the section "Constraints" on page 755 for details.

Reduced Costs

In linear programming problems, each variable has a corresponding reduced cost. To access the reduced cost of a variable, add the suffix .rc or .dual to the variable name. These two suffixes are interchangeable.

The reduced cost of a variable is the rate at which the objective value changes when the value of that variable changes. At optimality, basic variables have a reduced cost of zero; a nonbasic variable with zero reduced cost indicates the existence of multiple optimal solutions.

In nonlinear programming problems, the reduced cost interpretation does not apply. The .dual and .rc variable suffixes represent the gradient of the Lagrangian function, computed using the values returned by the solver.

The following example illustrates the use of the .rc suffix:

```
proc optmodel;
   var x >= 0, y >= 0, z >= 0;
   max cost = 4*x + 3*y - 5*z;
   con
      -x + y + 5*z <= 15,
      3*x - 2*y - z <= 12,
      2*x + 4*y + 2*z <= 16;
   solve;
   print x y z;
   print x.rc y.rc z.rc;
```

The PRINT statements generate the output shown in Figure 6.54.

```
The OPTMODEL Procedure

   x       y     z

   5     1.5     0


x.RC    y.RC    z.RC

   0       0    -6.5
```

Figure 6.54. Reduced Cost in Maximization Problem: Display

In this example, x and y are basic variables, while z is nonbasic. The reduced cost of z is –6.5, which implies that increasing z from 0 to 1 decreases the optimal value from 24.5 to 18.

Presolver

PROC OPTMODEL includes a simple presolver that processes linear constraints to produce tighter bounds on variables. The presolver can reduce the number of variables and constraints that are presented to the solver. These changes can result in reduced solution times.

Linear constraints that involve only a single variable are converted into variable bounds. The presolver then eliminates redundant linear constraints for which variable bounds force the constraint to always be satisfied. Tightly bounded variables where upper and lower bounds are within the VARFUZZ range (see the section "PROC OPTMODEL Statement" on page 684) are automatically fixed to the average of the bounds. The presolver also eliminates variables that are fixed by the user or by the presolver.

The presolver can infer tighter variable bounds from linear constraints when all variables in the constraint or all but one variable have known bounds. For example, when given the following PROC OPTMODEL declarations, the presolver can determine the bound $y \leq 4$:

```
proc optmodel;
   var x >= 3;
   var y;
   con c: x + y <= 7;
```

The presolver makes multiple passes and rechecks linear constraints after bounds are tightened for the referenced variables. The number of passes is controlled by the PRESOLVER= option. In some cases the solver might perform better without the presolve transformations, so almost all such transformations are unavailable when the option PRESOLVER=BASIC is specified. However, the presolver still eliminates variables that have values that have been fixed by the FIX statement. To disable the OPTMODEL presolver entirely, use PRESOLVER=NONE.

Model Update

The OPTMODEL modeling language provides several means of modifying a model after it is first specified. You can change the parameter values of the model. You can add new model components. The FIX and UNFIX statements can fix variables to specified values or rescind previously fixed values. The DROP and RESTORE statements can deactivate and reactivate constraints.

To illustrate how these statements work, reconsider the following example from the section "Constraints" on page 755:

```
proc optmodel;
   var x, y;
   min r = x**2 + y**2;
   con c: x+y >= 1;
   solve;
   print x y;
```

As described previously, the solver finds the optimal point x = y = 0.5 with r = 0.5. You can see the effect of the constraint c on the solution by temporarily removing it. You can add the following code:

```
drop c;
solve;
print x y;
```

This change produces the output in Figure 6.55.

```
                The OPTMODEL Procedure

                   Problem Summary

Objective Sense            Minimization
Objective Function                    r
Objective Type                Quadratic

Number of Variables                   2
Bounded Above                         0
Bounded Below                         0
Bounded Below and Above               0
Free                                  2
Fixed                                 0

Number of Constraints                 0


                The OPTMODEL Procedure

                   Solution Summary

   Solver                        L-BFGS
   Objective Function                 r
   Solution Status              Optimal
   Objective Value                    0
   Iterations                         0

   Optimality Error                   0

                     x     y

                     0     0
```

Figure 6.55. Solution with Dropped Constraint

Note that the SOLVE statement was able to use LBFGS, a solver for unconstrained problems (see "Broyden-Fletcher-Goldfarb-Shanno (BFGS) Algorithm" on page 946). The optimal point is $x = y = 0$, as expected.

You can restore the constraint `c` with the RESTORE statement, and you can also investigate the effect of forcing the value of variable `x` to 0.3. This requires the following statements:

```
restore c;
fix x=0.3;
solve;
print x y c.dual;
```

This produces the output in Figure 6.56.

```
                The OPTMODEL Procedure

                   Problem Summary

   Objective Sense             Minimization
   Objective Function                     r
   Objective Type                 Quadratic

   Number of Variables                    2
   Bounded Above                          0
   Bounded Below                          0
   Bounded Below and Above                0
   Free                                   1
   Fixed                                  1

   Number of Constraints                  1
   Linear LE (<=)                         0
   Linear EQ (=)                          0
   Linear GE (>=)                         1
   Linear Range                           0


                The OPTMODEL Procedure

                   Solution Summary

Solver                         NLPC/Trust Region
Objective Function                             r
Solution Status                          Optimal
Objective Value                             0.58
Iterations                                     1

Absolute Optimality Error                      0
Relative Optimality Error                      0
Absolute Infeasibility                         0
Relative Infeasibility                         0


                  x       y    c.DUAL

                0.3     0.7       1.4
```

Figure 6.56. Solution with Fixed Variable

The variable x still has the value that was defined in the FIX statement. The objective value has increased by 0.08 from its constrained optimum 0.5 (see Figure 6.47). The constraint c is active, as confirmed by the positive dual value.

You can return to the original optimization problem by allowing the solver to vary variable x with the UNFIX statement, as follows:

```
unfix x;
solve;
print x y c.dual;
```

This produces the output in Figure 6.57. The model was returned to its original conditions.

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	r
Objective Type	Quadratic
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	2
Fixed	0
Number of Constraints	1
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	1
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	NLPC/Trust Region
Objective Function	r
Solution Status	Optimal
Objective Value	0.5
Iterations	1
Absolute Optimality Error	0
Relative Optimality Error	0
Absolute Infeasibility	0
Relative Infeasibility	0

x	y	c.DUAL
0.5	0.5	1

Figure 6.57. Solution with Original Model

OPTMODEL Options

All PROC OPTMODEL options can be specified in the PROC statement (see the section "PROC OPTMODEL Statement" on page 684 for more information). However, it is sometimes necessary to change options after other OPTMODEL statements have been executed. For example, if an optimization technique had trouble with convergence, then it might be useful to vary the PRESOLVER= option value. This can be done with the RESET OPTIONS statement.

The RESET OPTIONS statement accepts options in the same form used by the PROC OPTMODEL statement. The RESET OPTIONS statement is also able to reset option values and to change options programmatically. For example, the following code prints the value of parameter n at various precisions:

```
proc optmodel;
   number n = 1/7;
   for {i in 1..9 by 4}
   do;
      reset options pdigits=(i);
      print n;
   end;
   reset options pdigits; /* reset to default */
```

```
The OPTMODEL Procedure

n

0.1


n

0.14286


n

0.142857143
```

Figure 6.58. Changing the PDIGITS Option Value

The output generated is shown in Figure 6.58. The RESET OPTIONS statement in the DO loop sets the PDIGITS option to the value of i. The final RESET OPTIONS statement restores the default option value, because the value was omitted.

Automatic Differentiation

PROC OPTMODEL automatically generates code to evaluate the derivatives for most objective expressions and nonlinear constraints. PROC OPTMODEL generates analytic derivatives for objective and constraint expressions written in terms of the procedure's mathematical operators and the following functions:

ABS	ATAN	COSH	LOG	SIGN	SQRT
ARCOS	CEIL	EXP	LOG10	SIN	TAN
ARSIN	COS	FLOOR	LOG2	SINH	TANH

CAUTION: Some of these functions, such as ABS, FLOOR, and SIGN, as well as some operators, such as IF-THEN, <> (element minimum operator), and >< (element maximum operator), must be used carefully in modeling expressions because functions including such components are not continuously differentiable or even continuous.

Expressions that use other SAS library functions might require numerical approximation of derivatives. PROC OPTMODEL uses either forward-difference approximation or central-difference approximation as specified by the FD= option (see the section "PROC OPTMODEL Statement" on page 684).

Note: The numerical gradient approximations are significantly slower than automatically generated derivatives when there are a large number of optimization variables.

Forward-Difference Approximations

The FD=FORWARD option requests the use of forward-difference derivative approximations. For a function f of n variables, the first-order derivatives are approximated by

$$g_i = \frac{\partial f}{\partial x_i} = \frac{f(x + e_i h_i) - f(x)}{h_i}$$

Notice that n additional function calls are needed here. The step lengths h_i, $i = 1, \ldots, n$, are based on the assumed function precision, *DIGITS*:

$$h_i = 10^{-DIGITS/2}(1 + |x_i|)$$

You can use the FDIGITS= option to specify the function precision, *DIGITS*, for the objective function. For constraints, use the CDIGITS= option.

The second-order derivatives are approximated using $n(n+3)/2$ extra function calls (Dennis and Schnabel 1983, pp. 80, 104):

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{f(x + h_i e_i) - 2f(x) + f(x - h_i e_i)}{h_i^2}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{f(x + h_i e_i + h_j e_j) - f(x + h_i e_i) - f(x + h_j e_j) + f(x)}{h_i h_j}$$

Notice that the diagonal of the Hessian uses a central-difference approximation (Abramowitz and Stegun 1972, p. 884). The step lengths are

$$h_i = 10^{-DIGITS/3}(1 + |x_i|)$$

Central-Difference Approximations

The FD=CENTRAL option requests the use of central-difference derivative approximations. Generally, central-difference approximations are more accurate than forward-difference approximations, but they require more function evaluations. For a function f of n variables, the first-order derivatives are approximated by

$$g_i = \frac{\partial f}{\partial x_i} = \frac{f(x + e_i h_i) - f(x - e_i h_i)}{2h_i}$$

Notice that $2n$ additional function calls are needed here. The step lengths h_i, $i = 1, \ldots, n$, are based on the assumed function precision, *DIGITS*:

$$h_i = 10^{-DIGITS/3}(1 + |x_i|)$$

You can use the FDIGITS= option to specify the function precision, *DIGITS*, for the objective function. For constraints, use the CDIGITS= option.

The second-order derivatives are approximated using $2n(n + 1)$ extra function calls (Abramowitz and Stegun 1972, p. 884):

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{-f(x + 2h_i e_i) + 16f(x + h_i e_i) - 30f(x) + 16f(x - h_i e_i) - f(x - 2h_i e_i)}{12h_i^2}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{f(x + h_i e_i + h_j e_j) - f(x + h_i e_i - h_j e_j) - f(x - h_i e_i + h_j e_j) + f(x - h_i e_i - h_j e_j)}{4h_i h_j}$$

The step lengths are

$$h_i = 10^{-DIGITS/3}(1 + |x_i|)$$

Conversions

Numeric values are implicitly converted to strings when needed for function arguments or operands to the string concatenation operator (||). A warning message is generated when the conversion is applied to a function argument. The conversion uses BEST12. format. Unlike the DATA step, the conversion trims blanks.

Implicit conversion of strings to numbers is not permitted. Use the INPUT function to explicitly perform such conversions.

More on Index Sets

Dummy parameters behave like parameters but are assigned values only when an index set is evaluated. You can reference the declared dummy parameters from index set expressions that follow the *index-set-item*. You can also reference the dummy parameters in the expression or statement controlled by the index set. As the members of an *index-set-item* set expression are enumerated, the element values of the members are assigned to the local dummy parameters.

The number of names in a dummy parameter declaration must match the element length of the corresponding set expression in the *index-set-item*. A single name is allowed when the set member type is scalar (numeric or string). If the set members are tuples that have $n > 1$ elements, then n names are required between the angle brackets (< >) that precede the IN keyword.

Multiple *index-set-items* in an index set are nominally processed in a left-to-right order. That is, a set expression from an *index-set-item* is evaluated as if the *index-set-items* that precede it have already been evaluated. The left-hand *index-set-items* can assign values to local dummy parameters that are used by the set expressions that follow them. After each member from the set expression is enumerated, any *index-set-items* to the right are reevaluated as needed. The actual order in which *index-set-items* are evaluated can vary, if necessary, to allow more efficient enumeration. PROC OPTMODEL will generate the same set of values in any case, although possibly in a different order than strict left-to-right evaluation.

You can view the element combinations that are generated from an index set as tuples. This is especially true for index set expressions (see the section "Index Set Expression" on page 732). However, in most cases no tuple set is actually formed and the element values are assigned only to local dummy parameters.

You can specify a selection expression following a colon (:). The index set generates only those combinations of values for which the selection expression is true. For example, the following statement produces a set of upper triangular indices:

```
proc optmodel;
   put (setof {i in 1..3, j in 1..3 : j >= i} <i, j>);
```

This code produces the output in Figure 6.59.

```
{<1,1>,<1,2>,<1,3>,<2,2>,<2,3>,<3,3>}
```

Figure 6.59. Upper Triangular Index Set

You can use the left-to-right evaluation of *index-set-items* to express the previous set more compactly. The following code produces the same output as the previous statement:

```
proc optmodel;
   put ({i in 1..3, i..3});
```

In this example, the first time the second index set item is evaluated, the value of the dummy parameter i is 1 so the item produces the set {1,2,3}. At the second evaluation the value of i is 2, so the second item produces the set {2,3}. At the final evaluation the value of i is 3, so the second item produces the set {3}.

In many cases it is useful to combine the SLICE operator with index sets. A special form of *index-set-item* uses the SLICE operator implicitly. Normally an index set item that is applied to a set of tuples of length greater than one must be of the form

< *name-1 [, . . . name-n]* **>** **IN** *set-expression*

In the special form, one or more of the name elements are replaced by expressions. The expressions select tuple elements by using the SLICE operator. Note that an expression consisting of a single name must be enclosed in parentheses to distinguish it from a dummy parameter. The remaining names are the dummy parameters for the index set item that is applied to the SLICE result. The following example demonstrates the use of implicit set slicing:

```
proc optmodel;
   number N = 3;
   set<num,str> S = {<1,'a'>,<2,'b'>,<3,'a'>,<4,'b'>};
   put ({i in 1..N, <(i),j> in S});
   put ({i in 1..N, j in slice(<i,*>, S)});
```

The two PUT statements in this example are equivalent.

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the OPTMODEL procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

Examples: OPTMODEL Procedure

Example 6.1. Matrix Square Root

This example demonstrates the use of PROC OPTMODEL array parameters and variables. The following code creates a randomized positive definite symmetric matrix and defines an optimization model to find the matrix square root of the generated matrix:

```
proc optmodel;
   number n = 5;  /* size of matrix */

   /* random original array */
   number A{1..n, 1..n} = 10 - 20*ranuni(-1);

   /* compute upper triangle of the
    * symmetric matrix A*transpose(A) */
   /* should be positive def unless A is singular */
   number P{i in 1..n, j in i..n};
   for {i in 1..n, j in i..n}
       P[i,j] = sum{k in 1..n} A[i,k]*A[j,k];

   /* coefficients of square root array
    * (upper triangle of symmetric matrix) */
   var q{i in 1..n, i..n};

   /* The default initial value q[i,j]=0 is
    * a local minimum of the objective,
    * so you must move it away from that point. */
   q[1,1] = 1;
```

```
/* minimize difference of square of q from P */
min r = sum{i in 1..n, j in i..n}
        (   sum{k in 1..i} q[k,i]*q[k,j]
          + sum{k in i+1..j} q[i,k]*q[k,j]
          + sum{k in j+1..n} q[i,k]*q[j,k]
          - P[i,j] )**2;

solve with nlpu / opttol=1e-5;
print q;
```

The code defines a random array **A** of size $n \times n$. The product **P** is defined as the matrix product AA^T. The product is symmetric, so the declaration of the parameter P gives it upper triangular indexing. The matrix represented by P should be positive definite unless **A** is singular. But singularity is unlikely because of the random generation of **A**. If **P** is positive definite, then it has a well-defined square root, **Q**, such that $P = QQ^T$.

The objective r simply minimizes the sum of squares of the coefficients, as follows:

$$r = \sum_{1 \le i \le j \le n} R_{i,j}^2$$

where $R = QQ^T - P$. (Note that this technique for computing matrix square roots is intended only for the demonstration of PROC OPTMODEL capabilities. Better methods exist.)

Output 6.1.1 shows part of the output from running this code. The values that are actually displayed depend on the random numbers generated.

Output 6.1.1. Matrix Square Root Results

The OPTMODEL Procedure

	q				
	1	2	3	4	5
1	6.15337	-12.49326	5.05941	-0.15411	7.20667
2		-5.21684	-0.48140	5.47795	-4.39379
3			-10.98701	1.95545	-7.50599
4				1.86863	-0.64200
5					3.54247

Example 6.2. Reading from and Creating a Data Set

This example demonstrates how to use the READ DATA statement to read parameters from a SAS data set. The objective is the Bard function, which is the following least-squares problem with $I = \{1, 2, \ldots, 15\}$:

$$f(x) = \frac{1}{2} \sum_{k \in I} \left[y_k - \left(x_1 + \frac{k}{v_k x_2 + w_k x_3} \right) \right]^2$$

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, \cdots, y_{15})$$

where $v_k = 16 - k$, $w_k = \min(k, v_k)$ $(k \in I)$, and

$$y = (0.14, 0.18, 0.22, 0.25, 0.29, 0.32, 0.35, 0.39, 0.37, 0.58, 0.73, 0.96, 1.34, 2.10, 4.39)$$

The minimum function value $f(x^*) = 4.107\text{E}{-3}$ is at the point $(0.08, 1.13, 2.34)$. The starting point $x^0 = (1, 1, 1)$ is used. This problem is identical to Example 4.1 on page 388 in the PROC NLP documentation. The following code uses the READ DATA statement to input parameter values and the CREATE DATA statement to save the solution in a SAS data set:

```
data bard;
   input y @@;
   datalines;
.14  .18  .22  .25  .29  .32  .35  .39
.37  .58  .73  .96 1.34 2.10 4.39
;
proc optmodel;
   set I = 1..15;
   number y{I};
   read data bard into [_n_] y;
   number v{k in I} = 16 - k;
   number w{k in I} = min(k, v[k]);
   var x{1..3} init 1;
   min f = 0.5*
      sum{k in I}
         (y[k] - (x[1] + k /
                  (v[k]*x[2] + w[k]*x[3])))**2;
   solve;
   print x;
   create data xdata from [i] xd=x;
```

In this code the values for parameter y are read from the BARD data set. The set I indexes the terms of the objective as well as the y array.

The code defines two utility parameters that contain coefficients used in the objective function. These coefficients could have been defined in the expression for the objective, f, but it was convenient to give them names and simplify the objective expression.

The result is shown in Output 6.2.1.

Output 6.2.1. Bard Function Solution

The OPTMODEL Procedure

[1]	x
1	0.082411
2	1.133125
3	2.343610

The final CREATE DATA statement saves the solution values determined by the solver into the data set XDATA. The data set contains an observation for each x index. Each observation contains two variables. The output variable i contains the index, while xd contains the value for the indexed entry in the array x. The resulting data can be seen by using the PRINT procedure as follows:

```
proc print data=xdata;
run;
```

The output from PROC PRINT is shown in Output 6.2.2.

Output 6.2.2. Output Data Set Contents

Obs	i	xd
1	1	0.08241
2	2	1.13312
3	3	2.34361

Example 6.3. Model Construction

This example uses PROC OPTMODEL features to simplify the construction of a mathematically formulated model. The model is based on Example 3.12 on page 267 in the PROC LP documentation. A single invocation of PROC OPTMODEL replaces several steps in the PROC LP code.

The model assigns production of various grades of cloth to a set of machines in order to maximize profit while meeting customer demand. Each machine has different capacities to produce the various grades of cloth. (See the PROC LP example for

more details.) The mathematical formulation, where x_{ijk} represents the amount of cloth of grade j to produce on machine k for customer i, follows:

$$\begin{array}{lll} \max & \sum_{ijk} r_{ijk} x_{ijk} & \\ \text{subject to} & \sum_{k} x_{ijk} = d_{ij} & \text{for all } i \text{ and } j \\ & \sum_{ij} c_{jk} x_{ijk} \leq a_k & \text{for all } k \\ & x_{ijk} \geq 0 & \text{for all } i, j, \text{ and } k \end{array}$$

The following code defines the same data sets used in the PROC LP example to specify the problem coefficients:

```
title 'An Assignment Problem';

data object;
   input machine customer
         grade1 grade2 grade3 grade4 grade5 grade6;
   datalines;
1 1 102 140 105  105  125  148
1 2 115 133 118  118  143  166
1 3  70 108  83   83   88   86
1 4  79 117  87   87  107  105
1 5  77 115  90   90  105  148
2 1 123 150 125  124  154   .
2 2 130 157 132  131  166   .
2 3 103 130 115  114  129   .
2 4 101 128 108  107  137   .
2 5 118 145 130  129  154   .
3 1  83  .   .    97  122  147
3 2 119  .   .   133  163  180
3 3  67  .   .    91  101  101
3 4  85  .   .   104  129  129
3 5  90  .   .   114  134  179
4 1 108 121  79   .   112  132
4 2 121 132  92   .   130  150
4 3  78  91  59   .    77   72
4 4 100 113  76   .   109  104
4 5  96 109  77   .   105  145
;

data demand;
   input customer
         grade1 grade2 grade3 grade4 grade5 grade6;
   datalines;
1 100 100 150  150  175  250
2 300 125 300  275  310  325
3 400   0 400  500  340    0
4 250   0 750  750    0    0
5   0 600 300    0  210  360
;

data resource;
```

```
   input machine
         grade1 grade2 grade3 grade4 grade5 grade6 avail;
   datalines;
1 .250 .275 .300  .350  .310  .295  744
2 .300 .300 .305  .315  .320  .     244
3 .350 .    .     .320  .315  .300  790
4 .280 .275 .260  .     .250  .295  672
;
```

The following PROC OPTMODEL code specifies the model, processes the input data sets, solves the optimization problem, and generates a solution data set:

```
proc optmodel;
   set CUSTOMERS;
   set GRADES = 1..6;
   set MACHINES;

   /* parameters */
   number return{CUSTOMERS, GRADES, MACHINES} init 0;
   number demand{CUSTOMERS, GRADES};
   number cost{GRADES,MACHINES} init 0;
   number avail{MACHINES} init 0;

   /* load the customer set and demands */
   read data demand
        into CUSTOMERS=[customer]
        {j in GRADES} <demand[customer,j]=col("grade"||j)>;

   /* load the machine set, time costs, and availability */
   read data resource nomiss
        into MACHINES=[machine]
        {j in GRADES} <cost[j,machine]=col("grade"||j)>
        avail;

   /* load objective data */
   read data object nomiss
        into [machine customer]
        {j in GRADES} <return[customer,j,machine]=col("grade"||j)>;

   /* the model */
   var x{CUSTOMERS, GRADES, MACHINES} >= 0;
   max obj = sum{i in CUSTOMERS, j in GRADES, k in MACHINES}
             return[i,j,k] * x[i,j,k];
   con req_demand{i in CUSTOMERS, j in GRADES}:
       sum{k in MACHINES} x[i,j,k] = demand[i,j];
   con req_avail{k in MACHINES}:
       sum{i in CUSTOMERS, j in GRADES}
          cost[j,k]*x[i,j,k] <= avail[k];

   /* fix x[i,j,k] to 0 if cost[j,k] = 0 */
   for {j in GRADES, k in MACHINES: cost[j,k] = 0} do;
      for {i in CUSTOMERS} fix x[i,j,k]=0;
   end;
```

```
/* call the solver and save the results */
solve with lp/solver=primal;
create data solution
       from [customer grade machine]
       ={i in CUSTOMERS, j in GRADES,k in MACHINES: x[i,j,k] NE 0}
       amount=x;
quit;
```

PROC OPTMODEL processes the data sets directly, using the READ DATA statements to load the data into suitably declared numeric parameters. The READ DATA statements use the iterated column syntax to transfer a range of data set variables into indexed parameters. The COL name expression is expanded in each case into the input data set variables grade1 to grade6. Missing values in the input data sets are handled by using the NOMISS option and initializing the parameters to zero.

For simplicity, the preceding code assumes a fixed set of cloth grades. However, the number of grades could be read from another data set or determined by using SAS functions to examine the variables in the data sets. The first and second READ DATA statements assign the customer and machine sets, respectively, with the set of index values read from the input data sets.

The model portion of the PROC OPTMODEL code parallels the mathematical formulation of the linear program. The solver produces the following output:

Output 6.3.1. LP Solver Result

An Assignment Problem

The OPTMODEL Procedure

Problem Summary

Objective Sense	Maximization
Objective Function	obj
Objective Type	Linear
Number of Variables	120
Bounded Above	0
Bounded Below	100
Bounded Below and Above	0
Free	0
Fixed	20
Number of Constraints	34
Linear LE (<=)	4
Linear EQ (=)	30
Linear GE (>=)	0
Linear Range	0

An Assignment Problem

The OPTMODEL Procedure

Solution Summary

Solver	Primal Simplex
Objective Function	obj
Solution Status	Optimal
Objective Value	871426
Iterations	36
Primal Infeasibility	3.51337E-13
Dual Infeasibility	0
Bound Infeasibility	0

The CREATE DATA statement writes the solution variables to a data set after the solver finishes executing. The explicit source index set is used to restrict the output to the nonzero solution variables. This data set can be processed by PROC TABULATE as follows to create a compact representation of the solution:

```
proc tabulate data=solution;
   class customer grade machine;
   var   amount;
   table (machine*customer), (grade*amount);
run;
```

This code produces the table shown in Output 6.3.2.

Output 6.3.2. An Assignment Problem

		grade			
		1	2	3	4
		amount	amount	amount	amount
		Sum	Sum	Sum	Sum
machine	customer				
1	1	.	100.00	150.00	150.00
	2	.	.	300.00	.
	3	.	.	256.72	210.31
	4	.	.	750.00	.
	5	.	92.27	.	.
2	3	.	.	143.28	.
	5	.	.	300.00	.
3	2	.	.	.	275.00
	3	.	.	.	289.69
	4	.	.	.	750.00
	5	.	.	.	.
4	1	100.00	.	.	.
	2	300.00	125.00	.	.
	3	400.00	.	.	.
	4	250.00	.	.	.
	5	.	507.73	.	.

(Continued)

Output 6.3.2. (continued)

		grade	
		5	6
		amount	amount
		Sum	Sum
machine	customer		
1	1	175.00	250.00
	2	.	.
	3	.	.
	4	.	.
	5	.	.
2	3	340.00	.
	5	.	.
3	2	310.00	325.00
	3	.	.
	4	.	.
	5	210.00	360.00
4	1	.	.
	2	.	.
	3	.	.
	4	.	.
	5	.	.

Example 6.4. Set Manipulation

This example demonstrates PROC OPTMODEL set manipulation operators. These operators are used to compute the set of primes up to a given limit. This example does not solve an optimization problem, but similar set manipulation could be used to set up an optimization model. Here is the code:

```
proc optmodel;
   number maxprime;  /* largest number to consider */
   set composites =
       union {i in 3..sqrt(maxprime) by 2} i*i..maxprime by 2*i;
   set primes = {2} union (3..maxprime by 2 diff composites);
   maxprime = 500;
   put primes;
```

The set composites contains the odd composite numbers up to the value of the parameter maxprime. The even numbers are excluded here to reduce execution time and memory requirements. The UNION aggregation operation is used in the definition to combine the sets of odd multiples of i for $i = 3, 5, \ldots$. Any composite number less than the value of the parameter maxprime has a divisor $\leq \sqrt{maxprime}$, so the

range of i can be limited. The set of multiples of i can also be started at $i \times i$ since smaller multiples are found in the set of multiples for a smaller index.

You can then define the set primes. The odd primes are determined by using the DIFF operator to remove the composites from the set of odd numbers no greater than the parameter maxprime. The UNION operator adds the single even prime, 2, to the resulting set of primes.

The PUT statement produces the result in Output 6.4.1.

Output 6.4.1. Primes ≤ 500

```
{2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,103,
107,109,113,127,131,137,139,149,151,157,163,167,173,179,181,191,193,197,199,211,
223,227,229,233,239,241,251,257,263,269,271,277,281,283,293,307,311,313,317,331,
337,347,349,353,359,367,373,379,383,389,397,401,409,419,421,431,433,439,443,449,
457,461,463,467,479,487,491,499}
```

Note that you were able to delay the definition of the value of the parameter maxprime until just before the PUT statement. Since the defining expressions of the SET declarations are handled symbolically, the value of maxprime is not necessary until you need the value of the set primes. Because the sets composites and primes are defined symbolically, their values reflect any changes to the parameter maxprime. You can see this update by appending the following statements to the preceding code:

```
maxprime = 50;
put primes;
```

The additional statements produce the results in Output 6.4.2. The value of the set primes has been recomputed to reflect the change to the parameter maxprime.

Output 6.4.2. Primes ≤ 50

```
{2,3,5,7,11,13,17,19,23,29,31,37,41,43,47}
```

Rewriting NLP Models for PROC OPTMODEL

This section covers techniques for converting NLP models to PROC OPTMODEL models.

To illustrate the basics, consider the following first version of the NLP model for Example 4.7 on page 409 in the NLP documentation:

```
proc nlp all;
   parms amountx amounty amounta amountb amountc
```

```
      pooltox pooltoy ctox ctoy pools = 1;
   bounds 0 <= amountx amounty amounta amountb amountc,
            amountx <= 100,
            amounty <= 200,
          0 <= pooltox pooltoy ctox ctoy,
          1 <= pools <= 3;
   lincon amounta + amountb = pooltox + pooltoy,
          pooltox + ctox = amountx,
          pooltoy + ctoy = amounty,
          ctox + ctoy    = amountc;
   nlincon nlc1-nlc2 >= 0.,
           nlc3 = 0.;
   max f;
   costa = 6; costb = 16; costc = 10;
   costx = 9; costy = 15;
   f = costx * amountx + costy * amounty
       - costa * amounta - costb * amountb - costc * amountc;
   nlc1 = 2.5 * amountx - pools * pooltox - 2. * ctox;
   nlc2 = 1.5 * amounty - pools * pooltoy - 2. * ctoy;
   nlc3 = 3 * amounta + amountb - pools * (amounta + amountb);
run;
```

This code defines a model that has bounds, linear constraints, nonlinear constraints, and a simple objective function. The following code is a straightforward conversion of the NLP code to PROC OPTMODEL form:

```
proc optmodel;
   var amountx init 1 >= 0 <= 100,
       amounty init 1 >= 0 <= 200;
   var amounta init 1 >= 0,
       amountb init 1 >= 0,
       amountc init 1 >= 0;
   var pooltox init 1 >= 0,
       pooltoy init 1 >= 0;
   var ctox init 1 >= 0,
       ctoy init 1 >= 0;
   var pools init 1 >=1 <= 3;
   con amounta + amountb = pooltox + pooltoy,
       pooltox + ctox = amountx,
       pooltoy + ctoy = amounty,
       ctox + ctoy    = amountc;
   number costa, costb, costc, costx, costy;
   costa = 6; costb = 16; costc = 10;
   costx = 9; costy = 15;
   max f = costx * amountx + costy * amounty
          - costa * amounta - costb * amountb - costc * amountc;
   con nlc1: 2.5 * amountx - pools * pooltox - 2. * ctox >= 0,
       nlc2: 1.5 * amounty - pools * pooltoy - 2. * ctoy >= 0,
       nlc3: 3 * amounta + amountb - pools * (amounta + amountb)
             = 0;
   solve;
   print amountx amounty amounta amountb amountc
         pooltox pooltoy ctox ctoy pools;
```

The PROC OPTMODEL variable declarations were split into individual declarations since PROC OPTMODEL does not permit name lists in its declarations. In the OPTMODEL procedure, variable bounds are part of the variable declaration instead of a separate BOUNDS statement. The PROC NLP statements are as follows:

```
parms amountx amounty amounta amountb amountc
      pooltox pooltoy ctox ctoy pools = 1;
bounds 0 <= amountx amounty amounta amountb amountc,
            amountx <= 100,
            amounty <= 200,
       0 <= pooltox pooltoy ctox ctoy,
       1 <= pools <= 3;
```

The following PROC OPTMODEL statements are equivalent to the PROC NLP statements:

```
var amountx init 1 >= 0 <= 100,
    amounty init 1 >= 0 <= 200;
var amounta init 1 >= 0,
    amountb init 1 >= 0,
    amountc init 1 >= 0;
var pooltox init 1 >= 0,
    pooltoy init 1 >= 0;
var ctox init 1 >= 0,
    ctoy init 1 >= 0;
var pools init 1 >=1 <= 3;
```

The linear constraints are declared in the NLP model with the following statement:

```
lincon amounta + amountb = pooltox + pooltoy,
       pooltox + ctox = amountx,
       pooltoy + ctoy = amounty,
       ctox + ctoy    = amountc;
```

The following linear constraint declarations in the PROC OPTMODEL model are quite similar to the NLP LINCON declarations:

```
con amounta + amountb = pooltox + pooltoy,
    pooltox + ctox = amountx,
    pooltoy + ctoy = amounty,
    ctox + ctoy    = amountc;
```

But PROC OPTMODEL provides much more flexibility in defining linear constraints. For example, coefficients can be named parameters or any other expression that evaluates to a constant.

The cost parameters are declared explicitly in the PROC OPTMODEL model. Unlike the DATA step or PROC NLP, PROC OPTMODEL requires names to be declared

before they are used. There are multiple ways to set the values of these parameters. The preceding example used assignments. The values could have been made part of the declaration by using the INIT *expression* clause or the = *expression* clause. The values could also have been read from a data set with the READ DATA statement.

Note in the original NLP code that the assignment to a parameter such as costa occurs every time the objective function is evaluated. However, the assignment occurs just once in the PROC OPTMODEL code, when the assignment statement is processed. This works because the values are constant. But the PROC OPTMODEL code permits the parameters to be reassigned later to interactively modify the model.

The following statements define the objective f in the NLP model:

```
max f;
. . .
f = costx * amountx + costy * amounty
    - costa * amounta - costb * amountb - costc * amountc;
```

The PROC OPTMODEL version of the objective is defined with the same expression text, as follows:

```
max f = costx * amountx + costy * amounty
       - costa * amounta - costb * amountb - costc * amountc;
```

But in PROC OPTMODEL the MAX statement and the assignment to the name f in the NLP code are combined. There are advantages and disadvantages to this approach. The PROC OPTMODEL formulation is much closer to the mathematical formulation of the model. However, PROC OPTMODEL currently has limited support for named intermediate variables, which can be used to structure the objective formula. The objective formula can refer to the name of other objectives. In that case the named objective expression is effectively substituted into the new objective formula.

In the NLP model the nonlinear constraints use the following syntax:

```
nlincon nlc1-nlc2 >= 0.,
        nlc3 = 0.;
. . .
nlc1 = 2.5 * amountx - pools * pooltox - 2. * ctox;
nlc2 = 1.5 * amounty - pools * pooltoy - 2. * ctoy;
nlc3 = 3 * amounta + amountb - pools * (amounta + amountb);
```

In the PROC OPTMODEL model the equivalent statements are as follows:

```
con nlc1: 2.5 * amountx - pools * pooltox - 2. * ctox >= 0,
    nlc2: 1.5 * amounty - pools * pooltoy - 2. * ctoy >= 0,
    nlc3: 3 * amounta + amountb - pools * (amounta + amountb)
           = 0;
```

The nonlinear constraints in PROC OPTMODEL use the same syntax as linear constraints. In fact, if the variable pools were declared as a parameter, then all the preceding constraints would be linear. The nonlinear constraint in PROC OPTMODEL combines the NLINCON statement of NLP with the assignment in the NLP code. As in objective expressions, objective names can be used in nonlinear constraint expressions to structure the formula.

The PROC OPTMODEL model does not use a RUN statement to invoke the solver. Instead the solver is invoked interactively by the SOLVE statement in PROC OPTMODEL. Note that by default the OPTMODEL procedure prints much less data about the optimization process. Generally this consists of messages from the solver (such as the termination reason) as well as a short status display. The PROC OPTMODEL code adds a PRINT statement in order to display the variable estimates from the solver.

The model for Example 4.8 on page 417 in the NLP documentation is used to illustrate how to convert NLP code that handles arrays into PROC OPTMODEL form. The NLP model is as follows:

```
proc nlp tech=tr pall;
   array c[10] -6.089 -17.164 -34.054  -5.914 -24.721
               -14.986 -24.100 -10.708 -26.662 -22.179;
   array x[10] x1-x10;
   min y;
   parms x1-x10 = .1;
   bounds 1.e-6 <= x1-x10;
   lincon 2. = x1 + 2. * x2 + 2. * x3 + x6 + x10,
          1. = x4 + 2. * x5 + x6 + x7,
          1. = x3 + x7  + x8 + 2. * x9 + x10;
   s = x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10;
   y = 0.;
   do j = 1 to 10;
      y = y + x[j] * (c[j] + log(x[j] / s));
   end;
run;
```

The model finds an equilibrium state for a mixture of chemicals. The following code shows a corresponding PROC OPTMODEL model:

```
proc optmodel;
   set CMP = 1..10;
   number c{CMP} = [-6.089 -17.164 -34.054  -5.914 -24.721
                    -14.986 -24.100 -10.708 -26.662 -22.179];
   var x{CMP} init 0.1 >= 1.e-6;
   con 2. = x[1] + 2. * x[2] + 2. * x[3] + x[6] + x[10],
       1. = x[4] + 2. * x[5] + x[6] + x[7],
       1. = x[3] + x[7]  + x[8] + 2. * x[9] + x[10];

   /* replace the variable s in the NLP model */
   /* min/max is irrelevant */
   min s = sum{i in CMP} x[i];
```

```
min y = sum{j in CMP} x[j] * (c[j] + log(x[j] / s));
solve with nlpc / tech=trureg;
print x y;
```

The PROC OPTMODEL model uses the set CMP to represent the set of compounds, which are numbered 1 to 10 in the example. The array c was initialized by using the equivalent PROC OPTMODEL syntax. Note that the individual array locations could also have been initialized by assignment or READ DATA statements.

The VAR declaration for variable x combines the VAR and BOUNDS statements of the NLP model. The index set of the array is based on the set of compounds CMP, to simplify changes to the model.

The linear constraints are similar in form to the NLP model. However, the PROC OPTMODEL version uses the array form of the variable names since the OPTMODEL procedure treats arrays as distinct variables, not as aliases of lists of scalar variables.

The objective s replaces the intermediate variable of the same name in the NLP model. The keyword (MAX or MIN) that was used to declare objective s is irrelevant here. This is an example of translating an intermediate variable from the other models to PROC OPTMODEL. An alternative way is to use an additional constraint for every intermediate variable. This is useful if you have a set of intermediate variables. In the preceding code, instead of declaring objective s, you can use the following statements:

```
. . .
var s;
con s = sum{i in CMP} x[i];
. . .
```

Note that this alternative formulation passes an extra variable and constraint to the solver.

The PROC OPTMODEL version uses a SUM operator over the set CMP, which enhances the flexibility of the model to accommodate possible changes in the set of compounds.

In the NLP model the objective function y is determined by an explicit loop. With PROC OPTMODEL the DO loop is replaced by a SUM aggregation operation. The accumulation in the NLP model is now performed by PROC OPTMODEL with the SUM operator.

This PROC OPTMODEL model can be further generalized. Note that the array initialization and constraints assume a fixed set of compounds. You can rewrite the model to handle an arbitrary number of compounds and chemical elements. The new model loads the linear constraint coefficients from a data set along with the objective coefficients for the parameter c, as follows:

```
data comp;
   input c a_1 a_2 a_3;
   datalines;
-6.089   1 0 0
-17.164  2 0 0
-34.054  2 0 1
-5.914   0 1 0
-24.721  0 2 0
-14.986  1 1 0
-24.100  0 1 1
-10.708  0 0 1
-26.662  0 0 2
-22.179  1 0 1
;
data atom;
   input b @@;
   datalines;
2. 1. 1.
;
proc optmodel;
   set CMP;
   set ELT;
   number c{CMP};
   number a{ELT,CMP};
   number b{ELT};
   read data atom into ELT=[_n_] b;
   read data comp into CMP=[_n_]
        c {i in ELT} < a[i,_n_]=col("a_"||i) >;
   var x{CMP} init 0.1 >= 1.e-6;
   con bal{i in ELT}: b[i] = sum{j in CMP} a[i,j]*x[j];
   min s = sum{i in CMP} x[i];
   min y = sum{j in CMP} x[j] * (c[j] + log(x[j] / s));
   print a b;
   solve with nlpc / tech=trureg;
   print x;
```

This version adds coefficients for the linear constraints to the COMP data set. The data set variable a_n represents the number of atoms in the compound for element n. The READ DATA statement for COMP uses the iterated column syntax to read each of the data set variables a_n into the appropriate location in the array a. In this example the expanded data set variable names are a_1, a_2, and a_3.

The preceding version also adds a new set, ELT, of chemical elements and a numeric parameter, b, that represents the left-hand side of the linear constraints. The data values for the parameters ELT and b are read from the data set ATOM. The model can handle varying sets of chemical elements because of this extra data set and the new parameters.

The linear constraints have been converted to a single indexed family of constraints. One constraint is applied for each chemical element in the set ELT. The constraint expression uses a simple form that applies generally to linear constraints. The following PRINT statement in the model shows the values read from the data sets to define the linear constraints:

```
print a b;
```

The PRINT statements in the model produce the results shown in Output 6.4.3.

Output 6.4.3. PROC OPTMODEL Output

```
                    The OPTMODEL Procedure

                                a
          1     2     3     4     5     6     7     8     9    10

  1       1     2     2     0     0     1     0     0     0     1
  2       0     0     0     1     2     1     1     0     0     0
  3       0     0     1     0     0     0     1     1     2     1


                          [1]     b

                            1     2
                            2     1
                            3     1


                       [1]             x

                         1     0.04066808
                         2     0.14773032
                         3     0.78315338
                         4     0.00141422
                         5     0.48524669
                         6     0.00069317
                         7     0.02739923
                         8     0.01794728
                         9     0.03731438
                        10     0.09687134
```

In the preceding model the chemical elements and compounds are designated by numbers. So in the PRINT output, for example, the row that is labeled "3" represents the amount of the compound H_2O. PROC OPTMODEL is capable of using more symbolic strings to designate array indices. The following version of the model uses strings to index arrays:

```
data comp;
   input name $ c a_h a_n a_o;
   datalines;
H     -6.089   1 0 0
H2    -17.164  2 0 0
H2O   -34.054  2 0 1
N     -5.914   0 1 0
N2    -24.721  0 2 0
NH    -14.986  1 1 0
NO    -24.100  0 1 1
O     -10.708  0 0 1
O2    -26.662  0 0 2
OH    -22.179  1 0 1
;
data atom;
   input name $ b;
   datalines;
H  2.
N  1.
O  1.
;
proc optmodel;
  set<string> CMP;
  set<string> ELT;
  number c{CMP};
  number a{ELT,CMP};
  number b{ELT};
  read data atom into ELT=[name] b;
  read data comp into CMP=[name]
       c {i in ELT} < a[i,name]=col("a_"||i) >;
  var x{CMP} init 0.1 >= 1.e-6;
  con bal{i in ELT}: b[i] = sum{j in CMP} a[i,j]*x[j];
  min s = sum{i in CMP} x[i];
  min y = sum{j in CMP} x[j] * (c[j] + log(x[j] / s));
  solve with nlpc / tech=trureg;
  print x;
```

In this model the sets CMP and ELT are now sets of strings. The data sets provide the names of the compounds and elements. The names of the data set variables for atom counts in the data set COMP now include the chemical element symbol as part of their spelling. For example, the atom count for element H (hydrogen) is named a_h. Note that these changes did not require any modification to the specifications of the linear constraints or the objective.

The PRINT statement in the preceding code produces the results shown in Output 6.4.4. The indices of variable x are now strings that represent the actual compounds.

Output 6.4.4. PROC OPTMODEL Output with Strings

```
The OPTMODEL Procedure

  [1]              x

  H        0.04066808
  H2       0.14773032
  H2O      0.78315338
  N        0.00141422
  N2       0.48524669
  NH       0.00069317
  NO       0.02739923
  O        0.01794728
  O2       0.03731438
  OH       0.09687134
```

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Chapter 7
The Interior Point Nonlinear Programming Solver (Experimental)

Chapter Contents

Chapter 7
The Interior Point Nonlinear Programming Solver (Experimental)

Overview

The interior point nonlinear programming (IPNLP) solver is a component of the OPTMODEL procedure. It can solve nonlinear programming (NLP) problems that contain both nonlinear equality and inequality constraints. The general NLP problem can be defined as follows:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_i(x) = 0, i \in \mathcal{E} = \{1, 2, ..., p\} \\ & g_i(x) \geq 0, i \in \mathcal{I} = \{1, 2, ..., q\} \\ & l \leq x \leq u \end{aligned}$$

where $x \in \mathbb{R}^n$ represents the vector of the decision variables; $f : \mathbb{R}^n \mapsto \mathbb{R}$ represents the objective function; $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ represents the vector of equality constraints, i.e., $h = (h_1, ..., h_p)$; $g : \mathbb{R}^n \mapsto \mathbb{R}^q$ represents the vector of inequality constraints, i.e., $g = (g_1, ..., g_q)$; and $l, u \in \mathbb{R}^n$ represent the vectors of the lower and upper bounds, respectively, on the decision variables.

It is assumed that the functions f, h_i, and g_i are twice continuously differentiable. Any point satisfying the constraints of the NLP problem is called a *feasible point*, and the set of all those points forms the feasible region of the NLP problem, i.e., $\mathcal{F} = \{x \in \mathbb{R}^n : h(x) = 0, g(x) \geq 0, l \leq x \leq u\}$.

The NLP problem can have a unique minimum or many different minima, depending on the type of functions involved in it. If the objective function is convex, the equality constraint functions are linear, and the inequality constraint functions are concave, then the NLP problem is called a convex program and has a unique minimum. All other types of NLP problems are called nonconvex and can contain more than one minimum, usually called *local minima*. The least of all those local minima is called the *global minimum* or global solution of the NLP problem. The IPNLP solver can find the unique solution of convex programs as well as a local minimum of a general NLP problem.

The IPNLP solver implements a primal-dual interior point algorithm that shares several powerful features from recent state-of-the-art algorithms (Akrotirianakis and Rustem 2000; Armand, Gilbert, and Jan-Jégou 2002; Vanderbei and Shanno 1999; Wächter and Biegler 2006; Yamashita 1998). It uses a line-search procedure and a merit function to ensure convergence of the iterates to a local minimum. The term

"primal-dual" means that the algorithm iteratively generates better approximations of the decision variables x (usually called "primal" variables) as well as the dual variables (also referred to as Lagrange multipliers). At every iteration, the algorithm solves a system of nonlinear equations by using Newton's method. The solution of that system provides the direction and the steps along which the next approximation of the local minimum will be searched. The algorithm also ensures that the primal iterates always remain strictly within their bounds—i.e., $l < x^k < u$, for every iteration k.

The current version of the solver is particularly suited for problems that contain many dense nonlinear inequality constraints, and it is expected to perform better than other nonlinear programming solvers in the SAS/OR suite.

Getting Started

Consider the following simple example of a nonlinear optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) = (x_1 + 3x_2 + x_3)^2 + 4(x_1 - x_2)^2 \\ \text{subject to} \quad & h_1(x) = x_1 + x_2 + x_3 = 1 \\ & g_1(x) = 6x_2 + 4x_3 - x_1^3 - 3 \geq 0 \\ & x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

The problem consists of a quadratic objective function, a linear equality, and a nonlinear inequality constraint. We are interested in finding a local minimum, starting from the point $x^0 = (0.1, 0.7, 0.2)$. To achieve this, we write the following SAS code:

```
proc optmodel;
   var x{1..3} >= 0;
   minimize obj = ( x[1] + 3*x[2] + x[3] )**2 + 4*(x[1] - x[2])**2;

   con constr1: sum{i in 1..3}x[i] = 1;
   con constr2: 6*x[2] + 4*x[3] - x[1]**3 -3 >= 0;

   /* starting point */
   x[1] = 0.1;
   x[2] = 0.7;
   x[3] = 0.2;

   solve with IPNLP;
   print x;
quit;
```

The SAS output displays a detailed summary of the problem, together with the status of the solver at termination, the total number of iterations required, and the value of the objective function at the local minimum. The summary and the optimal solution are shown in Figure 7.1.

```
                The OPTMODEL Procedure

                   Problem Summary

Objective Sense            Minimization
Objective Function                  obj
Objective Type                Quadratic

Number of Variables                   3
Bounded Above                         0
Bounded Below                         3
Bounded Below and Above               0
Free                                  0
Fixed                                 0

Number of Constraints                 2
Linear LE (<=)                        0
Linear EQ (=)                         1
Linear GE (>=)                        0
Linear Range                          0
Nonlinear LE (<=)                     0
Nonlinear EQ (=)                      0
Nonlinear GE (>=)                     1
Nonlinear Range                       0


                The OPTMODEL Procedure

                   Solution Summary

Solver                          IPNLP
Objective Function                obj
Solution Status               Optimal
Objective Value                     1
Iterations                         21

Infeasibility           2.810568E-10
Optimality Error         7.8706805E-7


                [1]                x

                 1        0.00071696
                 2        0.00000083
                 3        0.99928221
```

Figure 7.1. Problem Summary and the Optimal Solution

The SAS log shown in Figure 7.2 displays a brief summary of the problem being solved, followed by the iterations generated by the solver.

```
NOTE: The problem has 3 variables (0 free, 0 fixed).
NOTE: The problem has 1 linear constraints (0 LE, 1 EQ, 0 GE, 0 range).
NOTE: The problem has 3 linear constraint coefficients.
NOTE: The problem has 1 nonlinear constraints (0 LE, 0 EQ, 1 GE, 0 range).
NOTE: Using analytic derivatives for objective.
NOTE: Using analytic derivatives for nonlinear constraints.
NOTE: The IPNLP solver is called.
                        Objective                           Optimality
           Iter             Value     Infeasibility            Error
              0          25.00000           2.00000          70.04792
              1           8.25327           1.97152          36.56861
              2           6.39960           1.58946          20.83618
              3           2.04185           0.11218           3.08706
              4           1.54275           0.00562           1.06389
              5           1.28365           0.00029           0.43790
              6           1.14817           0.00006           0.19808
              7           1.07829           0.00007           0.11098
              8           1.04066           0.00007           0.08009
              9           1.02072           0.00005           0.02705
             10           1.01052           0.00002           0.00811
             11           1.00533           0.00001           0.00340
             12           1.00269           0.00000           0.00146
             13           1.00136           0.00000           0.00063
             14           1.00068           0.00000           0.00027
             15           1.00034           0.00000           0.00012
             16           1.00017           0.00000           0.00005
             17           1.00009           0.00000           0.00002
             18           1.00004           0.00000           0.00001
             19           1.00002           0.00000           0.00000
             20           1.00001           0.00000           0.00000
             21           1.00001           0.00000           0.00000
NOTE: Solver converged.
NOTE: Objective = 1.0000053879.
```

Figure 7.2. Progress of the Algorithm as Shown in the Log

Syntax

The following PROC OPTMODEL statement is available for the IPNLP solver:

SOLVE WITH IPNLP < / *options* > **;**

Functional Summary

Table 7.1 summarizes the options that can currently be used with the SOLVE WITH IPNLP statement.

Table 7.1. Options for the IPNLP Solver

Description	Option
Solver Options:	
specify the maximum number of iterations	MAXITER=
specify the maximum allowable real time	MAXTIME=
specify the convergence tolerance	OPTTOL=
Output Option:	
control the amount of printing by the solver	PRINTFREQ=

IPNLP Solver Options

This section describes the options recognized by the IPNLP solver. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the IPNLP solver is explicitly specified using a WITH clause.

Details of the currently available options are described as follows:

MAXITER=N
specifies that the solver take at most N major iterations to determine an optimum of the NLP problem. The value of this option is an integer greater than or equal to zero. A major iteration in IPNLP consists of finding a descent direction and a step size along which the next approximation of the optimum will reside. The default value of this option is 5000 iterations.

MAXTIME=T
specifies an upper limit of T seconds of real time for the solver to find a local optimum. Note that the time specified by the MAXTIME= option is checked only once at the end of each major iteration. The default value is 7200 seconds (2 hours).

OPTTOL=ϵ
defines the measure by which the user can decide whether the current iterate is an acceptable approximation of a local minimum. The value of this option is a positive real number. The IPNLP solver determines that the current iterate is a local minimum when the norm of the scaled vector of the optimality conditions is less than ϵ. The default value is ϵ=1E–6.

PRINTFREQ=N
specifies how often the iterations are to be displayed in the SAS log. N should be an integer number greater than or equal to zero. If $N \geq 1$, the solver prints only those iterations that are a multiple of N. If $N = 0$, no iteration is displayed in the log. The default value is PRINTFREQ=1.

Details

In this section, we present a brief discussion about the algorithmic details of the IPNLP solver. We start by defining the notation. Next, we present an introduction to the fundamental ideas in constrained optimization; the main point of the second section is to present the necessary and sufficient optimality conditions, which play a central role in all optimization algorithms. We conclude by discussing a general overview of primal-dual interior point algorithms for nonlinear optimization. A detailed treatment of the preceding topics can be found in Nocedal and Wright (1999), Wright (1997), and Forsgren, Gill, and Wright (2002).

Basic Definitions and Notation

The gradient of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is the vector of all the first partial derivatives of f, and is denoted by

$$\nabla f(x) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n})^{\mathrm{T}}$$

where the superscript T denotes the transpose of a vector.

The Hessian matrix of f, denoted by $\nabla^2 f(x)$, or simply by $H(x)$, is an $n \times n$ symmetric matrix whose (i, j) element is the second partial derivative of $f(x)$ with respect to x_i and x_j. That is, $H_{i,j}(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

Consider the vector function, $c : R^n \mapsto R^{p+q}$, whose first p elements are the equality constraint functions $h_i(x), i = 1, 2, .., p$, and whose last q elements are the inequality constraint functions $g_i(x), i = 1, 2, .., q$. That is,

$$c(x) = (h(x) : g(x))^{\mathrm{T}} = (h_1(x), ..., h_p(x) : g_1(x), ..., g_q(x))^{\mathrm{T}}$$

The $n \times (p + q)$ matrix whose ith column is the gradient of the ith element of $c(x)$ is called the Jacobian matrix of $c(x)$ (or simply the Jacobian of the NLP problem) and is denoted by $J(x)$. We can also use $J_h(x)$ to denote the $n \times p$ Jacobian matrix of the equality constraints and use $J_g(x)$ to denote the $n \times q$ Jacobian matrix of the inequality constraints. It is easy to see that $J(x) = (J_h(x) : J_g(x))$.

Overview of Constrained Optimization

A function that plays a pivotal role in establishing conditions that characterize a local minimum of an NLP problem is the Lagrangian function $\mathcal{L}(x, y, z)$, defined as

$$\mathcal{L}(x, y, z) = f(x) - \sum_{i \in \mathcal{E}} y_i h_i(x) - \sum_{i \in \mathcal{I}} z_i g_i(x)$$

Note that the Lagrangian function can be seen as a linear combination of the objective and constraint functions. The coefficients of the constraints, $y_i, i \in \mathcal{E}$, and $z_i,\ i \in \mathcal{I}$,

are called the Lagrange multipliers or dual variables. At a feasible point $\hat{x}$, an inequality constraint is called active if it is satisfied as an equality, i.e., $g_i(\hat{x}) = 0$. The set of active constraints at a feasible point $\hat{x}$ is then defined as the union of the index set of the equality constraints, $\mathcal{E}$, and the indices of those inequality constraints that are active at $\hat{x}$, i.e.,

$$\mathcal{A}(\hat{x}) = \mathcal{E} \cup \{i \in \mathcal{I} : g_i(\hat{x}) = 0\}$$

An important condition that is assumed to hold in the majority of optimization algorithms is the so-called linear independence constraint qualification (LICQ). The LICQ states that at any feasible point $\hat{x}$, the gradients of all the active constraints are linearly independent. The main purpose of the LICQ is to ensure that the set of constraints is well-defined, in a way that there are no redundant constraints, or in a way that there are no constraints defined such that their gradients are always equal to zero.

The First-Order Necessary Optimality Conditions

If x^* is a local minimum of the NLP problem and the LICQ holds at x^*, then there are vectors of Lagrange multipliers y^* and z^*, with components $y_i^*, i \in \mathcal{E}$, and $z_i^*, i \in \mathcal{I}$, respectively, such that the following conditions are satisfied:

$$\begin{aligned}
\nabla_x \mathcal{L}(x^*, y^*, z^*) &= 0 \\
h_i(x^*) &= 0, \quad i \in E \\
g_i(x^*) &\geq 0, \quad i \in I \\
z_i^* &\geq 0, \quad i \in I \\
z_i^* g_i(x^*) &= 0, \quad i \in I
\end{aligned}$$

where $\nabla_x \mathcal{L}(x^*, y^*, z^*)$ is the gradient of the Lagrangian function with respect to x, defined as

$$\nabla_x \mathcal{L}(x^*, y^*, z^*) = \nabla f(x) - \sum_{i \in \mathcal{E}} y_i \nabla h_i(x) - \sum_{i \in \mathcal{I}} z_i \nabla g_i(x)$$

The preceding conditions are often called the *Karush-Kuhn-Tucker (KKT) conditions*, named after their inventors. The last group of equations ($z_i g_i(x) = 0, i \in I$) is called the complementarity condition. Its main aim is to try to force the Lagrange multipliers, z_i^*, of the inactive inequalities (i.e., those inequalities with $g_i(x^*) > 0$) to zero.

The KKT conditions describe the way the first derivatives of the objective and constraints are related at a local minimum x^*. However, they are not enough to fully characterize a local minimum. The second-order optimality conditions attempt to fulfill this aim, by examining the curvature of the Hessian matrix of the Lagrangian function at a point that satisfies the KKT conditions.

The Second-Order Necessary Optimality Condition

Let x^* be a local minimum of the NLP problem, and let y^*, z^* be the corresponding Lagrange multipliers that satisfy the first-order optimality conditions. Then $d^{\mathrm{T}}\nabla_x^2\mathcal{L}(x^*, y^*, z^*)d \geq 0$ for all nonzero vectors d that satisfy the following conditions:

1. $\nabla h_i^{\mathrm{T}}(x^*)d = 0, \forall i \in \mathcal{E}$
2. $\nabla g_i^{\mathrm{T}}(x^*)d = 0, \forall i \in \mathcal{A}(x^*)$, such that $z_i^* > 0$
3. $\nabla g_i^{\mathrm{T}}(x^*)d \geq 0, \forall i \in \mathcal{A}(x^*)$, such that $z_i^* = 0$

The second-order necessary optimality condition states that, at a local minimum, the curvature of the Lagrangian function along the directions that satisfy the preceding conditions must be nonnegative.

Overview of Interior Point Methods

Primal-dual interior point methods can be classified into two categories: feasible and infeasible. The first category requires the starting point as well as all subsequent iterations of the algorithm to strictly satisfy all the inequality constraints. The second category relaxes those requirements and allows the iterates to violate some or all of the inequality constraints during the course of the minimization procedure. Currently, the IPNLP solver implements an infeasible algorithm. For this reason we will concentrate on that type of algorithm.

To make the notation less cluttered and the fundamentals of interior point methods easier to understand, we consider, without loss of generality, the following simpler NLP problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, i \in \mathcal{I} = \{1, 2, ..., q\} \end{aligned}$$

Note that the equality and bound constraints have been omitted from the preceding problem. Initially, slack variables are added to the inequality constraints, giving rise to the following problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) - s_i = 0, i \in \mathcal{I} \\ & s \geq 0 \end{aligned}$$

where $s = (s_1, ..., s_q)^{\mathrm{T}}$ represents the vector of the slack variables, which are required to be nonnegative. Next, all the nonnegativity constraints on the slack variables are eliminated by being incorporated into the objective function, by means of

a logarithmic function. This gives rise to the following equality-constrained NLP problem:

$$\begin{array}{ll} \text{minimize} & B(x,s) = f(x) - \mu \sum_{i \in \mathcal{I}} \ln(s_i) \\ \text{subject to} & g_i(x) - s_i = 0, i \in \mathcal{I} \end{array}$$

where μ is a positive parameter. The preceding problem is often called the *barrier problem*. The nonnegativity constraints on the slack variables are implicitly enforced by the logarithmic functions, since the logarithmic function prohibits s from taking zero or negative values.

Depending on the size of the parameter μ, a local minimum of the barrier problem provides an *approximation* to the local minimum of the original NLP problem. The smaller the size of μ, the better the approximation becomes. Infeasible primal-dual interior point algorithms repetitively solve the barrier problem for different values of μ that progressively go to zero, in order to get as close as possible to a local minimum of the original NLP problem.

To solve the barrier problem we first define its Lagrangian function:

$$\begin{aligned} \mathcal{L}_B(x,s,z) &= B(x,s) - z^{\mathrm{T}}(g(x) - s) \\ &= f(x) - \mu \sum_{i \in I} \ln(s_i) - z^{\mathrm{T}}(g(x) - s) \end{aligned}$$

Then we define its first-order optimality conditions:

$$\begin{array}{rcrcl} \nabla_x \mathcal{L}_B & = & \nabla_x f(x) - J(x)^{\mathrm{T}} z & = & 0 \\ \nabla_s \mathcal{L}_B & = & -\mu S^{-1} e + z & = & 0 \\ & & g(x) - s & = & 0 \end{array}$$

where $J(x)$ represents the Jacobian matrix of the vector function $g(x)$, S represents the diagonal matrix whose elements are the elements of the vector s (i.e., $S = diag\{s_1, ..., s_q\}$), and e represents a vector of all ones. By multiplying the second equation by S, we obtain the following equivalent nonlinear system:

$$\begin{array}{rcl} \nabla_x f(x) - J(x)^{\mathrm{T}} z & = & 0 \\ -\mu e + Sz & = & 0 \\ g(x) - s & = & 0 \end{array}$$

Note that if $\mu = 0$, the preceding conditions represent the optimality conditions of the original optimization problem, after adding slack variables. One of the main aims of the algorithm is to gradually reduce the value of μ to zero, so that it converges to a local optimum of the original NLP problem. The rate by which μ approaches zero affects the overall efficiency of the algorithm.

At an iteration k, the infeasible primal-dual interior point algorithm approximately solves the preceding system by using Newton's method. The Newton system is

$$\begin{bmatrix} H_{\mathcal{L}}(x^k, z^k) & 0 & -J(x^k)^{\mathrm{T}} \\ 0 & Z^k & S^k \\ J(x^k) & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta s^k \\ \Delta z^k \end{bmatrix} = - \begin{bmatrix} \nabla_x f(x^k) - J(x^k)^{\mathrm{T}} z \\ -\mu e + S^k z^k \\ g(x^k) - s^k \end{bmatrix}$$

where $H_{\mathcal{L}}$ is the Hessian matrix of the Lagrangian function $\mathcal{L}(x, z) = f(x) - z^{\mathrm{T}} g(x)$ of the original NLP problem, i.e.,

$$H_{\mathcal{L}}(x, z) = \nabla^2 f(x) - \sum_{i \in \mathcal{I}} z_i \nabla^2 g_i(x)$$

The solution $(\Delta x^k, \Delta s^k, \Delta z^k)$ of the Newton system provides a direction to move from the current iteration (x^k, s^k, z^k) to the next:

$$(x^{k+1}, s^{k+1}, z^{k+1}) = (x^k, s^k, z^k) + \alpha(\Delta x^k, \Delta s^k, \Delta z^k)$$

where α is the step length along the Newton direction. The step length is determined through a line-search procedure that ensures sufficient decrease of a merit function based on the augmented Lagrangian function of the barrier problem. The role of the merit function and the line-search procedure is to ensure that the objective and the infeasibility reduce sufficiently at every iteration and that the iterates approach a local minimum of the original NLP problem.

Solver Termination Criterion

The solver terminates when the norm of the optimality conditions of the original problem is less than the user-defined or default tolerance (see OPTTOL= option). More specifically, if

$$F(x, s, z) = (\nabla_x f(x) - J(x)^{\mathrm{T}} z, \quad Sz, \quad g(x) - s)^{\mathrm{T}}$$

is the vector of the optimality conditions of the original NLP problem, then the solver terminates when

$$\| F(x, s, z) \| \leq OPTTOL$$

Solver Termination Messages

Upon termination the solver produces the following messages in the log:

Solver converged
The solver has successfully found a local solution to the optimization problem.

Solver found a conditionally optimal solution
The solver is sufficiently close to a local solution, but it has difficulty in completely

satisfying the user-defined optimality tolerance. This can happen when the line search finds very small steps resulting in very slight progress of the algorithm. It can also happen when the prespecified tolerance is too strict for the optimization problem at hand.

Solver reached maximum number of iterations
The solver could not find a local optimum in the prespecified number of iterations.

Solver reached maximum allowed time
The solver could not find a local optimum in the prespecified maximum real time for the optimization process.

Solver did not converge
The solver could not satisfy the optimality conditions and failed to converge.

Problem is unbounded
The objective function takes arbitrarily large values, and the optimality conditions fail to be satisfied. This can happen when the problem is unconstrained or constrained and the feasible region is not bounded.

Solver out of memory
The problem is so large that the solver requires more memory to solve the problem.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put _OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the IPNLP solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit
FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY
indicates the infeasibility of the primal constraints at the solution.

OPTIMALITY_ERROR
indicate of the norm of the optimality conditions at the solution. See the section "Solver Termination Criterion" on page 810 for details.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Examples

In this section several examples are presented in order to show the capabilities and features of the IPNLP solver.

Example 7.1. Solving Highly Nonlinear Optimization Problems

This example demonstrates the use of the IPNLP solver to solve the following highly nonlinear optimization problem, which appears in Hock and Schittkowski (1981).

$$
\begin{aligned}
\text{minimize} \quad & f(x) = 0.4(x_1)^{0.67}(x_7)^{-0.67} + 0.4(x_2)^{0.67}(x_8)^{-0.67} \\
& \quad +10 - x_1 - x_2 \\
\text{subject to} \quad & 1 - 0.0588x_5x_7 - 0.1x_1 \geq 0 \\
& 1 - 0.0588x_6x_8 - 0.1x_1 - 0.1x_2 \geq 0 \\
& 1 - 4x_3(x_5)^{-1} - 2(x_3)^{-0.71}(x_5)^{-1} - 0.0588(x_3)^{-1.3}x_7 \geq 0 \\
& 1 - 4x_4(x_6)^{-1} - 2(x_4)^{-0.71}(x_6)^{-1} - 0.0588(x_4)^{-1.3}x_8 \geq 0 \\
& 0.1 \leq f(x) \leq 4.2 \\
& 0.1 \leq x_i \leq 10, i = 1, 2, ..., 8
\end{aligned}
$$

The initial point used is $x^0 = (6, 3, 0.4, 0.2, 6, 6, 1, 0.5)$. You can call the IPNLP solver within PROC OPTMODEL to solve the problem by writing the following SAS code:

```
proc optmodel;
   var x{1..8} >=.1 <=10;

   minimize obj = 0.4*x[1]^.67*x[7]^-.67+.4*x[2]^.67*x[8]^-.67
      +10-x[1]-x[2];

   con c1: 1-.0588*x[5]*x[7]-.1*x[1]>=0;
   con c2: 1-.0588*x[6]*x[8]-.1*x[1]-.1*x[2]>=0;
   con c3: 1-4*x[3]/x[5]-2/(x[3]^.71*x[5])-.0588*x[7]/x[3]^1.3>=0;
   con c4: 1-4*x[4]/x[6]-2/(x[4]^.71*x[6])-.0588*x[8]/x[4]^1.3>=0;
   con c5: .4*x[1]^.67*x[7]^-.67+.4*x[2]^.67*x[8]^-.67+10
      -x[1]-x[2]>=.1;
   con c6: .4*x[1]^.67*x[7]^-.67+.4*x[2]^.67*x[8]^-.67+10
      -x[1]-x[2]<=4.2;

   /* starting point */
   x[1] = 6;
   x[2] = 3;
   x[3] = .4;
   x[4] = .2;
   x[5] = 6;
   x[6] = 6;
   x[7] = 1;
   x[8] = .5;

   solve with ipnlp;
   print x;
quit;
```

The summaries and the optimal solution are shown in Output 7.1.1.

Output 7.1.1. Summaries and the Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	8
Bounded Above	0
Bounded Below	0
Bounded Below and Above	8
Free	0
Fixed	0
Number of Constraints	6
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	1
Nonlinear EQ (=)	0
Nonlinear GE (>=)	5
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	IPNLP
Objective Function	obj
Solution Status	Optimal
Objective Value	3.9512
Iterations	29
Infeasibility	5.049056E-12
Optimality Error	7.5708926E-7

[1]	x
1	6.46509
2	2.23272
3	0.66740
4	0.59576
5	5.93268
6	5.52724
7	1.01333
8	0.40067

Example 7.2. Solving Unconstrained Optimization Problems

Although the IPNLP solver is suited for solving constrained NLP problems, it can solve efficiently unconstrained problems too. We demonstrate its ability by considering the following relatively large unconstrained NLP problem:

$$\text{minimize } f(x) = \sum_{i=1}^{n-1}(-4x_i + 3.0) + \sum_{i=1}^{n-1}(x_i^2 + x_n^2)^2$$

where $n = 5000$. To solve this problem, you can write the following SAS code:

```
proc optmodel;
   number N=5000;
   var x{1..N} init 1.0;

   minimize obj = sum {i in 1..N - 1} ( - 4 * x[i] + 3.0)  +
                  sum {i in 1..N - 1} (x[i]^2 + x[N]^2)^2;

   solve with ipnlp;
quit;
```

The problem and solution summaries are shown in Output 7.2.1.

Output 7.2.1. Problem Summary and Solution Summary

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	5000
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	5000
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	IPNLP
Objective Function	obj
Solution Status	Optimal
Objective Value	0
Iterations	12
Infeasibility	0
Optimality Error	5.3537486E-8

Example 7.3. Solving NLP Problems with Range Constraints

Often there are constraints with both lower and upper bounds, i.e., $a \leq g(x) \leq b$. These constraints are called *range constraints*. The IPNLP solver can handle range constraints in an efficient way. Consider the following NLP problem:

$$
\begin{aligned}
\text{minimize} \quad & f(x) = 5.35(x_3)^2 + 0.83x_1x_5 + 37.29x_1 - 40792.141 \\
\text{subject to} \quad & 0 \leq a_1 + a_2x_2x_5 + a_3x_1x_4 - a_4x_3x_5 \leq 92 \\
& 0 \leq a_5 + a_6x_2x_5 + a_7x_1x_2 + a_8(x_3)^3 - 90 \leq 20 \\
& 0 \leq a_9 + a_10x_3x_5 + a_11x_1x_3 + a_12x_3x_4 - 20 \leq 5 \\
& 78 \leq x_1 \leq 102 \\
& 33 \leq x_2 \leq 45 \\
& 27 \leq x_i \leq 45, \quad i = 3, 4, 5
\end{aligned}
$$

where the values of the parameters $a_i, i = 1, 2, ..., 12$, are shown in Table 7.2.

Table 7.2. Data for Example 3

i	a_i	i	a_i	i	a_i
1	85.334407	5	80.51249	9	0.0047026
2	0.0056858	6	0.0071317	10	0.0012547
3	0.0006262	7	0.0029955	11	0.0019085
4	0.0022053	8	0.0021813	12	0.0019085

The initial point used is $x^0 = (78, 33, 27, 27, 27)$. You can call the IPNLP solver within PROC OPTMODEL to solve this problem by writing the following SAS code:

```
proc optmodel;
   number l {1..5} = [78 33 27 27 27];
   number u {1..5} = [102 45 45 45 45];

   number a {1..12} =
      [85.334407 0.0056858 0.0006262 0.0022053
      80.51249 0.0071317 0.0029955 0.0021813
      9.300961 0.0047026 0.0012547 0.0019085];

   var x {j in 1..5} >= l[j] <= u[j];

   minimize obj = 5.35*x[3]^2 + 0.83*x[1]*x[5] + 37.29*x[1]
                  - 40792.141;

   con constr1:
      0 <= a[1] + a[2]*x[2]*x[5] + a[3]*x[1]*x[4] -
         a[4]*x[3]*x[5] <= 92;
   con constr2:
      0 <= a[5] + a[6]*x[2]*x[5] + a[7]*x[1]*x[2] +
         a[8]*x[3]^2 - 90 <= 20;
   con constr3:
      0 <= a[9] + a[10]*x[3]*x[5] + a[11]*x[1]*x[3] +
         a[12]*x[3]*x[4] -20 <= 5;

   x[1] = 78;
   x[2] = 33;
   x[3] = 27;
   x[4] = 27;
   x[5] = 27;

   solve with ipnlp;
   print x;
quit;
```

The summaries and the optimal solution are shown in Output 7.3.1.

Output 7.3.1. Summaries and the Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Quadratic
Number of Variables	5
Bounded Above	0
Bounded Below	0
Bounded Below and Above	5
Free	0
Fixed	0
Number of Constraints	3
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	0
Nonlinear GE (>=)	0
Nonlinear Range	3

The OPTMODEL Procedure

Solution Summary

Solver	IPNLP
Objective Function	obj
Solution Status	Optimal
Objective Value	-30689
Iterations	72
Infeasibility	2.664535E-15
Optimality Error	5.9764635E-7

[1]	x
1	78.000
2	33.000
3	29.995
4	45.000
5	36.776

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Chapter 8
The Linear Programming Solver

Chapter Contents

Chapter 8
The Linear Programming Solver

Overview

The OPTMODEL procedure provides a framework for specifying and solving linear programs (LPs). A standard linear program has the following formulation:

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \; \{\geq, =, \leq\} \; \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

where

$\mathbf{x}$	$\in$	$\mathbb{R}^n$	is the vector of decision variables
$\mathbf{A}$	$\in$	$\mathbb{R}^{m \times n}$	is the matrix of constraints
$\mathbf{c}$	$\in$	$\mathbb{R}^n$	is the vector of objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{R}^m$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l}$	$\in$	$\mathbb{R}^n$	is the vector of lower bounds on variables
$\mathbf{u}$	$\in$	$\mathbb{R}^n$	is the vector of upper bounds on variables

The following LP solvers are available in the OPTMODEL procedure:

- primal simplex solver
- dual simplex solver
- interior point solver (experimental)

The simplex solvers implement the two-phase simplex method. In phase I, the solver tries to find a feasible solution. If no feasible solution is found, the LP is infeasible; otherwise, the solver enters phase II to solve the original LP. The interior point solver implements a primal-dual predictor-corrector interior point algorithm.

Getting Started

The following example illustrates how you can use the OPTMODEL procedure to solve linear programs. Suppose you want to solve the following problem:

$$\begin{array}{rrcrcrcl} \max & x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & - & x_3 & \leq & 1 \\ & -2x_1 & - & 3x_2 & + & 2x_3 & \leq & 1 \\ & & & x_1, & & x_2, \quad x_3 & \geq & 0 \end{array}$$

You can use the following statement to call the OPTMODEL procedure for solving linear programs:

```
proc optmodel;
  var x{i in 1..3} >= 0;

  max f = x[1] + x[2] + x[3] ;

  con c1: 3*x[1] + 2*x[2]  - x[3]       <= 1;
  con c2: -2*x[1]  - 3*x[2]  +  2*x[3]  <= 1;

  solve with lp / solver = ps presolver = none printfreq = 1;
  print x;
quit;
```

The iteration log displaying problem statistics, progress of the solution, and the optimal objective value is shown in Figure 8.1.

```
NOTE: The problem has 3 variables (0 free, 0 fixed).
NOTE: The problem has 2 linear constraints (2 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 6 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The PRIMAL SIMPLEX solver is called.
NOTE:                  Objective     Entering    Leaving
      Phase Iteration  Value         Variable    Variable
        2           1      0.500000  x[3]        c2      (S)
        2           2      8.000000  x[2]        c1      (S)
NOTE: Optimal.
NOTE: Objective = 8.
```

Figure 8.1. Log

The optimal solution and the optimal objective value are displayed in Figure 8.2.

The OPTMODEL Procedure

Problem Summary

Objective Sense	Maximization
Objective Function	f
Objective Type	Linear
Number of Variables	3
Bounded Above	0
Bounded Below	3
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	2
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	Primal Simplex
Objective Function	f
Solution Status	Optimal
Objective Value	8
Iterations	2
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0

[1]	x
1	0
2	3
3	5

Figure 8.2. Solution Summary

Syntax

The following statement is available in the OPTMODEL procedure:

SOLVE WITH LP < / *options* > **;**

Functional Summary

Table 8.1 summarizes the list of options available for the SOLVE WITH LP statement, classified by function.

Table 8.1. Options for the LP Solver

Description	Option
Solver Option:	
type of solver	SOLVER=
Presolve Option:	
type of presolve	PRESOLVER=
Control Options:	
feasibility tolerance	FEASTOL=
maximum number of iterations	MAXITER=
upper limit on real time used to solve the problem	MAXTIME=
optimality tolerance	OPTTOL=
frequency of printing solution progress	PRINTFREQ=
use CPU/real time	TIMETYPE=
Simplex Algorithm Options:	
type of initial basis	BASIS=
type of pricing strategy	PRICETYPE=
queue size for determining entering variable	QUEUESIZE=
enable or disable scaling of the problem	SCALE=
Interior Point Algorithm Options:	
stopping criterion based on duality gap	STOP_DG=
stopping criterion based on dual infeasibility	STOP_DI=
stopping criterion based on primal infeasibility	STOP_PI=

LP Solver Options

This section describes the options recognized by the LP solver. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the LP solver is explicitly specified using a WITH clause.

Solver Options

SOLVER=*option*
specifies one of the following LP solvers:

Option	Description
PRIMAL_SPX (PS)	Use primal simplex solver.
DUAL_SPX (DS)	Use dual simplex solver.
ITERATIVE_INTERIOR (II), experimental	Use interior point solver.

The valid abbreviated value for each option is indicated in parentheses. By default, the dual simplex solver is used.

Presolve Options

PRESOLVER=*option*
specifies one of the following presolve options:

Option	Description
NONE	Disable presolver.
AUTOMATIC	Apply presolver by using default setting.
BASIC	Perform basic presolve like removing empty rows, columns, and fixed variables.
MODERATE	Perform basic presolve and apply other inexpensive presolve techniques.
AGGRESSIVE	Perform moderate presolve and apply other aggressive (but expensive) presolve techniques.

The default option is AUTOMATIC. See the section "Presolve" on page 830 for details.

Control Options

FEASTOL=ϵ
specifies the feasibility tolerance, $\epsilon \in$[1E–9, 1E–4], for determining the feasibility of a variable. The default value is 1E–6.

MAXITER=k
specifies the maximum number of iterations. The value k can be any integer greater than or equal to one. If you do not specify this option, the procedure does not stop based on the number of iterations performed.

MAXTIME=k
specifies an upper limit of k seconds of time for the optimization process. The timer

used by this option is determined by the value of the TIMETYPE= option. If you do not specify this option, the procedure does not stop based on the amount of time elapsed.

OPTTOL=ϵ

specifies the optimality tolerance, $\epsilon \in$ [1E–9, 1E–4], for declaring optimality. The default value is 1E–6.

PRINTFREQ=k

specifies that the printing of the solution progress to the iteration log should occur after every k iterations. The print frequency, k, is an integer greater than or equal to zero.

The value $k = 0$ disables the printing of the progress of the solution. If the simplex algorithms are used, the default value of this option is determined dynamically according to the problem size. If the interior point algorithm is used, the default value of this option is 1.

TIMETYPE=*CPU* | *REAL*

sets the timer used by the MAXTIME= option, and controls the type of time reported by PRESOLVE_TIME and SOLUTION_TIME in the _OROPTMODEL_ macro variable. The "Optimization Statistics" table, an output of PROC OPTMODEL if option PRINTLEVEL=2 is specified in the PROC OPTMODEL statement, also includes the same time values for "Presolver Time" and "Solver Time." The other times (such as "Problem Generation Time") in this table are always CPU times. The default value of this option is CPU.

Simplex Algorithm Options

BASIS=*option*

specifies the following options for generating an initial basis:

Option	Description
CRASH	Generate an initial basis by using crash techniques (Maros 2003). The procedure creates a triangular basic matrix consisting of both decision variables and slack variables.
SLACK	Generate an initial basis by using all slack variables.
WARMSTART	Start the simplex solvers with available basis.

The default option is CRASH.

PRICETYPE=*option*

specifies one of the following pricing strategies for the simplex solvers:

Option	Description
HYBRID	Use hybrid Devex and steepest-edge pricing strategies. Available for primal simplex solver only.
PARTIAL	Use partial pricing strategy. Optionally, you can specify QUEUESIZE=. Available for primal simplex solver only.
FULL	Use the most negative reduced cost pricing strategy.
DEVEX	Use Devex pricing strategy.
STEEPESTEDGE	Use steepest-edge pricing strategy.

The default pricing strategy for the primal simplex solver is HYBRID and that for the dual simplex solver is STEEPESTEDGE. See the section "Pricing Strategies for the Primal Simplex Solver" on page 830 for details.

QUEUESIZE=k

specifies the queue size, $k \in [1, n]$, where n is the number of decision variables. This queue is used for finding an entering variable in the simplex iteration. The default value is chosen adaptively based on the number of decision variables. This option is used only when PRICETYPE=PARTIAL.

SCALE=*option*

specifies one of the following scaling options:

Option	Description
NONE	Disable scaling.
AUTOMATIC	Automatically apply scaling procedure if necessary.

The default option is AUTOMATIC.

Interior Point Algorithm Options

STOP_DG=δ

specifies the desired relative duality gap, $\delta \in$[1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 831 for details.

STOP_DI=β

specifies the maximum allowed relative dual constraints violation, $\beta \in$ [1E–9, 1E–4]. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 831 for details.

STOP_PI=α

specifies the maximum allowed relative bound and primal constraints violation,

$\alpha \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 831 for details.

Details

Presolve

Presolve in the simplex LP solvers of PROC OPTMODEL uses a variety of techniques to reduce the problem size, improve numerical stability, and detect infeasibility or unboundedness (Andersen and Andersen 1995; Gondzio 1997). During presolve, redundant constraints and variables are identified and removed. Presolve can further reduce the problem size by substituting variables. Variable substitution is a very effective technique, but it might occasionally increase the number of nonzero entries in the constraint matrix.

In most cases, using presolve is very helpful in reducing solution times. You can enable presolve at different levels or disable it by specifying the PRESOLVER= option.

Pricing Strategies for the Primal Simplex Solver

Several pricing strategies for the simplex solvers are available. Pricing strategies determine which variable enters the basis at each simplex pivot. These can be controlled by specifying the PRICETYPE= option.

The primal simplex solver has the following five pricing strategies:

PARTIAL	scans a queue of decision variables to find an entering variable. You can optionally specify the QUEUESIZE= option to control the length of this queue.
FULL	uses Dantzig's most violated reduced cost rule (Dantzig 1963). It compares the reduced cost of all decision variables, and selects the variable with the most violated reduced cost as the entering variable.
DEVEX	implements the Devex pricing strategy developed by Harris (1973).
STEEPESTEDGE	uses the steepest-edge pricing strategy developed by Forrest and Goldfarb (1992).
HYBRID	uses a hybrid of the Devex and steepest-edge pricing strategies.

The dual simplex solver has only three pricing strategies available: FULL, DEVEX, and STEEPESTEDGE.

The Interior Point Algorithm: Overview

The interior point LP solver (experimental) in PROC OPTMODEL implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following LP formulation (the primal):

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The corresponding dual is as follows:

$$\begin{aligned} \max \quad & \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of dual slack variables.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker) conditions, which can be stated as follows:

$$\begin{aligned} \mathbf{A}\mathbf{x} - \mathbf{s} &= \mathbf{b} \quad (\text{Primal Feasibility}) \\ \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} &= \mathbf{c} \quad (\text{Dual Feasibility}) \\ \mathbf{W}\mathbf{X}\mathbf{e} &= \mathbf{0} \quad (\text{Complementarity}) \\ \mathbf{S}\mathbf{Y}\mathbf{e} &= \mathbf{0} \quad (\text{Complementarity}) \\ \mathbf{x},\, \mathbf{y},\, \mathbf{w},\, \mathbf{s} &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal *slack* variables.

Note: Slack variables (the s vector) are automatically introduced by the solver when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the solver to handle slack variables much more efficiently.

The letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{S}$ denote matrices with corresponding x, y, w, and s on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations representing the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original LP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively; Θ and Ξ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. The interior point solver uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point solver is that it takes full advantage of the sparsity in the constraint matrix, thereby enabling it to efficiently solve large-scale linear programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore it is of interest to observe the following four measures:

- Relative primal infeasibility measure α:

$$\alpha = \frac{\|\mathbf{Ax} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- Relative dual infeasibility measure β:

$$\beta = \frac{\|\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- Relative duality gap δ:

$$\delta = \frac{|\mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- Absolute complementarity γ:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

where $\|v\|_2$ is the Euclidean norm of the vector v. These measures are displayed in the iteration log.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put _OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the LP solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit

FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY
indicates the (relative) primal infeasibility of the constraints at the solution.

DUAL_INFEASIBILITY
indicates the (relative) dual infeasibility of the constraints at the solution.

BOUND_INFEASIBILITY
indicates the (relative) violation by the solution of the lower and/or upper bounds.

DUALITY_GAP
indicates the (relative) duality gap. This term appears only if the option SOLVER= ITERATIVE_INTERIOR is specified in the SOLVE statement.

ITERATIONS
indicates the number of iterations taken to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken to solve the problem (seconds).

Note: The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time (default) or real time. The type is determined by the TIMETYPE= option.

Examples

Example 8.1. Diet Problem

Consider the problem of diet optimization. There are six different foods: bread, milk, cheese, potato, fish, and yogurt. The cost and nutrition values per unit are displayed in Table 8.2.

Table 8.2. Cost and Nutrition Values

	Bread	Milk	Cheese	Potato	Fish	Yogurt
Cost	2.0	3.5	8.0	1.5	11.0	1.0
Protein, g	4.0	8.0	7.0	1.3	8.0	9.2
Fat, g	1.0	5.0	9.0	0.1	7.0	1.0
Carbohydrates, g	15.0	11.7	0.4	22.6	0.0	17.0
Calories	90	120	106	97	130	180

The following SAS code creates the data set fooddata of Table 8.2:

```
data fooddata;
infile datalines;
input    name $  cost  prot  fat  carb  cal;
datalines;
         Bread   2     4     1    15    90
         Milk    3.5   8     5    11.7  120
         Cheese  8     7     9    0.4   106
         Potato  1.5   1.3   0.1  22.6  97
         Fish    11    8     7    0     130
         Yogurt  1     9.2   1    17    180
;
run;
```

The objective is to find a minimum-cost diet that contains at least 300 calories, not more than 10 grams of protein, not less than 10 grams of carbohydrates, and not less than 8 grams of fat. In addition, the diet should contain at least 0.5 unit of fish and no more than 1 unit of milk.

You can model the problem and solve it by using PROC OPTMODEL as follows:

```
proc optmodel;
   /* declare index set */
   set<str> FOOD;

   /* declare variables */
   var diet{FOOD} >= 0;

   /* objective function */
   num cost{FOOD};
   min f=sum{i in FOOD}cost[i]*diet[i];

   /* constraints */
   num prot{FOOD};
   num fat{FOOD};
   num carb{FOOD};
   num cal{FOOD};
   num min_cal, max_prot, min_carb, min_fat;
   con cal_con: sum{i in FOOD}cal[i]*diet[i] >= 300;
   con prot_con: sum{i in FOOD}prot[i]*diet[i] <= 10;
   con carb_con: sum{i in FOOD}carb[i]*diet[i] >= 10;
   con fat_con: sum{i in FOOD}fat[i]*diet[i] >= 8;

   /* read parameters */
   read data fooddata into FOOD=[name] cost prot fat carb cal;

   /* bounds on variables */
   diet['Fish'].lb = 0.5;
   diet['Milk'].ub = 1.0;

   /* solve and print the optimal solution */
   solve with lp/printfreq=1; /* print each iteration to log */
   print diet;
```

The optimal solution and the optimal objective value are displayed in Output 8.1.1.

Output 8.1.1. Optimal Solution to the Diet Problem

The OPTMODEL Procedure

Solution Summary

Solver	Dual Simplex
Objective Function	f
Solution Status	Optimal
Objective Value	12.081
Iterations	4
Primal Infeasibility	2.220446E-16
Dual Infeasibility	0
Bound Infeasibility	0

[1]	diet
Bread	0.000000
Cheese	0.449499
Fish	0.500000
Milk	0.053599
Potato	1.865168
Yogurt	0.000000

Example 8.2. Reoptimizing the Diet Problem Using BASIS=WARMSTART

After an LP is solved, you might want to change a set of the parameters of the LP and solve the problem again. This can be done efficiently in PROC OPTMODEL. The warm start technique uses the optimal solution of the solved LP as a starting point and solves the modified LP problem faster than it can be solved again from scratch. This example illustrates reoptimizing the diet problem described in Example 8.1.

Assume the optimal solution is found by the SOLVE statement. Instead of quitting the OPTMODEL procedure, we continue to solve several variations of the original problem.

Suppose the cost of cheese increases from 8 to 10 per unit and the cost of fish decreases from 11 to 7 per serving unit. We change the parameters and solve the modified problem by submitting the following code:

```
cost['Cheese']=10; cost['Fish']=7;
solve with lp/presolver=none
             basis=warmstart
             solver=ps
             printfreq=1;
print diet;
```

Note that the primal simplex solver is preferred because the primal solution to the last-solved LP is still feasible for the modified problem in this case. The solution is shown in Output 8.2.1.

Output 8.2.1. Optimal Solution to the Diet Problem with Modified Objective Function

The OPTMODEL Procedure

[1]	diet
Bread	0.000000
Cheese	0.449499
Fish	0.500000
Milk	0.053599
Potato	1.865168
Yogurt	0.000000

The following iteration log indicates that it takes the LP solver no more iterations to solve the modified problem by using BASIS=WARMSTART, since the optimal solution to the original problem remains optimal after the objective function is changed.

Output 8.2.2. Log

```
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The PRIMAL SIMPLEX solver is called.
NOTE: Optimal.
NOTE: Objective = 10.980335514.
```

Next we restore the original coefficients of the objective function and consider the case that you need a diet that supplies at least 150 calories. We change the parameters and solve the modified problem by submitting the following code:

```
cost['Cheese']=8; cost['Fish']=11;cal_con.lb=150;
solve with lp/presolver=none
             basis=warmstart
             solver=ds
             printfreq=1;
print diet;
```

Note that the dual simplex solver is preferred because the dual solution to the last-solved LP is still feasible for the modified problem in this case. The solution is shown in Output 8.2.3.

Output 8.2.3. Optimal Solution to the Diet Problem with Modified RHS

```
                    The OPTMODEL Procedure

                      [1]           diet

                      Bread      0.00000
                      Cheese     0.18481
                      Fish       0.50000
                      Milk       0.56440
                      Potato     0.14702
                      Yogurt     0.00000
```

The following iteration log indicates that it takes the LP solver just one more phase II iteration to solve the modified problem by using BASIS=WARMSTART.

Output 8.2.4. Log

```
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The DUAL SIMPLEX solver is called.
NOTE:                   Objective     Entering        Leaving
      Phase Iteration   Value         Variable        Variable
        2           1       9.174413  cal_con     (S) carb_con    (S)
NOTE: Optimal.
NOTE: Objective = 9.1744131985.
```

Next we restore the original constraint on calories and consider the case that you need a diet that supplies no more than 550 mg of sodium per day. The following row is appended to Table 8.2.

	Bread	Milk	Cheese	Potato	Fish	Yogurt
sodium, mg	148	122	337	186	56	132

We change the parameters, add the new constraint, and solve the modified problem by submitting the following code:

```
cal_con.lb=300;
num sod{FOOD}=[148 122 337 186 56 132];
con sodium: sum{i in FOOD}sod[i]*diet[i] <= 550;
solve with lp/presolver=none
                basis=warmstart
```

```
                    printfreq=1;
print diet;
```

The solution is shown in Output 8.2.5.

Output 8.2.5. Optimal Solution to the Diet Problem with Additional Constraint

```
                    The OPTMODEL Procedure

                     [1]            diet

                     Bread      0.000000
                     Cheese     0.449499
                     Fish       0.500000
                     Milk       0.053599
                     Potato     1.865168
                     Yogurt     0.000000
```

The following iteration log indicates that it takes the LP solver no more iterations to solve the modified problem by using the BASIS=WARMSTART option, since the optimal solution to the original problem remains optimal after one more constraint is added.

Output 8.2.6. Log

```
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 5 linear constraints (2 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 29 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The DUAL SIMPLEX solver is called.
NOTE: Optimal.
NOTE: Objective = 12.081337881.
```

Example 8.3. Two-Person Zero-Sum Game

Consider a two-person zero-sum game (where one person wins what the other person loses). The players make moves simultaneously, and each has a choice of actions. There is a *payoff* matrix that indicates the amount one player gives to the other under each combination of actions:

$$\begin{array}{cc} & \text{Player II plays } j \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Player I plays } i \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left(\begin{array}{cccc} -5 & 3 & 1 & 8 \\ 5 & 5 & 4 & 6 \\ -4 & 6 & 0 & 5 \end{array} \right) \end{array}$$

If player I makes move i and player II makes move j, then player I wins (and player II loses) a_{ij}. What is the best strategy for the two players to adopt? This example is simple enough to be analyzed from observation. Suppose player I plays 1 or 3; the best response of player II is to play 1. In both cases, player I loses and player II wins. So the best action for player I is to play 2. In this case, the best response for player II is to play 3, which minimizes the loss. In this case, (2, 3) is a *pure-strategy Nash equilibrium* in this game.

For illustration, consider the following mixed strategy case. Assume that player I selects i with probability p_i, $i = 1, 2, 3$, and player II selects j with probability q_j, $j = 1, 2, 3, 4$. Consider player II's problem of minimizing the maximum expected payout:

$$\min_{\mathbf{q}} \left\{ \max_i \sum_{j=1}^{4} a_{ij} q_j \right\} \quad \text{subject to} \quad \sum_{j=1}^{4} q_{ij} = 1, \quad \mathbf{q} \geq 0$$

This is equivalent to

$$\begin{array}{lrcll} \min\limits_{\mathbf{q},v} \; v & \text{subject to} & \sum\limits_{j=1}^{4} a_{ij} q_j & \leq & v \quad \forall\, i \\ & & \sum\limits_{j=1}^{4} q_j & = & 1 \\ & & \mathbf{q} & \geq & 0 \end{array}$$

We can transform the problem into a more standard format by making a simple change of variables: $x_j = q_j / v$. The preceding LP formulation now becomes

$$\begin{array}{lrcll} \min\limits_{\mathbf{x},v} \; v & \text{subject to} & \sum\limits_{j=1}^{4} a_{ij} x_j & \leq & 1 \quad \forall\, i \\ & & \sum\limits_{j=1}^{4} x_j & = & 1/v \\ & & \mathbf{q} & \geq & 0 \end{array}$$

which is equivalent to

$$\max_{\mathbf{x}} \sum_{j=1}^{4} x_j \quad \text{subject to} \quad A\mathbf{x} \leq \mathbf{1}, \quad \mathbf{x} \geq 0$$

where A is the payoff matrix and $\mathbf{1}$ is a vector of 1's. It turns out that the corresponding optimization problem from player I's perspective can be obtained by solving the dual problem, which can be written as

$$\min_{\mathbf{y}} \sum_{i=1}^{3} y_i \quad \text{subject to} \quad A^{\mathrm{T}}\mathbf{y} \geq \mathbf{1}, \quad \mathbf{y} \geq 0$$

You can model the problem and solve it by using PROC OPTMODEL as follows:

```
proc optmodel;
   num a{1..3, 1..4}=[-5 3 1 8
                       5 5 4 6
                      -4 6 0 5];
   var x{1..4} >= 0;
   max f = sum{i in 1..4}x[i];
   con c{i in 1..3}: sum{j in 1..4}a[i,j]*x[j] <= 1;
   solve with lp / solver = ps presolver = none printfreq = 1;
   print x;
   print c.dual;
quit;
```

The optimal solution is displayed in Output 8.3.1.

Output 8.3.1. Optimal Solutions to the Two-Person Zero-Sum Game

The OPTMODEL Procedure

Solution Summary

Solver	Primal Simplex
Objective Function	f
Solution Status	Optimal
Objective Value	0.25
Iterations	1
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0

[1]	x
1	0.00
2	0.00
3	0.25
4	0.00

[1]	c.DUAL
1	0.00
2	0.25
3	0.00

The optimal solution $\mathbf{x}^* = (0, 0, 0.25, 0)$ with an optimal value of 0.25. Therefore the optimal strategy for player II is $\mathbf{q}^* = \mathbf{x}^*/0.25 = (0, 0, 1, 0)$. You can check the optimal solution of the dual problem by using the constraint suffix ".dual". So $\mathbf{y}^* = (0, 0.25, 0)$ and player I's optimal strategy is (0, 1, 0). The solution is consistent with our intuition from observation.

References

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Chapter 9
The Mixed Integer Linear Programming Solver

Chapter Contents

Chapter 9
The Mixed Integer Linear Programming Solver

Overview

The OPTMODEL procedure provides a framework for specifying and solving mixed integer linear programs (MILPs). A standard mixed integer linear program has the following formulation:

$$\begin{array}{rl} \min & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \\ & \mathbf{x}_i \in \mathbb{Z} \quad \forall i \in \mathcal{S} \end{array} \qquad \text{(MILP)}$$

where

$\mathbf{x}$	$\in$	$\mathbb{R}^n$	is the vector of structural variables
$\mathbf{A}$	$\in$	$\mathbb{Q}^{m\times n}$	is the matrix of technological coefficients
$\mathbf{c}$	$\in$	$\mathbb{Q}^n$	is the vector of objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{Q}^m$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l}$	$\in$	$\mathbb{Q}^n$	is the vector of lower bounds on variables
$\mathbf{u}$	$\in$	$\mathbb{Q}^n$	is the vector of upper bounds on variables
$\mathcal{S}$			is a nonempty subset of the set $\{1\ldots, n\}$ of indices

The MILP solver, available in the OPTMODEL procedure, implements an LP-based branch-and-bound algorithm. This divide-and-conquer approach attempts to solve the original problem by solving linear programming relaxations of a sequence of smaller subproblems. The MILP solver also implements advanced techniques such as presolving, generating cutting planes, and applying primal heuristics to improve the efficiency of the overall algorithm.

The MILP solver provides various control options and solution strategies. In particular, you can enable, disable, or set levels for the advanced techniques previously mentioned. It is also possible to input an incumbent solution; see the section "Warm Start Option" on page 851 for details.

Getting Started

The following example illustrates how you can use the OPTMODEL procedure to solve mixed integer linear programs. For more examples, see the section "Examples" on page 867. Suppose you want to solve the following problem:

$$
\begin{array}{lrcrcrcrl}
\min & 2x_1 & - & 3x_2 & - & 4x_3 & & & \\
\text{s.t.} & & - & 2x_2 & - & 3x_3 & \geq & -5 & \text{(R1)} \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 4 & \text{(R2)} \\
& x_1 & + & 2x_2 & + & 3x_3 & \leq & 7 & \text{(R3)} \\
& & & x_1, & & x_2, \; x_3 & \geq & 0 & \\
& & & x_1, & & x_2, \; x_3 & \in & \mathbb{Z} &
\end{array}
$$

You can use the following code to call the OPTMODEL procedure for solving mixed integer linear programs:

```
proc optmodel;
   var x{1..3} >= 0 integer;

   min f = 2*x[1] - 3*x[2] - 4*x[3];

   con r1: -2*x[2] - 3*x[3] >= -5;
   con r2: x[1] + x[2] + 2*x[3] <= 4;
   con r3: x[1] + 2*x[2] + 3*x[3] <= 7;

   solve with milp / presolver = automatic heuristics = automatic;
   print x;
quit;
```

The PRESOLVER= and HEURISTICS= options specify the levels for presolving and applying heuristics, respectively. In this example, each option is set to its default value, AUTOMATIC, meaning that the solver determines the appropriate levels for presolve and heuristics automatically.

The optimal value of x is shown in Figure 9.1.

The OPTMODEL Procedure

[1]	x
1	0
2	1
3	1

Figure 9.1. Solution Output

The solution summary stored in the macro variable _OROPTMODEL_ can be viewed by issuing the following statement:

```
%put &_OROPTMODEL_;
```

This produces the output shown in Figure 9.2.

```
STATUS=OK SOLUTION_STATUS=OPTIMAL OBJECTIVE=-7 RELATIVE_GAP=0 ABSOLUTE_GAP=0 PRI
MAL_INFEASIBILITY=0 BOUND_INFEASIBILITY=0 INTEGER_INFEASIBILITY=0 NODES=1 ITERAT
IONS=2 PRESOLVE_TIME=0 SOLUTION_TIME=0
```

Figure 9.2. Macro Output

Syntax

The following statement is available in the OPTMODEL procedure:

SOLVE WITH MILP < / *options* > **;**

Functional Summary

Table 9.1 summarizes the options available for the SOLVE WITH MILP statement, classified by function.

Table 9.1. Options for the MILP Solver

Description	Option
Presolve Option	
type of presolve	PRESOLVER=
Warm Start Option	
primal solution input data set (warm start)	PRIMALIN
Control Options	
stopping criterion based on absolute objective gap	ABSOBJGAP=
cutoff value for node removal	CUTOFF=
emphasize feasibility or optimality	EMPHASIS=
maximum allowed difference between an integer variable's value and an integer	INTTOL=
maximum number of nodes to be processed	MAXNODES=
maximum number of solutions to be found	MAXSOLS=
maximum solution time	MAXTIME=
frequency of printing node log	PRINTFREQ=

Table 9.1. (continued)

Description	Option
detail of solution progress printed in log	PRINTLEVEL2=
stopping criterion based on relative objective gap	RELOBJGAP=
scale the problem matrix	SCALE=
stopping criterion based on target objective value	TARGET=
use CPU/real time	TIMETYPE=
Heuristics Option	
primal heuristics level	HEURISTICS=
Search Options	
node selection strategy	NODESEL=
use of variable priorities	PRIORITY=
number of simplex iterations performed on each variable in strong branching strategy	STRONGITER=
number of candidates for strong branching	STRONGLEN=
rule for selecting branching variable	VARSEL=
Cut Options	
overall cut level	ALLCUTS=
clique cut level	CUTCLIQUE=
flow cover cut level	CUTFLOWCOVER=
flow path cut level	CUTFLOWPATH=
Gomory cut level	CUTGOMORY=
generalized upper bound (GUB) cover cut level	CUTGUB=
implied bounds cut level	CUTIMPLIED=
knapsack cover cut level	CUTKNAPSACK=
lift-and-project cut level (experimental)	CUTLAP=
mixed integer rounding (MIR) cut level	CUTMIR=
row multiplier factor for cuts	CUTSFACTOR=

MILP Solver Options

This section describes the options recognize by the MILP solver in PROC OPTMODEL. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the MILP solver is explicitly specified using a WITH clause. For example, the following line could appear in PROC OPTMODEL code:

```
solve with milp / allcuts=aggressive maxnodes=10000 primalin;
```

Presolve Option

PRESOLVER=*option*

specifies a presolve *option* or its corresponding value *num*, as listed in Table 9.2.

Table 9.2. Values for PRESOLVER= Option

Number	Option	Description
–1	AUTOMATIC	Apply the default level of presolve processing.
0	NONE	Disable presolver.
1	BASIC	Perform minimal presolve processing.
2	MODERATE	Apply a higher level of presolve processing.
3	AGGRESSIVE	Apply the highest level of presolve processing.

The default value is AUTOMATIC.

Warm Start Option

PRIMALIN

enables you to input an integer feasible solution in PROC OPTMODEL before invoking the MILP solver. Adding the PRIMALIN option to the SOLVE statement requests that the MILP solver use the current variable values as a starting integer feasible solution (warm start). If the MILP solver finds that the input solution is valid, then the input solution provides an incumbent solution and a bound for the branch-and-bound algorithm. If the solution is not valid, then the PRIMALIN data are ignored. When it is difficult to find a good integer feasible solution for a problem, warm start can reduce solution time significantly.

Control Options

ABSOBJGAP=*num*

specifies a stopping criterion. When the absolute difference between the best integer objective and the objective of the best node remaining falls below the value of *num*, the solver stops. The value of *num* can be any positive number; the default value is 1E−6.

CUTOFF=*num*

cuts off any nodes in a minimization (maximization) problem with an objective value above (below) *num*. The value of *num* can be any number; the default value is the positive (negative) number that has the largest absolute value representable in your operating environment.

EMPHASIS=*option*

specifies a search emphasis *option* or its corresponding value *num* as listed in Table 9.3.

Table 9.3. Values for EMPHASIS= Option

Number	Option	Description
0	BALANCE	Perform a balanced search.
1	OPTIMAL	Emphasize optimality over feasibility.
2	FEASIBLE	Emphasize feasibility over optimality.

The default value is BALANCE.

INTTOL=*num*

specifies the amount by which an integer variable value can differ from an integer and still be considered integer feasible. The value of *num* can be any number between 0.0 and 1.0; the default value is 1E−5. The MILP solver attempts to find an optimal solution with integer infeasibility less than *num*. If you assign a value smaller than 1E−10 to *num* and the best solution found by the solver has integer infeasibility between *num* and 1E−10, then the solver terminates with a solution status of OPTIMAL_COND (see the section "Macro Variable _OROPTMODEL_ " on page 857).

MAXNODES=*num*

specifies the maximum number of branch-and-bound nodes to be processed. The value of *num* can be any nonnegative number; the default value is the positive integer that has the largest absolute value representable in your operating environment.

MAXSOLS=*num*

specifies a stopping criterion. If *num* solutions have been found, then the solver stops. The value of *num* can be any positive integer; the default value is the positive integer that has the largest absolute value representable in your operating environment.

MAXTIME=*num*

specifies the maximum time allowed for the MILP solver to find a solution. The type of time, either CPU time or real time, is determined by the value of the TIMETYPE= option. The value of *num* can be any positive number; the default value is the positive number that has the largest absolute value representable in your operating environment.

PRINTFREQ=*num*

specifies how often information is printed in the node log. The value of *num* can be any nonnegative number; the default value is 100. If *num* is set to 0, then the node log is disabled. If *num* is positive, then an entry will be made in the node log at the first node, at the last node, and at intervals dictated by the value of *num*. An entry will also be made each time a better integer solution is found.

PRINTLEVEL2=*option*

controls the amount of information displayed in the SAS log by the MILP solver, from a short description of presolve information and summary to details at each node. Table 9.4 describes the valid values for this option.

Table 9.4. Values for PRINTLEVEL2= Option

Number	Option	Description
0	NONE	Turn off all solver-related messages to SAS log.
1	BASIC	Display a solver summary after stopping.
2	MODERATE	Print a solver summary and a node log by using the interval dictated by the PRINTFREQ= option.
3	AGGRESSIVE	Print a detailed solver summary and a node log by using the interval dictated by the PRINTFREQ= option.

The default value is MODERATE.

RELOBJGAP=*num*

specifies a stopping criterion based on the best integer objective (BestInteger) and the objective of the best remaining node (BestBound). The relative objective gap is equal to

$$| \text{BestInteger} - \text{BestBound} | / (1\text{E}{-}10 + | \text{BestBound} |)$$

When this value becomes smaller than the specified gap size *num*, the solver stops. The value of *num* can be any number between 0 and 1; the default value is 1E−4.

SCALE=*option*

indicates whether or not to scale the problem matrix. SCALE= can take either of the values AUTOMATIC (–1) and NONE (0). SCALE=AUTOMATIC scales the matrix as determined by the MILP solver; SCALE=NONE disables scaling. The default value is AUTOMATIC.

TARGET=*num*

specifies a stopping criterion for minimization (maximization) problems. If the best integer objective is better than or equal to *num*, the solver stops. The value of *num* can be any number; the default value is the negative (positive) number that has the largest absolute value representable in your operating environment.

TIMETYPE=*CPU* | *REAL*

specifies the measurement of time used by the MILP solver. Numeric values of time can be specified in the MAXTIME= option or reported in the _OROPTMODEL_ macro variable. The value of the TIMETYPE= option determines whether CPU time or real time is used. The default value of this option is CPU.

Heuristics Option

HEURISTICS=*option*

enables the user to control the level of primal heuristics applied by the MILP solver. This level determines how frequently primal heuristics are applied during the branch-and-bound tree search. It also affects the maximum number of iterations allowed in iterative heuristics. Some computationally expensive heuristics might be disabled by

the solver at less aggressive levels. The values of *option* and the corresponding values of *num* are listed in Table 9.5.

Table 9.5. Values for HEURISTICS= Option

Number	Option	Description
–1	AUTOMATIC	Apply default level of heuristics, similar to MODERATE.
0	NONE	Disable all primal heuristics.
1	BASIC	Apply basic primal heuristics at low frequency.
2	MODERATE	Apply most primal heuristics at moderate frequency.
3	AGGRESSIVE	Apply all primal heuristics at high frequency.

The default value is AUTOMATIC. For details about primal heuristics, see the section "Primal Heuristics" on page 865.

Search Options

NODESEL=*option*

specifies the node selection strategy *option* or its corresponding value *num* as listed in Table 9.6.

Table 9.6. Values for NODESEL= Option

Number	Option	Description
–1	AUTOMATIC	Use automatic node selection.
0	BESTBOUND	Choose the node with the best relaxed objective (best-bound-first strategy).
1	BESTESTIMATE	Choose the node with the best estimate of the integer objective value (best-estimate-first strategy).
2	DEPTH	Choose the most recently created node (depth-first strategy).

The default value is AUTOMATIC. For details about node selection, see the section "Node Selection" on page 862.

PRIORITY=*0* | *1*

indicates whether or not to use specified branching priorities for integer variables. PRIORITY=0 ignores variable priorities; PRIORITY=1 uses priorities when they exist. The default value is 1. See the section "Branching Priorities" on page 863 for details.

STRONGITER=*num*
specifies the number of simplex iterations performed for each variable in the candidate list when the strong branching variable selection strategy is used. The value of *num* can be any positive number; the default value is automatically calculated by the MILP solver.

STRONGLEN=*num*
specifies the number of candidates used when the strong branching variable selection strategy is performed. The value of *num* can be any positive integer; the default value is 10.

VARSEL=*option*
specifies the rule for selecting the branching variable. The values of *option* and the corresponding values of *num* are listed in Table 9.7.

Table 9.7. Values for VARSEL= Option

Number	Option	Description
–1	AUTOMATIC	Use automatic branching variable selection.
0	MAXINFEAS	Choose the variable with maximum infeasibility.
1	MININFEAS	Choose the variable with minimum infeasibility.
2	PSEUDO	Choose a branching variable based on pseudocost.
3	STRONG	Use strong branching variable selection strategy.

The default value is AUTOMATIC. For details about variable selection, see the section "Variable Selection" on page 862.

Cut Options

Table 9.8 describes the *option* and *num* values for the cut options in the OPTMODEL procedure.

Table 9.8. Values for Individual Cut Options

Number	Option	Description
–1	AUTOMATIC	Generate cutting planes based on a strategy determined by the MILP solver.
0	NONE	Disable generation of cutting planes.
1	MODERATE	Use a moderate cut strategy.
2	AGGRESSIVE	Use an aggressive cut strategy.

You can use the ALLCUTS= option to set all cut types to the same level. You can override the ALLCUTS= value by using the options corresponding to particular cut types. For example, if you want PROC OPTMILP to generate only

Gomory cuts, specify ALLCUTS=NONE and CUTGOMORY=AUTOMATIC. If you want to generate all cuts aggressively but generate no lift-and-project cuts, set ALLCUTS=AGGRESSIVE and CUTLAP=NONE.

ALLCUTS=*option*

provides a shorthand way of setting all the cuts-related options in one setting. In other words, ALLCUTS=*num* is equivalent to setting each of the individual cuts parameters to the same value *num*. Thus, ALLCUTS=–1 has the effect of setting CUTCLIQUE=–1, CUTFLOWCOVER=–1, CUTFLOWPATH=–1, ..., CUTLAP=–1, and CUTMIR=–1. Table 9.8 lists the values that can be assigned to *option* and *num*. In addition, you can override levels for individual cuts with the CUTCLIQUE=, CUTFLOWCOVER=, CUTFLOWPATH=, CUTGOMORY=, CUTGUB=, CUTIMPLIED=, CUTKNAPSACK=, CUTLAP=, and CUTMIR= options. If the ALLCUTS= option is not specified, all the cuts-related options are either at their individually specified values (if the corresponding option is specified) or at their default values (if that option is not specified).

CUTCLIQUE=*option*

specifies the level of clique cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTCLIQUE= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTFLOWCOVER=*option*

specifies the level of flow cover cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTFLOWCOVER= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTFLOWPATH=*option*

specifies the level of flow path cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTFLOWPATH= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTGOMORY=*option*

specifies the level of Gomory cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTGOMORY= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTGUB=*option*

specifies the level of generalized upper bound (GUB) cover cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTGUB= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTIMPLIED=*option*

specifies the level of implied bound cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTIMPLIED= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTKNAPSACK=*option*

specifies the level of knapsack cover cuts generated by the MILP solver. Table 9.8

lists the values that can be assigned to *option* and *num*. The CUTKNAPSACK= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTLAP=*option* **(experimental)**
specifies the level of lift-and-project (LAP) cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTLAP= option overrides the ALLCUTS= option. The default value is NONE.

CUTMIR=*option*
specifies the level of mixed integer rounding (MIR) cuts generated by the MILP solver. Table 9.8 lists the values that can be assigned to *option* and *num*. The CUTMIR= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTSFACTOR=*num*
specifies a row multiplier factor for cuts. The number of cuts added is limited to *num* times the original number of rows. The value of *num* can be any nonnegative number less than or equal to 100; the default value is 3.0.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure defines a macro variable named _OROPTMODEL_. This variable contains a character string that indicates the status of the solver upon termination. The contents of the macro variable depend on which solver was invoked. For the MILP solver, the various terms of _OROPTMODEL_ are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the solver
IO_ERROR	problem in reading or writing data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
OPTIMAL_AGAP	optimal solution within absolute gap specified using ABSOBJGAP= option

OPTIMAL_RGAP	optimal solution within relative gap specified using RELOBJGAP= option
OPTIMAL_COND	solution is optimal, but some infeasibilities (primal, bound, or integer) exceed tolerances due to scaling or choice of small INTTOL= value
TARGET	solution not worse than target specified using TARGET= option
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	problem is infeasible or unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
SOLUTION_LIM	solver reached maximum number of solutions specified using option MAXSOLS=
NODE_LIM_SOL	solver reached maximum number of nodes specified using MAXNODES= option and found a solution
NODE_LIM_NOSOL	solver reached maximum number of nodes specified using MAXNODES= option and did not find a solution
TIME_LIM_SOL	solver reached the execution time limit specified using MAXTIME= option and found a solution
TIME_LIM_NOSOL	solver reached the execution time limit specified using MAXTIME= option and did not find a solution
ABORT_SOL	solver was stopped by user but still found a solution
ABORT_NOSOL	solver was stopped by user and did not find a solution
OUTMEM_SOL	solver ran out of memory but still found a solution
OUTMEM_NOSOL	solver ran out of memory and either did not find a solution or failed to output solution due to insufficient memory
FAIL_SOL	solver stopped due to errors but still found a solution
FAIL_NOSOL	solver stopped due to errors and did not find a solution

OBJECTIVE
indicates the objective value obtained by the solver at termination.

RELATIVE_GAP
specifies the relative gap between the best integer objective (BestInteger) and the objective of the best remaining node (BestBound) upon termination of the MILP solver. The relative gap is equal to

$$| \text{BestInteger} - \text{BestBound} | / (1\text{E}{-}10 + | \text{BestBound} |)$$

ABSOLUTE_GAP
specifies the absolute gap between the best integer objective (BestInteger) and the objective of the best remaining node (BestBound) upon termination of the MILP solver. The absolute gap is equal to $| \text{BestInteger} - \text{BestBound} |$.

PRIMAL_INFEASIBILITY
indicates the (relative) primal infeasibility of the constraints at the solution.

BOUND_INFEASIBILITY
indicates the maximum violation by the solution of the lower and/or upper bounds.

INTEGER_INFEASIBILITY
specifies the maximum violation of the integrality of an integer variable returned by the MILP solver.

NODES
specifies the number of nodes enumerated by the MILP solver by using the branch-and-bound algorithm.

ITERATIONS
indicates the number of simplex iterations taken to solve the problem.

PRESOLVE_TIME
indicates the time (in seconds) used in preprocessing.

SOLUTION_TIME
indicates the time (in seconds) taken to solve the problem.

Note: The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time (default) or real time. The type is determined by the TIMETYPE= option.

Details

This section includes brief overviews of the following:

- the branch-and-bound algorithm
- controlling the branch-and-bound algorithm
- the types of presolve techniques available
- cutting planes
- primal heuristics
- node log

The Branch-and-Bound Algorithm

The branch-and-bound algorithm, first proposed by Land and Doig (1960), is an effective approach to solving mixed integer linear programs. The following discussion outlines the approach and explains how to enhance its progress by using several advanced techniques.

The branch-and-bound algorithm solves a mixed integer linear program by dividing the search space and generating a sequence of subproblems. The search space of a mixed integer linear program can be represented by a tree. Each node in the tree is identified with a subproblem derived from previous subproblems on the path leading to the root of the tree. The subproblem (MILP^0) associated with the root is identical to the original problem, which we will call (MILP), given in the section "Overview" on page 847.

The linear programming relaxation (LP^0) of (MILP^0) can be written as

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

The branch-and-bound algorithm generates subproblems along the nodes of the tree by using the following scheme. Consider $\bar{x}^0$, the optimal solution to (LP^0), which is usually obtained using the dual simplex algorithm. If $\bar{x}_i^0$ is integer for all $i \in \mathcal{S}$, then $\bar{x}^0$ is an optimal solution to (MILP). Suppose that for some $i \in \mathcal{S}$, $\bar{x}_i^0$ is nonintegral. In that case the algorithm defines two new subproblems (MILP^1) and (MILP^2), descendants of the parent subproblem (MILP^0). The subproblem (MILP^1) is identical to (MILP^0) except for the additional constraint

$$x_i \leq \lfloor \bar{x}_i^0 \rfloor$$

and the subproblem (MILP^2) is identical to (MILP^0) except for the additional constraint

$$x_i \geq \lceil \bar{x}_i^0 \rceil$$

The notation $\lfloor y \rfloor$ represents the largest integer less than or equal to y, and the notation $\lceil y \rceil$ represents the smallest integer greater than or equal to y. The two preceding constraints can be handled by modifying the bounds of the variable x_i rather than by explicitly adding the constraints to the constraint matrix. The two new subproblems do not have $\bar{x}^0$ as a feasible solution, but the integer solution to (MILP) must satisfy one of the preceding constraints. The two subproblems thus defined are called *active nodes* in the branch-and-bound tree, and the variable x_i is called the *branching variable*.

In the next step the branch-and-bound algorithm chooses one of the active nodes and attempts to solve the linear programming relaxation of that subproblem. The relaxation might be infeasible, in which case the subproblem is dropped (fathomed).

If the subproblem can be solved and the solution is *integer feasible* (that is, x_i is an integer for all $i \in \mathcal{S}$), then its objective value provides an *upper bound* for the objective value in the minimization problem (MILP); if the solution is not integer feasible, then it defines two new subproblems. Branching continues in this manner until there are no active nodes. At this point the best integer solution found is an optimal solution for (MILP). If no integer solution has been found, then (MILP) is integer infeasible. You can specify other criteria to stop the branch-and-bound algorithm before it processes all the active nodes; see the section "Controlling the Branch-and-Bound Algorithm" on page 861 for details.

Upper bounds from integer feasible solutions can be used to *fathom* or *cut off* active nodes. Since the objective value of an optimal solution cannot be greater than an upper bound, active nodes with lower bounds higher than an existing upper bound can be safely deleted. In particular, if z is the objective value of the current best integer solution, then any active subproblems whose relaxed objective value is greater than or equal to z can be discarded.

It is important to realize that mixed integer linear programs are NP-hard. Roughly speaking, this means that the effort required to solve a mixed integer linear program grows exponentially with the size of the problem. For example, a problem with 10 binary variables can in the worst case generate $2^{10} = 1,024$ nodes in the branch-and-bound tree. A problem with 20 binary variables can in the worst case generate $2^{20} = 1,048,576$ nodes in the branch-and-bound tree. Although it is unlikely that the branch-and-bound algorithm will have to generate every single possible node, the need to explore even a small fraction of the potential number of nodes for a large problem can be resource intensive.

A number of techniques can speed up the search progress of the branch-and-bound algorithm. Heuristics are used to find feasible solutions, which can improve the upper bounds on solutions of mixed integer linear programs. Cutting planes can reduce the search space and thus improve the lower bounds on solutions of mixed integer linear programs. When using cutting planes, the branch-and-bound algorithm is also called the *branch-and-cut algorithm*. Preprocessing can reduce problem size and improve problem solvability. The MILP solver in PROC OPTMODEL employs various heuristics, cutting planes, preprocessing, and other techniques, which you can control through corresponding options.

Controlling the Branch-and-Bound Algorithm

There are numerous strategies that can be used to control the branch-and-bound search (see Linderoth and Savelsbergh 1998, Achterberg, Koch, and Martin 2005). The MILP solver in PROC OPTMODEL implements the most widely used strategies and provides several options that enable you to direct the choice of the next active node and of the branching variable. In the discussion that follows, let (LP^k) be the linear programming relaxation of subproblem (MILP^k). Also, let

$$f_i(k) = \bar{x}_i^k - \lfloor \bar{x}_i^k \rfloor$$

where $\bar{x}^k$ is the optimal solution to the relaxation problem (LP^k) solved at node k.

Node Selection

The NODESEL= option specifies the strategy used to select the next active node. The valid keywords for this option are AUTOMATIC, BESTBOUND, BESTESTIMATE, and DEPTH. The following list describes the strategy associated with each keyword.

AUTOMATIC enables the MILP solver to choose the best node selection strategy based on problem characteristics and search progress. This is the default setting.

BESTBOUND chooses the node with the smallest (or largest, in the case of a maximization problem) relaxed objective value. The best-bound strategy tends to reduce the number of nodes to be processed and can improve lower bounds quickly. If there is no good upper bound, however, the number of active nodes can be large. This can result in the solver running out of memory.

BESTESTIMATE chooses the node with the smallest (or largest, in the case of a maximization problem) objective value of the estimated integer solution. Besides improving lower bounds, the best-estimate strategy also attempts to process nodes that can yield good feasible solutions.

DEPTH chooses the node that is deepest in the search tree. Depth-first search is effective in locating feasible solutions, since such solutions are usually deep in the search tree. Compared to the costs of the best-bound and best-estimate strategies, the cost of solving LP relaxations is less in the depth-first strategy. The number of active nodes is generally small, but it is possible that the depth-first search will remain in a portion of the search tree with no good integer solutions. This occurrence is computationally expensive.

Variable Selection

The VARSEL= option specifies the strategy used to select the next branching variable. The valid keywords for this option are AUTOMATIC, MAXINFEAS, MININFEAS, PSEUDO, and STRONG. The following list describes the action taken in each case when $\bar{x}^k$, a relaxed optimal solution of (MILP^k), is used to define two active subproblems. In the following list, "INTTOL" refers to the value assigned using the INTTOL= option. For details about the INTTOL= option, see the section "Control Options" on page 851.

AUTOMATIC enables the MILP solver to choose the best variable selection strategy based on problem characteristics and search progress. This is the default setting.

MAXINFEAS chooses as the branching variable the variable x_i such that i maximizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and}$$

$$\text{INTTOL} \leq f_i(k) \leq 1 - \text{INTTOL}\}$$

MININFEAS chooses as the branching variable the variable x_i such that i minimizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and}$$

$$\text{INTTOL} \leq f_i(k) \leq 1 - \text{INTTOL}\}$$

PSEUDO chooses as the branching variable the variable x_i such that i maximizes the weighted up and down pseudocosts. Pseudocost branching attempts to branch on significant variables first, quickly improving lower bounds. Pseudocost branching estimates significance based on historical information; however, this approach might not be accurate for future search.

STRONG chooses as the branching variable the variable x_i such that i maximizes the estimated improvement in the objective value. Strong branching first generates a list of candidates, then branches on each candidate and records the improvement in the objective value. The candidate with the largest improvement is chosen as the branching variable. Strong branching can be effective for combinatorial problems, but it is usually computationally expensive.

Branching Priorities

In some cases, it is possible to speed up the branch-and-bound algorithm by branching on variables in a specific order. You can accomplish this in PROC OPTMODEL by attaching branching priorities to the integer variables in your model by using the .priority suffix. More information about this suffix is available in the section "Integer Variable Suffixes." For an example in which branching priorities are used, see Example 9.3 on page 875.

Presolve

The MILP solver in PROC OPTMODEL includes a variety of presolve techniques to reduce problem size, improve numerical stability, and detect infeasibility or unboundedness (Andersen and Andersen 1995; Gondzio 1997). During presolve, redundant constraints and variables are identified and removed. Presolve can further reduce the problem size by substituting variables. Variable substitution is a very effective technique, but it might occasionally increase the number of nonzero entries in the constraint matrix. Presolve might also modify the constraint coefficients to tighten the formulation of the problem.

In most cases, using presolve is very helpful in reducing solution times. You can enable presolve at different levels by specifying the PRESOLVER= option.

Cutting Planes

The feasible region of every linear program forms a *polyhedron*. Every polyhedron in n-space can be written as a finite number of half-spaces (equivalently, inequalities). In our notation, this polyhedron is defined by the set $\mathcal{Q} = \{x \in \mathbb{R}^n \mid Ax \leq b, l \leq x \leq u\}$. After we add the restriction that some variables must be integral, the set of feasible solutions, $\mathcal{F} = \{x \in \mathcal{Q} \mid x_i \in \mathbb{Z}\, \forall i \in \mathcal{S}\}$, no longer forms a polyhedron.

The *convex hull* of a set X is the minimal convex set containing X. In solving a mixed integer linear program, in order to take advantage of LP-based algorithms we want to find the convex hull, conv$(\mathcal{F})$, of $\mathcal{F}$. If we can find conv$(\mathcal{F})$ and describe it compactly, then we can solve a mixed integer linear program with a linear programming solver. This is generally very difficult, so we must be satisfied with finding an approximation. Typically, the better the approximation, the more efficiently the LP-based branch-and-bound algorithm can perform.

As described in the section "The Branch-and-Bound Algorithm" on page 860, the branch-and-bound algorithm begins by solving the linear programming relaxation over the polyhedron $\mathcal{Q}$. Clearly, $\mathcal{Q}$ contains the convex hull of the feasible region of the original integer program; that is, conv$(\mathcal{F}) \subseteq \mathcal{Q}$.

Cutting plane techniques are used to tighten the linear relaxation to better approximate conv$(\mathcal{F})$. Assume we are given a solution $\bar{x}$ to some intermediate linear relaxation during the branch-and-bound algorithm. A cut, or valid inequality ($\pi x \leq \pi^0$), is some half-space with the following characteristics:

- The half-space contains conv$(\mathcal{F})$; that is, every integer feasible solution is feasible for the cut ($\pi x \leq \pi^0, \forall x \in \mathcal{F}$).
- The half-space does not contain the current solution $\bar{x}$; that is, $\bar{x}$ is not feasible for the cut ($\pi \bar{x} > \pi^0$).

Cutting planes were first made popular by Dantzig, Fulkerson, and Johnson (1954) in their work on the traveling salesman problem. The two major classifications of cutting planes are *generic cuts* and *structured cuts*. The first class of cuts is based solely on algebraic arguments and can be applied to any relaxation of any integer program. The second class of cuts is specific to certain structures that can be found in some relaxations of the mixed integer linear program. These structures are automatically discovered during the cut initialization phase of the MILP solver. Table 9.9 lists the various types of cutting planes that are built into the MILP solver. Included in each type are algorithms for numerous variations based on different relaxations and lifting techniques. For a survey of cutting plane techniques for mixed integer programming, see Marchand et al. (1999). For a survey of lifting techniques, see Atamturk (2004).

Table 9.9. Cutting Planes in the MILP Solver

Generic Cutting Planes	Structured Cutting Planes
Gomory Mixed Integer	Cliques
Lift-and-Project	Flow Cover

Table 9.9. (continued)

Generic Cutting Planes	Structured Cutting Planes
Mixed Integer Rounding	Flow Path
	Generalized Upper Bound Cover
	Implied Bound
	Knapsack Cover

You can set levels for individual cuts by using the CUTCLIQUE=, CUTFLOWCOVER=, CUTFLOWPATH=, CUTGOMORY=, CUTGUB=, CUTIMPLIED=, CUTKNAPSACK=, CUTLAP=, and CUTMIR= options.

The valid levels for these options are listed in Table 9.8.

The cut level determines the internal strategy used by the MILP solver for generating the cutting planes. The strategy consists of several factors, including how frequently the cut search is called, the number of cuts allowed, and the aggressiveness of the search algorithms.

Sophisticated cutting planes, such as those included in the MILP solver, can take a great deal of CPU time. Typically the additional tightening of the relaxation helps to speed up the overall process, because it provides better bounds for the branch-and-bound tree and helps guide the LP solver toward integer solutions. In rare cases shutting off cutting planes completely might lead to faster overall run times.

The default settings of the MILP solver have been tuned to work well for most instances. However, problem-specific expertise might suggest adjusting one or more of the strategies. These options give you that flexibility.

Primal Heuristics

Primal heuristics, an important component of the MILP solver in PROC OPTMODEL, are applied during the branch-and-bound algorithm. They are used to find integer feasible solutions early in the search tree, thereby improving the upper bound for a minimization problem. Primal heuristics play a role complementary to cutting planes in reducing the gap between the upper and lower bounds, thus reducing the size of the branch-and-bound tree.

Applying primal heuristics in the branch-and-bound algorithm assists in the following areas:

- finding a good upper bound early in the tree search; this can lead to earlier fathoming, resulting in fewer subproblems to be processed.
- locating a reasonably good feasible solution when that is sufficient; sometimes a good feasible solution is the best the solver can produce within certain time or resource limits.

- tightening bounds on integer variables by using reduced cost fixing.

The MILP solver implements several heuristic methodologies. Some algorithms, such as rounding and iterative rounding (diving) heuristics, attempt to construct an integer feasible solution by using fractional solutions to the continuous relaxation at each node of the branch-and-cut tree. Other algorithms start with an incumbent solution and attempt to find a better solution within a neighborhood of the current best solution.

The HEURISTICS= option enables you to control the level of primal heuristics applied by the MILP solver. This level determines how frequently primal heuristics are applied during the tree search. Some expensive heuristics might be disabled by the solver at less aggressive levels. Setting the HEURISTICS= option to a lower level also reduces the maximum number of iterations allowed in iterative heuristics.

The valid values for this option are listed in Table 9.5.

Node Log

The following information about the status of the branch-and-bound algorithm is printed in the node log:

Node	indicates the sequence number of the current node in the search tree.
Active	indicates the current number of active nodes in the branch-and-bound tree.
Sols	indicates the number of feasible solutions found so far.
BestInteger	indicates the best upper bound (assuming minimization) found so far.
BestBound	indicates the best lower bound (assuming minimization) found so far.
Gap	indicates the relative gap between BestInteger and BestBound, displayed as a percentage. If the relative gap is larger than 1000, then the absolute gap is displayed. If there are no active nodes remaining, the value of Gap is 0.
Time	indicates the elapsed real time.

The PRINTFREQ= option can be used to control the amount of information printed in the node log. By default a new entry is included in the log at the first node, at the last node, and at 100-node intervals. A new entry is also included each time a better integer solution is found. The PRINTFREQ= option enables you to change the interval between entries in the node log. Figure 9.3 shows a sample node log.

```
NOTE: The problem has 10 variables (0 free, 0 fixed).
NOTE: The problem has 0 binary and 10 integer variables.
NOTE: The problem has 2 linear constraints (2 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 20 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 10 variables, 2 constraints, and 20 constraint
      coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active     Sols    BestInteger      BestBound      Gap     Time
             0       1        1              0     88.0955497  100.00%        0
             0       1        2     83.0000000     88.0626822    5.75%        0
             0       1        2     83.0000000     87.9665871    5.65%        0
             0       1        2     83.0000000     87.9660825    5.65%        0
             0       1        3     85.0000000     87.9331742    3.34%        0
             0       1        3     85.0000000     87.9140538    3.31%        0
NOTE: OPTMILP added 3 cuts with 30 cut coefficients at the root.
             5       2        4     86.0000000     87.6821242    1.92%        0
             8       3        5     87.0000000     87.6821242    0.78%        0
NOTE: Optimal.
NOTE: Objective = 87.
NOTE: PROCEDURE OPTMODEL used (Total process time):
      real time           0.03 seconds
      cpu time            0.03 seconds
```

Figure 9.3. Sample Node Log

Examples

This section contains examples intended to illustrate the options and syntax of the MILP solver in PROC OPTMODEL. Example 9.1 illustrates the use of PROC OPTMODEL to solve an employee scheduling problem. Example 9.2 discusses a multicommodity transshipment problem with fixed charges. Example 9.3 demonstrates how to warm start PROC OPTMILP. Example 9.4 shows the solution of an instance of the traveling salesman problem in PROC OPTMODEL. Other examples of mixed integer linear programs, along with example SAS code, are given in Chapter 16.

Example 9.1. Scheduling

The following application has been adapted from Example 3.13.

Scheduling is a common application area where mixed integer linear programming techniques are used. In this example, you have eight one-hour time slots in each of five days. You have to assign four employees to these time slots so that each slot is covered on every day. You allow the employees to specify preference data for each slot on each day. In addition, the following constraints must be satisfied:

- Each employee has some time slots for which he or she is unavailable (OneEmpPerSlot).
- Each employee must have either time slot 4 or time slot 5 off for lunch (EmpMustHaveLunch).
- Each employee can work at most two time slots in a row (AtMost2ConSlots).
- Each employee can work only a specified number of hours in the week (WeeklyHoursLimit).

To formulate this problem, let i denote a person, j denote a time slot, and k denote a day. Then, let $x_{ijk} = 1$ if person i is assigned to time slot j on day k, and 0 otherwise. Let p_{ijk} denote the preference of person i for slot j on day k. Let h_i denote the number of hours in a week that person i will work. The formulation of this problem follows:

$$
\begin{array}{lllll}
\max & \displaystyle\sum_{ijk} p_{ijk} x_{ijk} & & & \\
\text{s.t.} & \displaystyle\sum_{i} x_{ijk} = 1 & \forall j,k & & \text{(OneEmpPerSlot)} \\
& x_{i4k} + x_{i5k} \le 1 & \forall i,k & & \text{(EmpMustHaveLunch)} \\
& x_{i,\ell,k} + x_{i,\ell+1,k} + x_{i,\ell+2,k} \le 2 & \forall i,k, \text{ and } l \le 6 & & \text{(AtMost2ConSlots)} \\
& \displaystyle\sum_{jk} x_{ijk} \le h_i & \forall i & & \text{(WeeklyHoursLimit)} \\
& x_{ijk} = 0 & \forall i,j,k \text{ s.t. } p_{ijk} > 0 & & \\
& x_{ijk} \in \{0,1\} & \forall i,j,k & &
\end{array}
$$

The following data set preferences gives the preferences for each individual, time slot, and day. A 10 represents the most desirable time slot, and a 1 represents the least desirable time slot. In addition, a 0 indicates that the time slot is not available. The data set maxhours gives the maximum number of hours each employee can work per week.

```
data preferences;
   input name $ slot mon tue wed thu fri;
   datalines;
marc  1    10 10 10 10 10
marc  2     9  9  9  9  9
marc  3     8  8  8  8  8
marc  4     1  1  1  1  1
marc  5     1  1  1  1  1
marc  6     1  1  1  1  1
marc  7     1  1  1  1  1
marc  8     1  1  1  1  1
mike  1    10  9  8  7  6
mike  2    10  9  8  7  6
mike  3    10  9  8  7  6
mike  4    10  3  3  3  3
mike  5     1  1  1  1  1
mike  6     1  2  3  4  5
mike  7     1  2  3  4  5
mike  8     1  2  3  4  5
bill  1    10 10 10 10 10
bill  2     9  9  9  9  9
bill  3     8  8  8  8  8
bill  4     0  0  0  0  0
bill  5     1  1  1  1  1
bill  6     1  1  1  1  1
bill  7     1  1  1  1  1
bill  8     1  1  1  1  1
bob   1    10  9  8  7  6
bob   2    10  9  8  7  6
bob   3    10  9  8  7  6
bob   4    10  3  3  3  3
bob   5     1  1  1  1  1
bob   6     1  2  3  4  5
bob   7     1  2  3  4  5
bob   8     1  2  3  4  5
;

data maxhours;
   input name $ hour;
   datalines;
marc  20
mike  20
bill  20
bob   20
;
```

Using PROC OPTMODEL, you can model and solve the scheduling problem as follows.

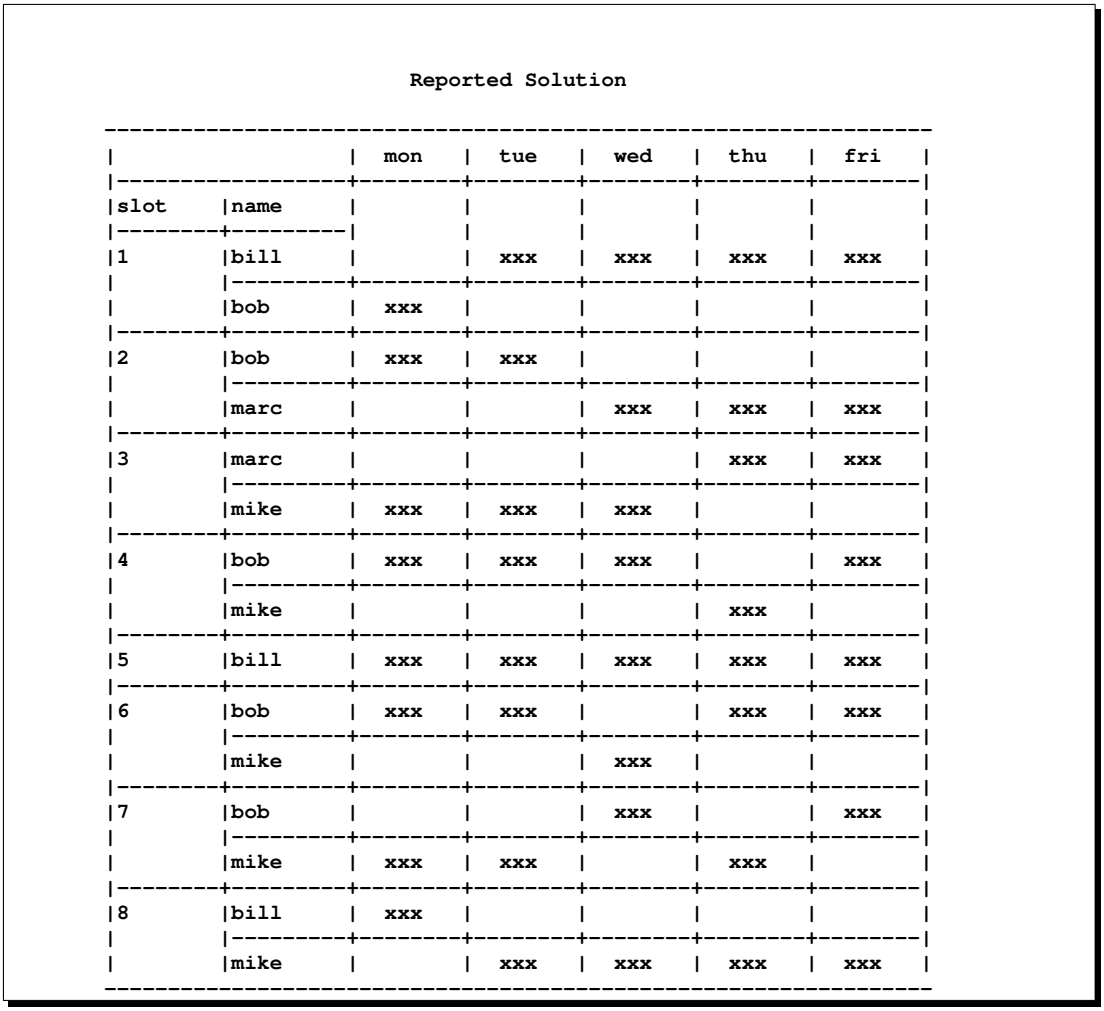
```
proc optmodel presolver=none;

   /* read in the preferences and max hours from the data sets */
   set <string,num> DailyEmployeeSlots;
   set <string>     Employees;

   set <num>    TimeSlots = (setof {<name,slot> in DailyEmployeeSlots} slot);
   set <string> WeekDays  = {"mon","tue","wed","thu","fri"};

   num WeeklyMaxHours{Employees};
   num PreferenceWeights{DailyEmployeeSlots,Weekdays};
   num NSlots = card(TimeSlots);

   read data preferences into DailyEmployeeSlots=[name slot]
        {day in Weekdays} <PreferenceWeights[name,slot,day] = col(day)>;
   read data maxhours into Employees=[name] WeeklyMaxHours=hour;

   /* declare the binary assignment variable x[i,j,k] */
   var Assign{<name,slot> in DailyEmployeeSlots, day in Weekdays} binary;

   /* for each p[i,j,k] = 0, fix x[i,j,k] = 0 */
   for {<name,slot> in DailyEmployeeSlots, day in Weekdays:
       PreferenceWeights[name,slot,day] = 0}
         fix Assign[name,slot,day] = 0;

   /* declare the objective function */
   max TotalPreferenceWeight =
      sum{<name,slot> in DailyEmployeeSlots, day in Weekdays}
         PreferenceWeights[name,slot,day] * Assign[name,slot,day];

   /* declare the constraints */
   con OneEmpPerSlot{slot in TimeSlots, day in Weekdays}:
      sum{name in Employees} Assign[name,slot,day] = 1;

   con EmpMustHaveLunch{name in Employees, day in Weekdays}:
      Assign[name,4,day] + Assign[name,5,day] <= 1;

   con AtMost2ConsSlots{name in Employees, start in 1..NSlots-2,
                        day in Weekdays}:
      Assign[name,start,day] + Assign[name,start+1,day]
            + Assign[name,start+2,day] <= 2 ;

   con WeeklyHoursLimit{name in Employees}:
      sum{slot in TimeSlots, day in Weekdays} Assign[name,slot,day]
           <= WeeklyMaxHours[name];

   /* solve the model */
   solve with milp;

   /* clean up the solution */
   for {<name,slot> in DailyEmployeeSlots, day in Weekdays}
      Assign[name,slot,day] = round(Assign[name,slot,day],1e-6);

   create data report from [name slot]={<name,slot> in DailyEmployeeSlots:
      max {day in Weekdays} Assign[name,slot,day] > 0}
         {day in Weekdays} <col(day)=(if Assign[name,slot,day] > 0
         then Assign[name,slot,day] else .)>;
quit;
```

The following code demonstrates how to use the TABULATE procedure to display a schedule showing how the eight time slots are covered for the week.

```
title 'Reported Solution';
proc format;
   value xfmt 1='  xxx   ';
run;
proc tabulate data=report;
   class name slot;
   var mon--fri;
   table (slot * name), (mon tue wed thu fri)*sum=' '*f=xfmt.
     /misstext=' ';
run;
```

The output from the preceding code is displayed in Output 9.1.1.

Output 9.1.1. Scheduling Reported Solution



Reported Solution

		mon	tue	wed	thu	fri
slot	name					
1	bill		xxx	xxx	xxx	xxx
	bob	xxx				
2	bob	xxx	xxx			
	marc			xxx	xxx	xxx
3	marc				xxx	xxx
	mike	xxx	xxx	xxx		
4	bob	xxx	xxx	xxx		xxx
	mike				xxx	
5	bill	xxx	xxx	xxx	xxx	xxx
6	bob	xxx	xxx		xxx	xxx
	mike			xxx		
7	bob			xxx		xxx
	mike	xxx	xxx		xxx	
8	bill	xxx				
	mike		xxx	xxx	xxx	xxx

Example 9.2. Multicommodity Transshipment Problem with Fixed Charges

The following application has been adapted from Example 3.14.

The following example illustrates the use of PROC OPTMODEL to generate a mixed integer linear program to solve a multicommodity network flow model with fixed charges. Consider a network with nodes N, arcs A, and a set C of commodities to be shipped between the nodes. There is a variable shipping cost s_{ijc} for each of the four commodities c across each of the arcs (i, j). In addition, there is a fixed charge f_{ij} for the use of each arc (i, j). The shipping costs and fixed charges are defined in the data set arcdata, as follows:

```
data arcdata;
   array c  c1-c4;
   input from $ to $ c1 c2 c3 c4 fx;
   datalines;
farm-a  Chicago 20 15 17 22 100
farm-b  Chicago 15 15 15 30  75
farm-c  Chicago 30 30 10 10 100
farm-a  StLouis 30 25 27 22 150
farm-c  StLouis 10  9 11 10  75
Chicago NY      75 75 75 75 200
StLouis NY      80 80 80 80 200
;
run;
```

The supply (positive numbers) at each of the nodes and the demand (negative numbers) at each of the nodes d_{ic} for each commodity c are shown in the data set nodedata, as follows:

```
data nodedata;
   array sd sd1-sd4;
   input node $ sd1 sd2 sd3 sd4;
   datalines;
farm-a  100  100  40   .
farm-b  100  200  50  50
farm-c   40  100  75 100
NY     -150 -200 -50 -75
;
run;
```

Let x_{ijc} define the flow of commodity c across arc (i, j). Let $y_{ij} = 1$ if arc (i, j) is used, and 0 otherwise. Since the total flow on an arc (i, j) must be less than the total demand across all nodes $k \in N$, we can define the trivial upper bound u_{ijc} as

$$x_{ijc} \leq u_{ijc} = \sum_{k \in N \mid d_{kc} < 0} (-d_{kc})$$

This model can be represented using the following mixed integer linear program:

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A}\sum_{c\in C} s_{ijc}x_{ijc} + \sum_{(i,j)\in A} f_{ij}y_{ij} & & & & \\
\text{s.t.} \quad & \sum_{j\in N\,|\,(i,j)\in A} x_{ijc} - \sum_{j\in N\,|\,(j,i)\in A} x_{jic} & \leq & d_{ic} & \forall i \in N, c \in C & \text{(balance\_con)} \\
& x_{ijc} & \leq & u_{ijc}y_{ij} & \forall (i,j) \in A, c \in C & \text{(fixed\_charge\_con)} \\
& x_{ijc} & \geq & 0 & \forall (i,j) \in A, c \in C & \\
& y_{ij} \in \{0,1\} & & & \forall (i,j) \in A &
\end{aligned}
$$

Constraint (balance_con) ensures conservation of flow for both supply and demand. Constraint (fixed_charge_con) models the fixed charge cost by forcing $y_{ij} = 1$ if $x_{ijc} > 0$ for any commodity $c \in C$.

The PROC OPTMODEL code follows.

```
proc optmodel presolver=none;
   set COMMODITIES = 1..4;
   set <str,str> ARCS;
   set <str> NODES = (setof {<i,j> in ARCS} i)
       union (setof {<i,j> in ARCS} j);

   num shipping_cost {ARCS, COMMODITIES};
   num fixed_charge {ARCS};
   num supply_demand {NODES, COMMODITIES} init 0;
   num upper_bound {ARCS, comm in COMMODITIES} init
       sum {i in NODES: supply_demand[i,comm] < 0} (-supply_demand[i,comm]);

   read data arcdata into ARCS=[from to] {comm in COMMODITIES}
       <shipping_cost[from,to,comm] = col("c"||comm)> fixed_charge=fx;
   read data nodedata nomiss into [node] {comm in COMMODITIES}
       <supply_demand[node,comm] = col("sd"||comm)>;

   var Flow {<i,j> in ARCS, comm in COMMODITIES} >= 0 <= upper_bound[i,j,comm];
   var UseArc {ARCS} binary;

   /* minimize shipping costs plus fixed charges */
   min TotalCost =
      sum {<i,j> in ARCS, comm in COMMODITIES}
          shipping_cost[i,j,comm] * Flow[i,j,comm]
    + sum {<i,j> in ARCS} fixed_charge[i,j] * UseArc[i,j];

   /* flow balance constraints: outflow - inflow <= supply_demand */
   con balance_con {i in NODES, comm in COMMODITIES}:
       sum {j in NODES: <i,j> in ARCS} Flow[i,j,comm]
     - sum {j in NODES: <j,i> in ARCS} Flow[j,i,comm]
    <= supply_demand[i,comm];

   /* fixed charge constraints: if Flow > 0 for some commodity then UseArc = 1 */
   con fixed_charge_con {<i,j> in ARCS, comm in COMMODITIES}:
       Flow[i,j,comm] <= upper_bound[i,j,comm] * UseArc[i,j];
```

```
   solve with milp;
   print {<i,j> in ARCS, comm in COMMODITIES: Flow[i,j,comm] > 1.0e-5} Flow;
   for {<i,j> in ARCS} UseArc[i,j] = round(UseArc[i,j].sol);
   print UseArc;
quit;
```

The solution summary, as well as the output from the two PRINT statements, are shown in Output 9.2.1.

Output 9.2.1. Multicommodity Transshipment Problem with Fixed Charges Solution Summary

The OPTMODEL Procedure

Solution Summary

Solver	MILP
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	42825
Iterations	29
Nodes	1
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.409463E-16
Bound Infeasibility	7.047314E-19
Integer Infeasibility	9E-8

[1]	[2]	[3]	Flow
Chicago	NY	1	110
Chicago	NY	2	100
Chicago	NY	3	50
Chicago	NY	4	75
StLouis	NY	1	40
StLouis	NY	2	100
farm-a	Chicago	1	10
farm-a	Chicago	2	100
farm-b	Chicago	1	100
farm-c	Chicago	3	50
farm-c	Chicago	4	75
farm-c	StLouis	1	40
farm-c	StLouis	2	100

[1]	[2]	Use Arc
Chicago	NY	1
StLouis	NY	1
farm-a	Chicago	1
farm-a	StLouis	0
farm-b	Chicago	1
farm-c	Chicago	1
farm-c	StLouis	1

Example 9.3. Facility Location

Consider the classic facility location problem. Given a set L of customer locations and a set F of candidate facility sites, you must decide which sites to build facilities on and assign coverage of customer demand to these sites so as to minimize cost. All customer demand d_i must be satisfied, and each facility has a demand capacity limit C. The total cost is the sum of the distances c_{ij} between facility j and its assigned customer i, plus a fixed charge f_j for building a facility at site j. Let $y_j = 1$ represent choosing site j to build a facility, and 0 otherwise. Also, let $x_{ij} = 1$ represent the assignment of customer i to facility j, and 0 otherwise. This model can be formulated as the following integer linear program:

$$
\begin{array}{lllll}
\min & \displaystyle\sum_{i\in L}\sum_{j\in F} c_{ij}x_{ij} + \sum_{j\in F} f_j y_j & & & \\
\text{s.t.} & \displaystyle\sum_{j\in F} x_{ij} = 1 & \forall i \in L & \text{(assign\_def)} \\
& x_{ij} \le y_j & \forall i \in L, j \in F & \text{(link)} \\
& \displaystyle\sum_{i\in L} d_i x_{ij} \le C y_j & \forall j \in F & \text{(capacity)} \\
& x_{ij} \in \{0,1\} & \forall i \in L, j \in F & \\
& y_j \in \{0,1\} & \forall j \in F &
\end{array}
$$

Constraint (assign_def) ensures that each customer is assigned to exactly one site. Constraint (link) forces a facility to be built if any customer has been assigned to that facility. Finally, constraint (capacity) enforces the capacity limit at each site.

Let us also consider a variation of this same problem where there is no cost for building a facility. This problem is typically easier to solve than the original problem. For this variant, let the objective be

$$
\min \quad \sum_{i\in L}\sum_{j\in F} c_{ij}x_{ij}
$$

First, let us construct a random instance of this problem by using the following DATA steps:

```
%let NumCustomers  = 50;
%let NumSites      = 10;
%let SiteCapacity  = 35;
%let MaxDemand     = 10;
%let xmax          = 200;
%let ymax          = 100;
%let seed          = 1234;

/* generate random customer locations */
```

```
data cdata(drop=i);
   length name $8;
   do i = 1 to &NumCustomers;
      name = compress('C'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      demand = ranuni(&seed) * &MaxDemand;
      output;
   end;
run;

/* generate random site locations and fixed charge */
data sdata(drop=i);
   length name $8;
   do i = 1 to &NumSites;
      name = compress('SITE'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      fixed_charge = (abs(&xmax/2-x) + abs(&ymax/2-y)) / 2;
      output;
   end;
run;
```

In the following PROC OPTMODEL code, we first generate the model with the fixed-charge variant of the cost function and solve the problem for the fixed-charge model. Next, we remove the fixed costs from the objective function and solve the problem for the model with no fixed charge. Finally, we solve the fixed-charge model again. Note that the solution to the model with no fixed charge is feasible for the fixed-charge model and should provide a good starting point for the MILP solver. We use the PRIMALIN option to provide an incumbent solution (warm start).

```
proc optmodel;
   set <str> CUSTOMERS;
   set <str> SITES;

   /* x and y coordinates of CUSTOMERS and SITES */
   num x {CUSTOMERS union SITES};
   num y {CUSTOMERS union SITES};
   num demand {CUSTOMERS};
   num fixed_charge {SITES};

   /* distance from customer i to site j */
   num dist {i in CUSTOMERS, j in SITES}
       = sqrt((x[i] - x[j])^2 + (y[i] - y[j])^2);

   read data cdata into CUSTOMERS=[name] x y demand;
   read data sdata into SITES=[name] x y fixed_charge;

   var Assign {CUSTOMERS, SITES} binary;
   var Build {SITES} binary;

   min CostNoFixedCharge
       = sum {i in CUSTOMERS, j in SITES} dist[i,j] * Assign[i,j];
   min CostFixedCharge
       = CostNoFixedCharge + sum {j in SITES} fixed_charge[j] * Build[j];
```

```
   /* each customer assigned to exactly one site */
   con assign_def {i in CUSTOMERS}:
      sum {j in SITES} Assign[i,j] = 1;

   /* if customer i assigned to site j, then facility must be built at j */
   con link {i in CUSTOMERS, j in SITES}:
       Assign[i,j] <= Build[j];

   /* each site can handle at most &SiteCapacity demand */
   con capacity {j in SITES}:
       sum {i in CUSTOMERS} demand[i] * Assign[i,j] <= &SiteCapacity * Build[j];

   /* solve the MILP with fixed charges */
   solve obj CostFixedCharge;

   /* solve the MILP with no fixed charges */
   solve obj CostNoFixedCharge;

   /* clean up the solution */
   for {i in CUSTOMERS, j in SITES} Assign[i,j] = round(Assign[i,j]);
   for {j in SITES} Build[j] = round(Build[j]);

   call symput('varcostNo',put(CostNoFixedCharge,6.1));

   /* create a data set for use by GPLOT */
   create data CostNoFixedCharge_Data from
       [customer site]={i in CUSTOMERS, j in SITES: Assign[i,j] = 1}
       xi=x[i] yi=y[i] xj=x[j] yj=y[j];

   /* solve the MILP, with fixed charges with warm start */
   solve obj CostFixedCharge with milp / primalin;

   /* clean up the solution */
   for {i in CUSTOMERS, j in SITES} Assign[i,j] = round(Assign[i,j]);
   for {j in SITES} Build[j] = round(Build[j]);

   num varcost = sum {i in CUSTOMERS, j in SITES} dist[i,j] * Assign[i,j].sol;
   num fixcost = sum {j in SITES} fixed_charge[j] * Build[j].sol;
   call symput('varcost', put(varcost,6.1));
   call symput('fixcost', put(fixcost,5.1));
   call symput('totalcost', put(CostFixedCharge,6.1));

   /* create a data set for use by GPLOT */
   create data CostFixedCharge_Data from
       [customer site]={i in CUSTOMERS, j in SITES: Assign[i,j] = 1}
       xi=x[i] yi=y[i] xj=x[j] yj=y[j];
quit;
```

The information printed in the log for the fixed-charge model is displayed in Output 9.3.1.

Output 9.3.1. OPTMODEL Log for Facility Location with Fixed Charges

```
NOTE: There were 50 observations read from the data set WORK.CDATA.
NOTE: There were 10 observations read from the data set WORK.SDATA.
NOTE: The problem has 510 variables (0 free, 0 fixed).
NOTE: The problem has 510 binary and 0 integer variables.
NOTE: The problem has 560 linear constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .   1870.6874826        .       0
             0       1       0              .   1887.7583984        .       0
             0       1       2   2030.5371397   1892.7586646    7.28%       0
             0       1       5   1921.6837559   1895.6168522    1.38%       0
             0       1       5   1921.6837559   1896.3025743    1.34%       0
             0       1       5   1921.6837559   1898.2934630    1.23%       0
             0       1       5   1921.6837559   1898.3693562    1.23%       0
             0       1       5   1921.6837559   1898.4146301    1.23%       0
NOTE: OPTMILP added 19 cuts with 1970 cut coefficients at the root.
            15      16       7   1911.7680097   1898.4146301    0.70%       1
            17      18       8   1911.7679632   1898.4146301    0.70%       1
           100      72       8   1911.7679632   1902.2341995    0.50%       3
           200     149       8   1911.7679632   1902.6557849    0.48%       4
           300     201       8   1911.7679632   1902.8809984    0.47%       5
           321     202       9   1911.5023765   1903.0622409    0.44%       6
           322     202      10   1911.5023718   1903.0622409    0.44%       6
           400     257      10   1911.5023718   1903.2157397    0.44%       7
           500     331      10   1911.5023718   1903.4391858    0.42%       8
           600     374      10   1911.5023718   1904.0767221    0.39%       9
           653     381      11   1910.9293121   1904.6205473    0.33%      10
           654     382      12   1910.9293120   1904.6205473    0.33%      10
           655     382      13   1910.9293116   1904.6205473    0.33%      10
           700     396      13   1910.9293116   1904.7170348    0.33%      11
           800     406      13   1910.9293116   1905.3160494    0.29%      14
           900     413      13   1910.9293116   1905.5409424    0.28%      16
           984     353      14   1908.7057490   1905.8021302    0.15%      18
           985     354      15   1908.7057291   1905.8021302    0.15%      18
          1000     342      16   1908.7057291   1905.8999397    0.15%      19
          1100     304      16   1908.7057291   1906.2754335    0.13%      22
          1200     240      16   1908.7057291   1906.6943952    0.11%      26
          1300     198      16   1908.7057291   1907.0531758    0.09%      30
          1400     114      16   1908.7057291   1907.7219764    0.05%      34
          1500      22      16   1908.7057291   1908.4264757    0.01%      39
          1510      12      16   1908.7057291   1908.5245060    0.01%      39
NOTE: Optimal within relative gap.
NOTE: Objective = 1908.7057291.
```

The information printed in the log for the no-fixed-charge model is displayed in Output 9.3.2.

Output 9.3.2. OPTMODEL Log for Facility Location with No Fixed Charges

```
NOTE: The problem has 510 variables (0 free, 0 fixed).
NOTE: The problem has 510 binary and 0 integer variables.
NOTE: The problem has 560 linear constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 10 variables and 500 constraints.
NOTE: The OPTMILP presolver removed 1010 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 500 variables, 60 constraints, and 1000
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .   1520.1872420        .       0
             0       1       0              .   1539.2006720        .       0
             0       1       0              .   1544.8618998        .       0
             0       1       2   1568.4968729   1545.8084732    1.47%       0
             0       1       2   1568.4968729   1547.5231740    1.36%       0
             0       1       2   1568.4968729   1548.8648329    1.27%       0
             0       1       3   1563.3984995   1549.2087898    0.92%       0
             0       1       3   1563.3984995   1549.2087915    0.92%       0
NOTE: OPTMILP added 14 cuts with 1015 cut coefficients at the root.
            12      10       4   1563.3719799   1549.2087915    0.91%       0
            14       9       5   1554.1656132   1551.0382988    0.20%       0
            19      11       6   1554.0107314   1551.0382988    0.19%       0
            30       1       7   1553.8285559   1553.1324191    0.04%       1
NOTE: Optimal.
NOTE: Objective = 1553.8285559.
```

The results from the warm start approach are shown in Output 9.3.3. The number of nodes is smaller than in the direct solve; in this case, warm starting the MILP solver results in computational savings.

Output 9.3.3. OPTMODEL Log for Facility Location with Fixed Charges, Using Warm Start

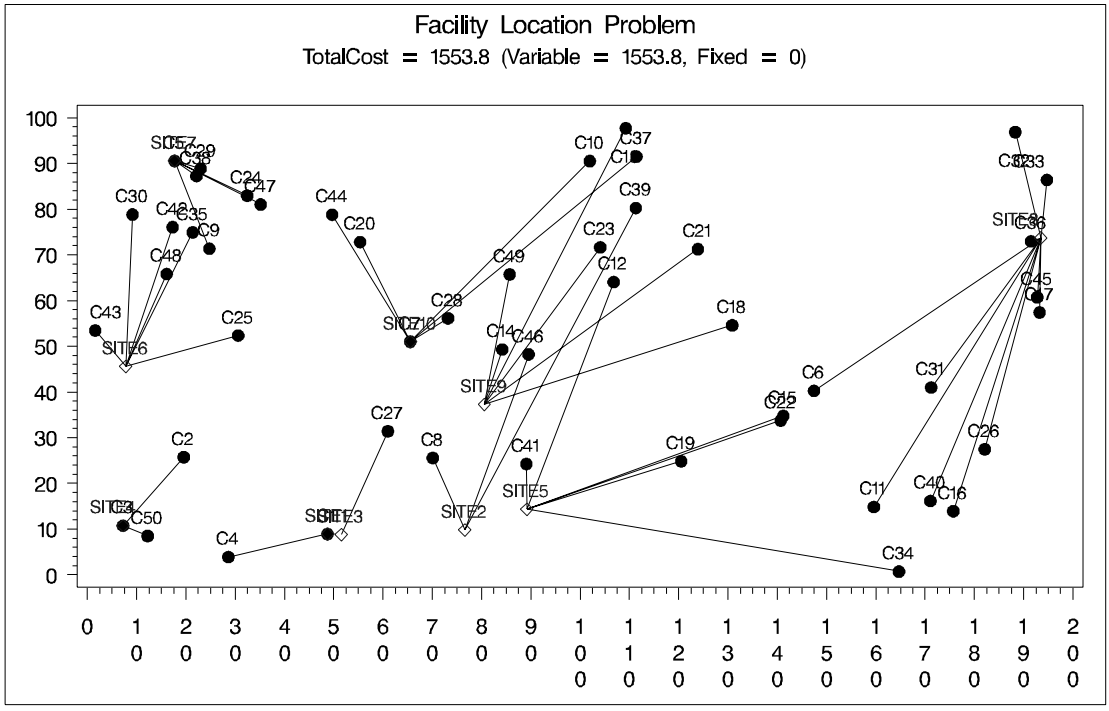
```
NOTE: The data set WORK.COSTNOFIXEDCHARGE_DATA has 50 observations and 6
      variables.
NOTE: The problem has 510 variables (0 free, 0 fixed).
NOTE: The problem has 510 binary and 0 integer variables.
NOTE: The problem has 560 linear constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
           Node  Active    Sols    BestInteger      BestBound      Gap    Time
              0       1       1   1967.9350441   1870.6874826    5.20%       0
              0       1       1   1967.9350441   1887.7584203    4.25%       0
              0       1       1   1967.9350441   1892.7586866    3.97%       0
              0       1       4   1921.6837648   1895.0700047    1.40%       0
              0       1       4   1921.6837648   1895.9244295    1.36%       0
              0       1       4   1921.6837648   1898.8816348    1.20%       0
              0       1       4   1921.6837648   1899.0762206    1.19%       0
              0       1       4   1921.6837648   1899.1900845    1.18%       0
              0       1       4   1921.6837648   1899.1953627    1.18%       0
NOTE: OPTMILP added 14 cuts with 1095 cut coefficients at the root.
             11      12       6   1915.1518488   1899.1953627    0.84%       1
             13      14       7   1914.2727588   1899.1953627    0.79%       1
             19      19       8   1913.5664405   1899.1953627    0.76%       1
             97      71       9   1912.0491494   1903.1016611    0.47%       3
            100      74       9   1912.0491494   1903.1016611    0.47%       3
            148     102      10   1911.7679486   1904.4133622    0.39%       4
            150     104      11   1911.7679302   1904.4133622    0.39%       4
            200     111      11   1911.7679302   1905.5157753    0.33%       6
            211      58      12   1908.7057172   1905.6621305    0.16%       6
            300      13      12   1908.7057172   1908.2118351    0.03%      11
            309       8      12   1908.7057172   1908.5557179    0.01%      11
NOTE: Optimal within relative gap.
NOTE: Objective = 1908.7057172.
```

The following two SAS programs produce a plot of the solutions for both variants of the model, using data sets produced by PROC OPTMODEL.

```
title1 "Facility Location Problem";
title2 "TotalCost = &varcostNo (Variable = &varcostNo, Fixed = 0)";
data csdata;
    set cdata(rename=(y=cy)) sdata(rename=(y=sy));
run;
/* create Annotate data set to draw line between customer and assigned site */
%annomac;
data anno(drop=xi yi xj yj);
   %SYSTEM(2, 2, 2);
   set CostNoFixedCharge_Data(keep=xi yi xj yj);
   %LINE(xi, yi, xj, yj, *, 1, 1);
run;
proc gplot data=csdata anno=anno;
   axis1 label=none order=(0 to &xmax by 10);
   axis2 label=none order=(0 to &ymax by 10);
   symbol1 value=dot interpol=none
      pointlabel=("#name" height=0.7) cv=black;
   symbol2 value=diamond interpol=none
      pointlabel=("#name" color=blue height=0.7) cv=blue;
   plot cy*x sy*x / overlay haxis=axis1 vaxis=axis2;
run;
quit;
```

The output of this program is shown in Output 9.3.4.

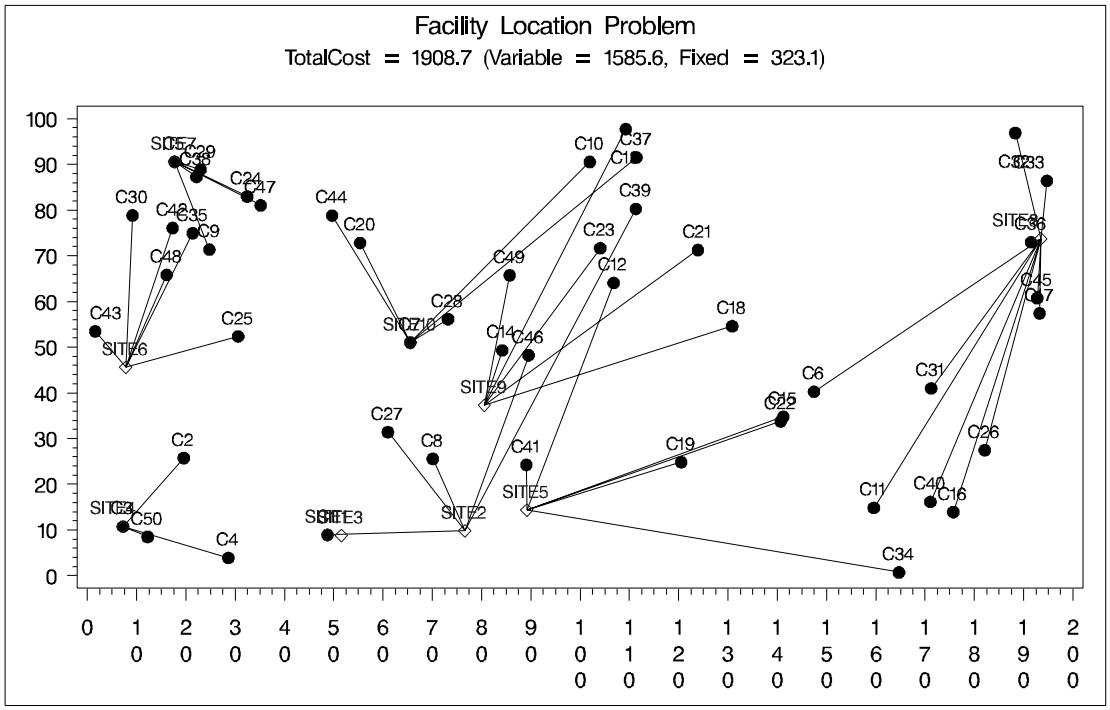
Output 9.3.4. Solution Plot for Facility Location with No Fixed Charges



```
title1 "Facility Location Problem";
title2 "TotalCost = &totalcost (Variable = &varcost, Fixed = &fixcost)";
/* create Annotate data set to draw line between customer and assigned site */
data anno(drop=xi yi xj yj);
   %SYSTEM(2, 2, 2);
   set CostFixedCharge_Data(keep=xi yi xj yj);
   %LINE(xi, yi, xj, yj, *, 1, 1);
run;
proc gplot data=csdata anno=anno;
   axis1 label=none order=(0 to &xmax by 10);
   axis2 label=none order=(0 to &ymax by 10);
   symbol1 value=dot interpol=none
      pointlabel=("#name" height=0.7) cv=black;
   symbol2 value=diamond interpol=none
      pointlabel=("#name" color=blue height=0.7) cv=blue;
   plot cy*x sy*x / overlay haxis=axis1 vaxis=axis2;
run;
quit;
```

The output of this program is shown in Output 9.3.5.

Output 9.3.5. Solution Plot for Facility Location with Fixed Charges



The economic trade-off for the fixed-charge model forces us to build fewer sites and push more demand to each site.

It is possible to expedite the solution of the fixed-charge facility location problem by choosing appropriate branching priorities for the decision variables. Recall that for each site j, the value of the variable y_j determines whether or not a facility is built on that site. Suppose you decide to branch on the variables y_j before the variables x_{ij}. You can set a higher branching priority for y_j by using the .priority suffix for the Build variables in PROC OPTMODEL, as follows:

```
for{j in SITES} Build[j].priority=10;
```

Setting higher branching priorities for certain variables is not guaranteed to speed up the MILP solver, but it can be helpful in some instances. The following program creates and solves an instance of the facility location problem in which giving higher priority to y_j causes the MILP solver to find the optimal solution more quickly. We use the PRINTFREQ= option to abbreviate the node log.

```
%let NumCustomers  = 50;
%let NumSites      = 10;
%let SiteCapacity  = 35;
%let MaxDemand     = 10;
%let xmax          = 200;
%let ymax          = 100;
%let seed          = 2345;

/* generate random customer locations */
data cdata(drop=i);
   length name $8;
   do i = 1 to &NumCustomers;
      name = compress('C'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      demand = ranuni(&seed) * &MaxDemand;
      output;
   end;
run;

/* generate random site locations and fixed charge */
data sdata(drop=i);
   length name $8;
   do i = 1 to &NumSites;
      name = compress('SITE'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      fixed_charge = (abs(&xmax/2-x) + abs(&ymax/2-y)) / 2;
      output;
   end;
run;
```

```
proc optmodel;

   set <str> CUSTOMERS;
   set <str> SITES;

   /* x and y coordinates of CUSTOMERS and SITES */
   num x {CUSTOMERS union SITES};
   num y {CUSTOMERS union SITES};
   num demand {CUSTOMERS};
   num fixed_charge {SITES};

   /* distance from customer i to site j */
   num dist {i in CUSTOMERS, j in SITES}
       = sqrt((x[i] - x[j])^2 + (y[i] - y[j])^2);

   read data cdata into CUSTOMERS=[name] x y demand;
   read data sdata into SITES=[name] x y fixed_charge;

   var Assign {CUSTOMERS, SITES} binary;
   var Build {SITES} binary;

   min CostFixedCharge
       = sum {i in CUSTOMERS, j in SITES} dist[i,j] * Assign[i,j]
         + sum {j in SITES} fixed_charge[j] * Build[j];

   /* each customer assigned to exactly one site */
   con assign_def {i in CUSTOMERS}:
      sum {j in SITES} Assign[i,j] = 1;

   /* if customer i assigned to site j, then facility must be built at j */
   con link {i in CUSTOMERS, j in SITES}:
       Assign[i,j] <= Build[j];

   /* each site can handle at most &SiteCapacity demand */
   con capacity {j in SITES}:
       sum {i in CUSTOMERS} demand[i] * Assign[i,j] <= &SiteCapacity * Build[j];

   /* assign priority to Build variables (y) */
   for{j in SITES} Build[j].priority=10;

   /* solve the MILP with fixed charges, using branching priorities */
   solve obj CostFixedCharge with milp / printfreq=1000;

quit;
```

The resulting output is shown in Output 9.3.6.

Output 9.3.6. PROC OPTMODEL Log for Facility Location with Branching Priorities

```
NOTE: There were 50 observations read from the data set WORK.CDATA.
NOTE: There were 10 observations read from the data set WORK.SDATA.
NOTE: The problem has 510 variables (0 free, 0 fixed).
NOTE: The problem has 510 binary and 0 integer variables.
NOTE: The problem has 560 linear constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
           Node  Active    Sols    BestInteger      BestBound      Gap    Time
              0       1       0              .   1738.1031175        .       0
              0       1       0              .   1758.5003556        .       0
              0       1       2   1866.5681081   1762.3876827    5.91%       0
              0       1       2   1866.5681081   1763.3066050    5.86%       0
              0       1       2   1866.5681081   1763.4070613    5.85%       0
              0       1       2   1866.5681081   1763.4117513    5.85%       0
              0       1       2   1866.5681081   1763.4381384    5.85%       0
              0       1       2   1866.5681081   1763.4395129    5.85%       0
NOTE: OPTMILP added 21 cuts with 1024 cut coefficients at the root.
            719     412       5   1819.5015954   1789.9268442    1.65%      13
            720     374       6   1817.4626728   1789.9268442    1.54%      13
            727     379       7   1817.0482087   1789.9268442    1.52%      13
            728     380       8   1817.0482084   1789.9268442    1.52%      13
            729     373       9   1816.6377022   1789.9268442    1.49%      13
            731     374      10   1816.6377020   1789.9268442    1.49%      13
           1000     568      10   1816.6377020   1791.6912973    1.39%      17
           1433     434      11   1805.1743109   1793.9598843    0.63%      23
           1439     440      12   1805.1743107   1793.9598843    0.63%      23
           1440     440      13   1805.1743106   1793.9598843    0.63%      23
           1578     415      14   1803.1353883   1795.1262957    0.45%      26
           1585     419      15   1803.1353882   1795.1262957    0.45%      26
           2000     363      15   1803.1353882   1798.1139089    0.28%      38
           2543       7      15   1803.1353882   1802.9786948    0.01%      60
NOTE: Optimal within relative gap.
NOTE: Objective = 1803.1353882.
NOTE: PROCEDURE OPTMODEL used (Total process time):
      real time           1:04.40
      cpu time            1:04.28
```

The output in Output 9.3.7 is generated by running the same program without the line that assigns higher branching priorities to the Build variables. We again use the PRINTFREQ= option to abbreviate the node log.

Output 9.3.7. PROC OPTMODEL Log for Facility Location without Branching Priorities

```
NOTE: There were 50 observations read from the data set WORK.CDATA.
NOTE: There were 10 observations read from the data set WORK.SDATA.
NOTE: The problem has 510 variables (0 free, 0 fixed).
NOTE: The problem has 510 binary and 0 integer variables.
NOTE: The problem has 560 linear constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .   1738.1031175        .       0
             0       1       0              .   1758.5003556        .       0
             0       1       2   1866.5681081   1762.3876827    5.91%       0
             0       1       2   1866.5681081   1763.3066050    5.86%       0
             0       1       2   1866.5681081   1763.4070613    5.85%       0
             0       1       2   1866.5681081   1763.4117513    5.85%       0
             0       1       2   1866.5681081   1763.4381384    5.85%       0
             0       1       2   1866.5681081   1763.4395129    5.85%       0
NOTE: OPTMILP added 21 cuts with 1024 cut coefficients at the root.
            62      57       5   1816.9874272   1770.2238747    2.64%       2
            70      65       6   1812.0960927   1770.2238747    2.37%       3
           669     269       7   1809.4577938   1789.3549161    1.12%      15
           673     270       8   1809.4577936   1789.3549161    1.12%      15
           715     258       9   1805.3052821   1789.4199472    0.89%      16
           729     268      10   1805.1743110   1789.4432745    0.88%      16
           730     255      11   1803.1353884   1789.4432745    0.77%      16
          1707     436      12   1803.1353881   1795.4046228    0.43%      38
          3078       4      12   1803.1353881   1802.9717862    0.01%      82
NOTE: Optimal within relative gap.
NOTE: Objective = 1803.1353881.
NOTE: PROCEDURE OPTMODEL used (Total process time):
      real time           1:28.35
      cpu time            1:28.18
```

By comparing Output 9.3.6 and Output 9.3.7 you can see that in this instance, increasing the branching priorities of the Build variables results in computational savings.

Example 9.4. Traveling Salesman Problem

The traveling salesman problem (TSP) is that of finding a minimum cost *tour* in an undirected graph G with vertex set $V = \{1, ..., |V|\}$ and edge set E. A tour is a connected subgraph for which each vertex has degree two. The goal is then to find a tour of minimum total cost, where the total cost is the sum of the costs of the edges in the tour. With each edge $e \in E$ we associate a binary variable x_e, indicating whether edge e is part of the tour, and a cost $c_e \in \mathbb{R}$. Let $\delta(S) = \{\{i, j\} \in E \mid i \in S,\ j \notin S\}$. Then an integer linear programming (ILP) formulation of the TSP is as follows:

$$
\begin{array}{lllll}
\min & \displaystyle\sum_{e \in E} c_e x_e & & & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta(i)} x_e = 2 & \forall i \in V & & (\text{two\_match}) \\
& \displaystyle\sum_{e \in \delta(S)} x_e \geq 2 & \forall S \subset V,\ 2 \leq |S| \leq |V| - 1 & & (\text{subtour\_elim}) \\
& x_e \in \{0, 1\} & \forall e \in E & &
\end{array}
$$

The equations (two_match) are the *matching constraints*, which ensure that each vertex has degree two in the subgraph, while the inequalities (subtour_elim) are known as the *subtour elimination constraints* (SECs) and enforce connectivity.

Since there is an exponential number $O(2^{|V|})$ of SECs, it is impossible to explicitly construct the full TSP formulation for large graphs. An alternative formulation of polynomial size was introduced by Miller, Tucker, and Zemlin (1960):

$$
\begin{array}{lllll}
\min & \displaystyle\sum_{(i,j) \in E} c_{ij} x_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{j \in V} x_{ij} = 1 & \forall i \in V & & (\text{assign\_i}) \\
& \displaystyle\sum_{i \in V} x_{ij} = 1 & \forall j \in V & & (\text{assign\_j}) \\
& u_i - u_j + 1 \leq (|V| - 1)(1 - x_{ij}) & \forall (i, j) \in V, i \neq 1, j \neq 1 & & (\text{mtz}) \\
& 2 \leq u_i \leq |V| & \forall i \in \{2, .., |V|\}, & & \\
& x_{ij} \in \{0, 1\} & \forall (i, j) \in E & &
\end{array}
$$

In this formulation, we use a directed graph. Constraints (assign_i) and (assign_j) now enforce that each vertex has degree two (one edge in, one edge out). The MTZ constraints (mtz) enforce that no subtours exist.

TSPLIB, located at http://elib.zib.de/pub/Packages/mp-testdata/tsp/tsplib/tsplib.html, is a set of benchmark instances for the TSP. The following DATA step converts a TSPLIB instance of type EUC_2D into a SAS data set containing the coordinates of the vertices:

```
/* convert the TSPLIB instance into a data set */
data tspData(drop=H);
   infile "st70.tsp";
   input H $1. @;
   if H not in ('N','T','C','D','E');
   input @1 var1-var3;
run;
```

The following PROC OPTMODEL code attempts to solve the TSPLIB instance st70.tsp by using the MTZ formulation:

```
/* direct solution using the MTZ formulation */
proc optmodel;
   set VERTICES;
   set EDGES = {i in VERTICES, j in VERTICES: i ne j};
   num xc {VERTICES};
   num yc {VERTICES};

   /* read in the instance and customer coordinates (xc, yc) */
   read data tspData into VERTICES=[_n_] xc=var2 yc=var3;

   /* the cost is the euclidean distance rounded to the nearest integer */
   num c {<i,j> in EDGES}
       init floor( sqrt( ((xc[i]-xc[j])**2 + (yc[i]-yc[j])**2)) + 0.5);

   var x {EDGES} binary;
   var u {i in 2..card(VERTICES)} >= 2 <= card(VERTICES);

   /* each vertex has exactly one in-edge and one out-edge */
   con assign_i {i in VERTICES}:
       sum {j in VERTICES: i ne j} x[i,j] = 1;
   con assign_j {j in VERTICES}:
       sum {i in VERTICES: i ne j} x[i,j] = 1;

   /* minimize the total cost */
   min obj
       = sum {<i,j> in EDGES} (if i > j then c[i,j] else c[j,i]) * x[i,j];

   /* no subtours */
   con mtz {<i,j> in EDGES : (i ne 1) and (j ne 1)}:
       u[i] - u[j] + 1 <= (card(VERTICES) - 1) * (1 - x[i,j]);

   solve;
quit;
```

It is well known that the MTZ formulation is much weaker than the subtour formulation. The exponential number of SECs makes it impossible, at least in large instances, to use a direct call to the MILP solver with the subtour formulation. For this reason, if you want to solve the TSP with one SOLVE statement, you must use the MTZ formulation and rely strictly on generic cuts and heuristics. Except for very small instances, this is unlikely to be a good approach.

A much more efficient way to tackle the TSP is to dynamically generate the subtour inequalities as *cuts*. Typically this is done by solving the LP relaxation of the 2-matching problem, finding violated subtour cuts, and adding them iteratively. The

problem of finding violated cuts is known as the *separation problem*. In this case, the separation problem takes the form of a minimum cut problem, which is nontrivial to implement efficiently. Therefore, for the sake of illustration, we solve an integer program at each step of the process.

The initial formulation of the TSP will be the integral 2-matching problem. We solve this by using PROC OPTMODEL to obtain an integral matching, which is not necessarily a tour. In this case, the separation problem is trivial. If the solution is a connected graph, then it is a tour, so we have solved the problem. If the solution is a disconnected graph, then each component forms a violated subtour constraint. These constraints are added to the formulation and the integer program is re-solved. This process is repeated until the solution defines a tour.

The following PROC OPTMODEL code solves the TSP by using the subtour formulation and iteratively adding subtour constraints.

```
%global numiters;
%do i = 1 %to 100; %global obj&i; %end;

/* iterative solution using the subtour formulation */
proc optmodel;
   set VERTICES;
   set EDGES = {i in VERTICES, j in VERTICES: i > j};
   num xc {VERTICES};
   num yc {VERTICES};

   num numsubtour init 0;
   set SUBTOUR {1..numsubtour};

   /* read in the instance and customer coordinates (xc, yc) */
   read data tspData into VERTICES=[var1] xc=var2 yc=var3;

   /* the cost is the euclidean distance rounded to the nearest integer */
   num c {<i,j> in EDGES}
       init floor( sqrt( ((xc[i]-xc[j])**2 + (yc[i]-yc[j])**2)) + 0.5);

   var x {EDGES} binary;

   /* minimize the total cost */
   min obj =
       sum {<i,j> in EDGES} c[i,j] * x[i,j];

   /* each vertex has exactly one in-edge and one out-edge */
   con two_match {i in VERTICES}:
       sum {j in VERTICES: i > j} x[i,j]
     + sum {j in VERTICES: i < j} x[j,i] = 2;

   /* no subtours (these constraints are generated dynamically) */
   con subtour_elim {s in 1..numsubtour}:
       sum {<i,j> in EDGES: (i in SUBTOUR[s] and j not in SUBTOUR[s])
          or (i not in SUBTOUR[s] and j in SUBTOUR[s])} x[i,j] >= 2;

   /* this starts the algorithm to find violated subtours */
   set <num,num> EDGES1;
   set INITVERTICES = setof{<i,j> in EDGES1} i;
   set VERTICES1;
```

```
   set NEIGHBORS;
   set <num,num> CLOSURE;
   num component {INITVERTICES};
   num numcomp  init 2;
   num iter     init 1;
   num numiters init 1;
   set ITERS = 1..numiters;
   num sol {ITERS, EDGES};

   /* initial solve with just matching constraints */
   solve;
   call symput(compress('obj'||put(iter,best.)),
              trim(left(put(round(obj),best.))));
   for {<i,j> in EDGES} sol[iter,i,j] = round(x[i,j]);

   /* while the solution is disconnected, continue */
   do while (numcomp > 1);
      iter = iter + 1;
      /* find connected components of support graph   */
      EDGES1 = {<i,j> in EDGES: round(x[i,j].sol) = 1};
      EDGES1 = EDGES1 union {setof {<i,j> in EDGES1} <j,i>};
      VERTICES1 = INITVERTICES;
      CLOSURE = EDGES1;
      for {i in INITVERTICES} component[i] = 0;
         for {i in VERTICES1} do;
            NEIGHBORS = slice(<i,*>,CLOSURE);
            CLOSURE = CLOSURE union (NEIGHBORS cross NEIGHBORS);
         end;
         numcomp = 0;
         do while (card(VERTICES1) > 0);
            numcomp = numcomp + 1;
            for {i in VERTICES1} do;
               NEIGHBORS = slice(<i,*>,CLOSURE);
               for {j in NEIGHBORS} component[j] = numcomp;
               VERTICES1 = VERTICES1 diff NEIGHBORS;
               leave;
            end;
         end;

         if numcomp = 1 then leave;
         numiters = iter;
         numsubtour = numsubtour + numcomp;
         for {comp in 1..numcomp} do;
            SUBTOUR[numsubtour-numcomp+comp]
             = {i in VERTICES: component[i] = comp};
         end;

         solve;
         call symput(compress('obj'||put(iter,best.)),
                    trim(left(put(round(obj),best.))));
         for {<i,j> in EDGES} sol[iter,i,j] = round(x[i,j]);
      end;

   /* create a data set for use by gplot */
   create data solData from
      [iter i j]={it in ITERS, <i,j> in EDGES: sol[it,i,j] = 1}
      xi=xc[i] yi=yc[i] xj=xc[j] yj=yc[j];
   call symput('numiters',put(numiters,best.));
quit;
```

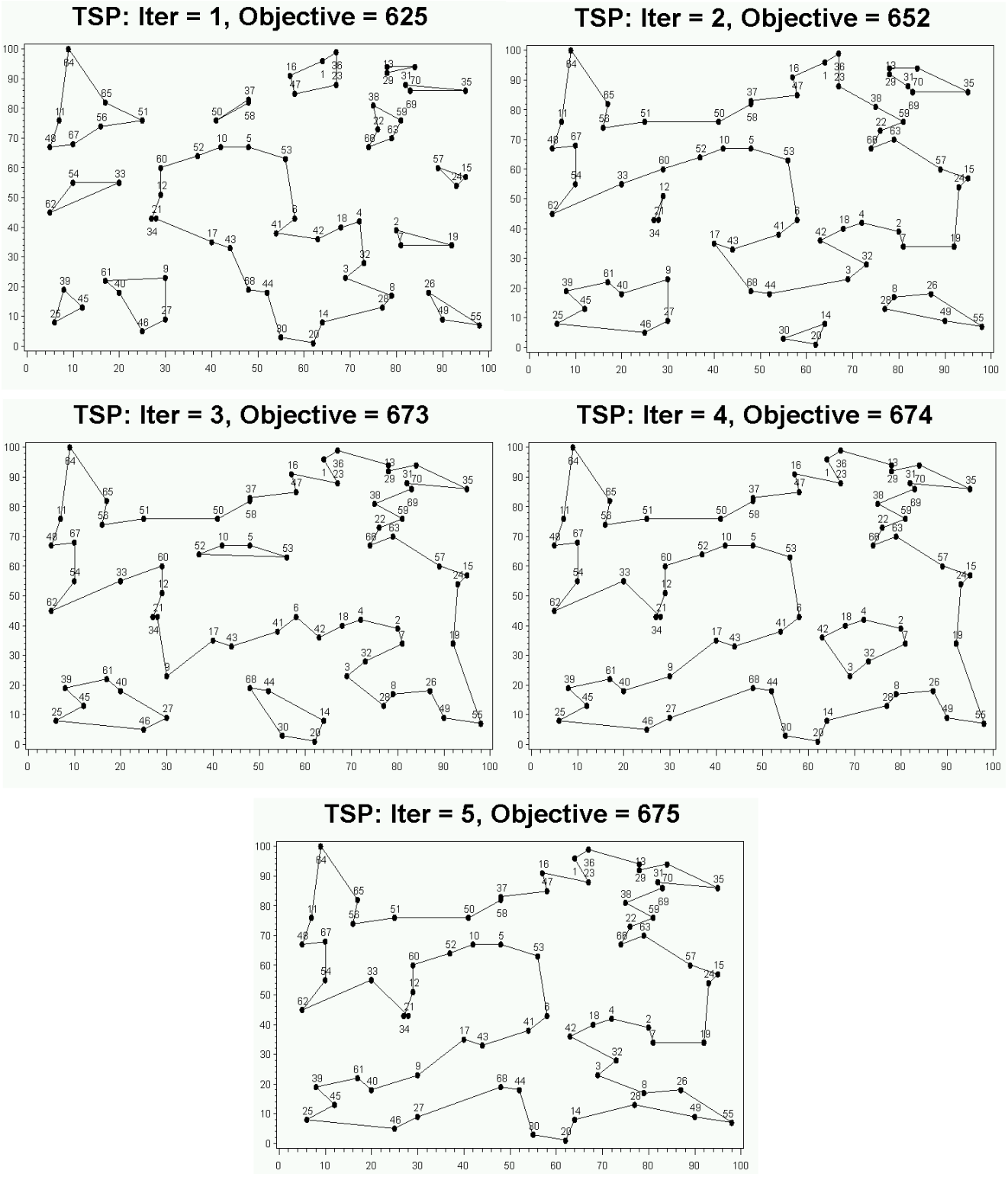
We can generate plots of the solution and objective value at each stage by using the following code:

```
%macro plotTSP;
%annomac;
%do i = 1 %to &numiters;
/* create annotate data set to draw subtours */
data anno(drop=iter xi yi xj yj);
   %SYSTEM(2, 2, 2);
   set solData(keep=iter xi yi xj yj);
   where iter = &i;
   %LINE(xi, yi, xj, yj, *, 1, 1);
run;

title1 "TSP: Iter = &i, Objective = &&obj&i";
title2;
proc gplot data=tspData anno=anno;
   axis1 label=none;
   symbol1 value=dot interpol=none
   pointlabel=("#var1") cv=black;
   plot var3*var2 / haxis=axis1 vaxis=axis1;
run;
quit;
%end;
%mend plotTSP;
%plotTSP;
```

The plots in Output 9.4.1 show the solution and objective value at each stage. Notice that at each stage, we restrict some subset of subtours. When we reach the final stage, we have a valid tour.

Note: An alternative way of approaching the TSP is to use a genetic algorithm. See Example 1.1 (Chapter 1, *SAS/OR User's Guide: Local Search Optimization*) for an example of how to use PROC GA to solve the TSP.

Output 9.4.1. Traveling Salesman Problem Iterative Solution



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Chapter 10
The NLPC Nonlinear Optimization Solver

Chapter Contents

Chapter 10
The NLPC Nonlinear Optimization Solver

Overview

In nonlinear optimization, we try to minimize or maximize an objective function that can be subject to a set of constraints. The objective function is typically nonlinear in terms of the decision variables. If the problem is constrained, it can be subject to bound, linear, or nonlinear constraints. In general, we can classify nonlinear optimization (minimization or maximization) problems into the following four categories:

- unconstrained
- bound constrained
- linearly constrained
- nonlinearly constrained

Since a maximization problem is equivalent to minimizing the negative of the same objective function, the general form of nonlinear optimization problems can, without loss of generality, be mathematically described as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c(x) \left\{ \leq \mid = \mid \geq \right\} b \\ & l \leq x \leq u \end{aligned}$$

where $f(x)$: $\mathbb{R}^n \mapsto \mathbb{R}$ is the nonlinear objective function; $c(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ are the functions of general nonlinear equality and inequality constraints, also referred to as the *body of constraints*; $b \in \mathbb{R}^m$ are the constant terms of the constraints, also referred to as the *right-hand side* (RHS); and l and u are lower and upper bounds on the decision variable x. If $c(x)$ are all linear in x, the nonlinear optimization problem becomes a *linearly constrained* problem, which can be expressed as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & Ax \left\{ \leq \mid = \mid \geq \right\} b \\ & l \leq x \leq u \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$. If $Ax \left\{ \leq \mid = \mid \geq \right\} b$ is not present, we have a *bound constrained* problem. If it is also true that $l_i = -\infty$ and $u_i = \infty$ for all $i = 1, \ldots, n$, we have an *unconstrained* problem in which x can take values in the entire $\mathbb{R}^n$ space.

These different problem classes typically call for different types of algorithms to solve them. The algorithm(s) devised specifically to solve a particular class of problem might not be suitable for solving problems in a different class. For instance, there are algorithms that specifically solve unconstrained and bound constrained problems. For linearly constrained problems, the fact that the Jacobian of the constraints is constant enables us to design algorithms that are more efficient for that class.

Optimization Techniques and Types of Problems Solved

The algorithms in the NLPC solver take advantage of the problem characteristics and automatically select an appropriate variant of an algorithm for a problem. Each of the optimization techniques implemented in the NLPC solver can handle unconstrained, bound constrained, linearly constrained, and nonlinearly constrained problems without your explicitly requesting which variant of the algorithm should be used. The NLPC solver is also designed for backward compatibility with PROC NLP, enabling you to migrate from PROC NLP to the more versatile PROC OPTMODEL modeling language. See Chapter 6, "The OPTMODEL Procedure" for details. You can access several optimization techniques in PROC NLP or their modified versions through the new interface.

The NLPC solver implements the following optimization techniques:

- conjugate gradient method
- Newton-type method with line search
- trust region method
- quasi-Newton method (experimental)

These techniques assume the objective and constraint functions to be twice continuously differentiable. The derivatives of the objective and constraint functions, which are provided to the solver by using the PROC OPTMODEL modeling language, are computed using one of the following two methods:

- automatic differentiation
- finite-difference approximation

For details about automatic differentiation and finite-difference approximation, see the section "Automatic Differentiation" on page 774.

Getting Started

The NLPC solver solves unconstrained nonlinear optimization problems and problems with a nonlinear objective function subject to bound, linear, or nonlinear constraints. It provides several optimization techniques that effectively handle these classes of problems. Guidelines for choosing a particular optimization technique for a problem can be found in the section "Optimization Algorithms" on page 912.

To solve a particular problem with a supported optimization technique in the NLPC solver, you need to specify the problem by using the PROC OPTMODEL modeling language. The problem specification typically includes the MIN/MAX statement for the objective, the CON statement for the constraints, and the VAR statement for declaring the decision variables and defining bounds. Hence familiarity with the PROC OPTMODEL modeling language is assumed for requesting a particular optimization technique in the NLPC solver to solve the optimization problem.

After you have specified the nonlinear optimization problem by using the PROC OPTMODEL modeling language, you can specify the NLPC solver by using the SOLVE statement as follows:

SOLVE WITH NLPC [/ *OPTIONS*]**;**

where OPTIONS can specify the optimization technique, termination criteria, and/or whether to display the iteration log. For details about these options, see the section "NLPC Solver Options" on page 910.

Introductory Examples

The following introductory examples illustrate how to get started using the NLPC solver and also provide basic information about the use of PROC OPTMODEL.

An Unconstrained Problem

Consider the following example of minimizing the Rosenbrock function (Rosenbrock 1960):

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

where $x = (x_1, x_2)$. The minimum function value is $f(x^*) = 0$ at $x^* = (1, 1)$. Note that this problem has no constraints.

The following PROC OPTMODEL statements can be used to solve this problem:

```
proc optmodel;
   number a = 100;
   var x{1..2};
   min f = a*(x[2] - x[1]^2)^2 + (1 - x[1])^2;

   solve with nlpc / tech=newtyp;
   print x;
quit;
```

The VAR statement declares the decision variables x_1 and x_2. The MIN statement identifies the symbol f that defines the objective function in terms of x_1 and x_2. The TECH=NEWTYP option in the SOLVE statement specifies that the Newton-type method with line search is used to solve this problem. Finally, the PRINT statement is specified to display the solution to this problem.

The output that summarizes the problem characteristics and the solution obtained by the solver are displayed in Figure 10.1. Note that the solution has $x_1 = 1$ and $x_2 = 1$, and an objective value very close to 0.

```
                Problem Summary

Objective Sense             Minimization
Objective Function                     f
Objective Type                 Nonlinear

Number of Variables                    2
Bounded Above                          0
Bounded Below                          0
Bounded Below and Above                0
Free                                   2
Fixed                                  0

Number of Constraints                  0


              Solution Summary

Solver                         NLPC/Newton-Type
Objective Function                             f
Solution Status                          Optimal
Objective Value                        6.4296E-18
Iterations                                     14

Absolute Optimality Error              4.8615356E-8
Relative Optimality Error              4.8615356E-8

                    [1]       x

                     1        1
                     2        1
```

Figure 10.1. Minimization of the Rosenbrock Function

Bound Constraints on the Decision Variables

Decision variables often have bound constraints of the form

$$l_i \leq x_i \leq u_i \quad \text{for } i = 1, \ldots, n$$

where n is the number of decision variables. The bounds on the variables can be specified with the symbols "**>=**" and "**<=**" in the VAR statement.

Consider the following bound constrained problem (Hock and Schittkowski 1981, Example 5):

$$\begin{aligned} \text{minimize} \quad & f(x) = \sin(x_1 + x_2) + (x_1 - x_2)^2 - 1.5x_1 + 2.5x_2 + 1 \\ \text{subject to} \quad & -1.5 \leq x_1 \leq 4 \\ & -3 \leq x_2 \leq 3 \end{aligned}$$

Given a starting point at $x^0 = (0, 0)$, there is a local minimum at

$$x^* = (-\frac{\pi}{3} + \frac{1}{2}, -\frac{\pi}{3} - \frac{1}{2}) \approx (-0.5472, -1.5472)$$

with the objective value $f(x^*) = -\sqrt{3}/2 - \pi/3 \approx 1.9132$. This problem can be formulated and solved by the following statements:

```
proc optmodel;
   set S = 1..2;
   number lb{S} = [-1.5 -3];
   number ub{S} = [4 3];
   number x0{S} = [0 0];
   var x{i in S} >= lb[i] <= ub[i] init x0[i];

   min obj = sin(x[1] + x[2]) + (x[1] - x[2])^2
             - 1.5*x[1] + 2.5*x[2] + 1;

   solve with nlpc / printfreq=1;
   print x;
quit;
```

The starting point is specified with the keyword INIT in the VAR statement. As usual, the MIN statement identifies the objective function. Since there is no explicit optimization technique specified (using the TECH= option), the NLPC solver uses the trust region method, which is the default algorithm for problems of this size. The PRINTFREQ= option is used to display the iteration log during the optimization process.

In Figure 10.2, the problem is summarized at the top. Then, the details of the iterations are displayed. A message is printed to indicate that the default optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1.0E−6) were satisfied. (See the section "Optimality Control" on page 918 for more information.) A summary of the solution shows that the trust region method was used for the optimization and the solution found is optimal. It also shows the optimal objective value and the number of iterations taken to find the solution. The optimal solution is displayed at the end.

```
                          Problem Summary

                 Objective Sense            Minimization
                 Objective Function                  obj
                 Objective Type                Nonlinear

                 Number of Variables                   2
                 Bounded Above                         0
                 Bounded Below                         0
                 Bounded Below and Above               2
                 Free                                  0
                 Fixed                                 0

                 Number of Constraints                 0


                    Iteration Log: Trust Region Method

                                                          Trust
                       Function    Objective    Optimality  Region
           Iter         Calls        Value         Error    Radius

              0             0       1.0000        1.0000         .
              1             3      -1.5758        0.8905     2.231
              2             5      -1.8962        0.2414     0.546
              3             6      -1.9131        0.0127     0.554
              4             7      -1.9132     0.0000521     0.149
              5             8      -1.9132     9.061E-10    0.0103

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.


                          Solution Summary

                 Solver                       NLPC/Trust Region
                 Objective Function                         obj
                 Solution Status                        Optimal
                 Objective Value                        -1.9132
                 Iterations                                   5

                 Absolute Optimality Error          9.061385E-10
                 Relative Optimality Error          9.061385E-10
                 Absolute Infeasibility                        0
                 Relative Infeasibility                        0


                                [1]          x

                                 1     -0.5472
                                 2     -1.5472
```

Figure 10.2. A Bound Constrained Problem

Linear Constraints on the Decision Variables

More general linear equality or inequality constraints have the form

$$\sum_{j=1}^{n} a_{ij} x_j \; \{\leq \mid = \mid \geq\} \; b_i \quad \text{for } i = 1, \ldots, m$$

where n is the number of decision variables and m is the number of constraints, which can be specified with the CON statement.

Consider, for example, Rosenbrock's post office problem (Schittkowski 1987, p. 74):

$$\begin{aligned} \text{minimize} \quad & f(x) = -x_1 x_2 x_3 \\ \text{subject to} \quad & c_1(x) = x_1 + 2x_2 + 2x_3 \geq 0 \\ & c_2(x) = 72 - x_1 - 2x_2 - 2x_3 \geq 0 \\ & 0 \leq x_1 \leq 20 \\ & 0 \leq x_2 \leq 11 \\ & 0 \leq x_3 \leq 42 \end{aligned}$$

Starting from $x^0 = (10, 10, 10)$, you can reach a minimum at $x^* = (20, 11, 15)$, with a corresponding objective value $f(x^*) = -3300$. You can use the following SAS code to formulate and solve this problem:

```
proc optmodel;
   number ub{1..3} = [20 11 42];
   var x{i in 1..3} >= 0 <= ub[i] init 10;

   min f = -1*x[1]*x[2]*x[3];
   con c1: x[1] + 2*x[2] + 2*x[3] >= 0;
   con c2: 72 - x[1] - 2*x[2] - 2*x[3] >= 0;

   solve with nlpc / tech=congra printfreq=1;
   print x;
quit;
```

As usual, the VAR statement specifies the bounds on the variables, and the starting point; the MIN statement identifies the objective function. In addition, the two CON statements describe the linear constraints $c_1(x)$ and $c_2(x)$. To solve this problem, select the conjugate gradient optimization technique by using the TECH=CONGRA option. The PRINTFREQ= option is used to display the iteration log.

In Figure 10.3, you can find a problem summary and the iteration log. The solution summary and the solution are printed at the bottom.

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Nonlinear
Number of Variables	3
Bounded Above	0
Bounded Below	0
Bounded Below and Above	3
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	2
Linear Range	0

Iteration Log: Conjugate Gradient Method

Iter	Function Calls	Objective Value	Optimality Error	Step Size	Slope of Search Direction	Restarts
0	0	−1000	1.0000	.	.	0
1	3	−1331	1.0000	0.0100	−30000	0
2	5	−3056	0.2952	0.0468	−29282	1
3	7	−3300	0	0.0455	−6722.2	2

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E−6) are satisfied.

Solution Summary

Solver	NLPC/Conjugate Gradient
Objective Function	f
Solution Status	Optimal
Objective Value	−3300
Iterations	3
Absolute Optimality Error	0
Relative Optimality Error	0
Absolute Infeasibility	3.552714E−15
Relative Infeasibility	4.866731E−17

[1]	x
1	20
2	11
3	15

Figure 10.3. Rosenbrock's Post Office Problem

You can formulate linear constraints in a more compact manner. Consider the following example (Hock and Schittkowski 1981, test example 24):

$$\begin{aligned} \text{minimize} \quad & f(x) = \tfrac{1}{27\sqrt{3}}((x_1 - 3)^2 - 9)x_2^3 \\ \text{subject to} \quad & x_1/\sqrt{3} - x_2 \geq 0 \\ & x_1 + \sqrt{3}x_2 \geq 0 \\ & -x_1 - \sqrt{3}x_2 \geq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The minimum function value is $f(x^*) = -1$ at $x^* = (3, \sqrt{3})$. Assume a feasible starting point, $x^0 = (1, 0.5)$.

You can specify this model by using the following PROC OPTMODEL statements:

```
proc optmodel;
   number a{1..3, 1..2} = [ .57735     -1
                                1   1.732
                               -1 -1.732 ];
   number b{1..3} = [ 0 0 -6 ];
   number x0{1..2} = [ 1 .5 ];
   var x{i in 1..2} >= 0 init x0[i];

   min f = ((x[1] - 3)^2 - 9) * x[2]^3 / (27*sqrt(3));
   con cc {i in 1..3}: sum{j in 1..2} a[i,j]*x[j] >= b[i];

   solve with nlpc / printfreq=1;
   print x;
quit;
```

Note that instead of writing three individual linear constraints as in Rosenbrock's post office problem, we use a two-dimensional array a to represent the coefficient matrix of the linear constraints and a one-dimensional array b for the right-hand side. Consequently, all three linear constraints are represented in a single CON statement. This method is especially useful for larger models and for models in which the constraint coefficients are subject to change.

The output showing the problem summary, the iteration log, the solution summary, and the solution is displayed in Figure 10.4.

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Nonlinear
Number of Variables	2
Bounded Above	0
Bounded Below	2
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	3
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	3
Linear Range	0

Iteration Log: Trust Region Method

Iter	Function Calls	Objective Value	Optimality Error	Trust Region Radius
0	0	-0.0134	0.0802	.
1	2	-0.0218	0.0706	1.000
2	9	-1.0000	0	9.166

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.

Solution Summary

Solver	NLPC/Trust Region
Objective Function	f
Solution Status	Optimal
Objective Value	-1
Iterations	2
Absolute Optimality Error	0
Relative Optimality Error	0
Absolute Infeasibility	7.361733E-16
Relative Infeasibility	1.051676E-16

[1]	x
1	3.0000
2	1.7321

Figure 10.4. A Second Linearly Constrained Problem

Nonlinear Constraints on the Decision Variables

General nonlinear equality or inequality constraints have the form

$$c_i(x) \{\leq | = | \geq\} b_i \quad \text{for } i = 1, \ldots, m$$

where $c_i(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is a general nonlinear constraint, which can be specified with the CON statement.

Consider the following nonlinearly constrained problem (Avriel 1976, p. 456):

$$\begin{aligned} \text{minimize} \quad & f(x) = (x_1 - 4)^2 + (x_2 - 4)^2 \\ \text{subject to} \quad & c_1(x) = 3x_1^2 + x_2^2 - 2x_1x_2 - 4x_1 \leq 12 \\ & c_2(x) = 3x_1 + 4x_2 \leq 28 \end{aligned}$$

An initial point is given at $x^0 = (2, 0)$. You can use the following SAS code to formulate and solve this problem:

```
proc optmodel;
   num x0{1..2} = [2 0];
   var x{i in 1..2} init x0[i];

   min f = (x[1] - 4)^2 + (x[2] - 4)^2;
   con c1: 3*x[1]^2 + x[2]^2 - 2*x[1]*x[2] - 4*x[1] <= 12;
   con c2: 3*x[1] + 4*x[2] <= 28;

   solve with nlpc / tech=qne printfreq=1;
   print x;
quit;
```

Note that $c_1(x)$ is a nonlinear constraint and $c_2(x)$ is a linear constraint. Both can be specified by using the CON statement. The PROC OPTMODEL modeling language automatically recognizes types of constraints to which they belong. The experimental quasi-Newton method is requested to solve this problem.

A problem summary is shown in Figure 10.5. Figure 10.6 displays the iteration log. Note that the quasi-Newton method is an infeasible point algorithm; i.e., the iterates remain infeasible to the nonlinear constraints until the optimal solution is found. The column "Maximum Constraint Violation" displays the infeasibility. The solution summary and the solution are shown in Figure 10.7.

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Quadratic
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	2
Fixed	0
Number of Constraints	2
Linear LE (<=)	1
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	1
Nonlinear EQ (=)	0
Nonlinear GE (>=)	0
Nonlinear Range	0

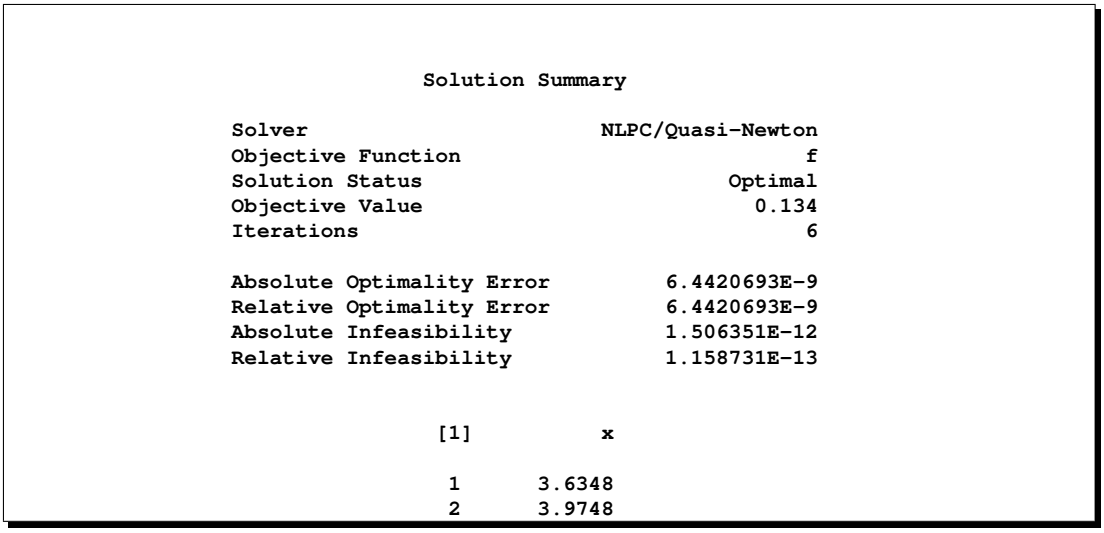
Figure 10.5. Nonlinearly Constrained Problem: Problem Summary

Iteration Log: Quasi-Newton Method with BFGS Update

Iter	Function Calls	Objective Value	Infeasibility	Optimality Error	Step Size	Predicted Function Reduction
0	0	20.0000	0	1.0000	.	.
1	4	0.9570	0	1.0275	1.000	1.7256
2	5	0.0942	0.0480	0.1134	1.000	0.0783
3	6	0.1335	0.000538	0.004636	1.000	0.00103
4	7	0.1340	2.9482E-7	0.000336	1.000	5.148E-7
5	8	0.1340	2.1822E-9	2.4734E-6	1.000	4.2E-9
6	9	0.1340	1.159E-13	6.4421E-9	1.000	2.101E-9

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.

Figure 10.6. Nonlinearly Constrained Problem: Iteration Log



Solution Summary

Solver	NLPC/Quasi-Newton
Objective Function	f
Solution Status	Optimal
Objective Value	0.134
Iterations	6
Absolute Optimality Error	6.4420693E-9
Relative Optimality Error	6.4420693E-9
Absolute Infeasibility	1.506351E-12
Relative Infeasibility	1.158731E-13

[1]	x
1	3.6348
2	3.9748

Figure 10.7. Nonlinearly Constrained Problem: Solution

Syntax

The following PROC OPTMODEL statement is available for the NLPC solver:

SOLVE WITH NLPC < / *options* > **;**

Functional Summary

Table 10.1 outlines the options that can be used with the SOLVE WITH NLPC statement.

Table 10.1. Options for the NLPC Solver

Description	Option
Optimization Option:	
optimization technique	TECH=
Termination Criterion Options:	
maximum number of iterations	MAXITER=
maximum number of function calls	MAXFUNC=
upper limit on optimization time	MAXTIME=
upper limit on magnitude of objective value	OBJLIMIT=
tolerance for absolute optimality error	ABSOPTTOL=
tolerance for relative optimality error	RELOPTTOL=
Printed Output Option:	
frequency at which to display iteration log	PRINTFREQ=

NLPC Solver Options

This section describes the options recognized by the NLPC solver. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the NLPC solver is explicitly specified using a WITH clause.

The following notation is used in this section and the section "Details" on page 912:

M	a nonnegative floating-point number
N	a nonnegative integer
δ	a small nonnegative floating-point number
m	number of general nonlinear constraints, including the linear constraints but not the bound constraints
n	dimension of x, i.e., the number of decision variables
x, x^k	iterate, i.e., the vector of n decision variables; x^k denotes the iterate at the kth iteration.
$f(x)$	objective function
$\nabla f(x)$	gradient of the objective function
$\nabla^2 f(x)$	Hessian of the objective function
λ, λ^k	Lagrange multiplier vector, $\lambda \in \mathbb{R}^m$; λ^k denotes the estimate of the Lagrange multipliers at the kth iteration
$L(x, \lambda)$	Lagrangian function of constrained problems
$\nabla_x L(x, \lambda)$	gradient of the Lagrangian function with respect to x
$\nabla_x^2 L(x, \lambda)$	Hessian of the Lagrangian function with respect to x
$\|\cdot\|_\infty$	infinity norm of a vector; for example, $\|y\|_\infty = \max_{i=1,\ldots,n} \{\|y_i\|\}$ for $y \in \mathbb{R}^n$

ABSOPTTOL=δ

specifies the tolerance for the absolute optimality error. For the solver to terminate,

$$\|\nabla f(x^k)\|_\infty \leq \delta$$

must be satisfied for unconstrained problems, or

$$\|\nabla_x L(x^k, \lambda^k)\|_\infty \leq \delta$$

must be satisfied for constrained problems. The default value is $\delta = 1.0\text{E}{-3}$ for all four optimization techniques. The range of valid values for δ is between 0 and 1.

Note: For more information about this termination criterion, see the section "Optimality Control" on page 918.

MAXFUNC=N
MAXFEVAL=N

specifies that the optimization process stop after a maximum of N function calls. The default values are as follows:

- for TECH=TRUREG or NEWTYP, $N = 3000$
- for TECH=QUANEW, $N = 5000$
- for TECH=CONGRA, $N = 6000$

Note that the optimization process can be terminated only after completing a full iteration. Therefore, the number of function calls that are actually performed can exceed the number that is specified by the MAXFUNC= option.

MAXITER=*N*
specifies that the optimization process stop after a maximum of N iterations. The default values are as follows:

- for TECH=TRUREG or NEWTYP, $N = 500$
- for TECH=QUANEW, $N = 800$
- for TECH=CONGRA, $N = 1000$

MAXTIME=*M*
specifies an upper limit of M seconds of real time for the optimization process. If you do not specify this option, the optimization process does not stop based on the amount of time elapsed. Note that the time specified by the MAXTIME= option is checked only at the end of each iteration. The optimization terminates when the actual running time is greater than or equal to M.

OBJLIMIT=*M*
specifies an upper limit on the magnitude of the objective value. For a minimization problem, the algorithm terminates when the objective value becomes less than $-M$; for a maximization problem, the algorithm stops when the objective value exceeds M. When this happens, it implies that either the problem is unbounded or the algorithm diverges. If optimization were allowed to continue, numerical difficulty might be encountered. The default value is $M = 1.0\text{E}{+}20$. The range of valid values for M is $M \geq 1.0\text{E}{+}8$.

PRINTFREQ=*j*
specifies that the printing of the solution progress to the iteration log should occur after every j iterations. The print frequency, j, is an integer greater than or equal to zero. The value $j = 0$ disables the printing of the progress of the solution. Note that iteration 0 and the last iteration are always displayed for $j > 0$.

By default, the NLPC solver does not display the iteration log.

RELOPTTOL=*δ*
specifies the tolerance for the relative optimality error. For the solver to terminate,

$$\|\nabla f(x^k)\|_\infty \leq \delta \max\{1, |f(x^k)|\}$$

must be satisfied for unconstrained problems, or

$$\|\nabla_x L(x^k, \lambda^k)\|_\infty \leq \delta \max\{1, \|\nabla f(x^k)\|_\infty\}$$

must be satisfied for constrained problems. The default value is $\delta = 1.0\text{E}{-6}$ for all four optimization techniques. The range of valid values for δ is between 0 and 1.

Note: For more information about this termination criterion, see the section "Optimality Control" on page 918.

TECH=*keyword*
TECHNIQUE=*keyword*
SOLVER=*keyword*

specifies the optimization technique. Valid keywords are as follows:

- CONGRA or CGR
  uses a conjugate gradient method. For $n > 1000$, CONGRA is the default optimization technique in the NLPC solver.
- NEWTYP or NTY
  uses a Newton-type method with line search.
- TRUREG or TRE
  uses a trust region method. For $n \leq 1000$, TRUREG is the default optimization technique in the NLPC solver.
- QUANEW or QNE (experimental)
  uses a quasi-Newton method with the BFGS update. QUANEW is the optimization technique in the NLPC solver to solve problems with nonlinear constraints.

Details

Optimization Algorithms

There are four optimization algorithms available in the NLPC solver. A particular algorithm can be selected by using the TECH= option in the SOLVE statement.

Table 10.2. Algorithmic Options for the NLPC Solver

Algorithm	TECH=
Newton-type method with line search	NEWTYP
Trust region method	TRUREG
Conjugate gradient method	CONGRA
Quasi-Newton method (experimental)	QUANEW

Different optimization techniques require different derivatives, and computational efficiency can be improved depending on the kind of derivatives needed. Table 10.3 summarizes, for each optimization technique, which derivatives are needed (FOD: first-order derivatives; SOD: second-order derivatives) and what types of constraints (UNC: unconstrained; BC: bound constraints; LIC: linear constraints; NLC: nonlinear constraints) are supported.

Table 10.3. Types of Derivatives Required and Constraints Supported by Algorithm

Algorithm	Derivatives Needed		Constraints Supported			
	FOD	SOD	UNC	BC	LIC	NLC
TRUREG	x	x	x	x	x	-
NEWTYP	x	x	x	x	x	-
CONGRA	x	-	x	x	x	-
QUANEW	x	-	x	x	x	x

Choosing an Optimization Algorithm

Several factors play a role in choosing an optimization technique for a particular problem. First, the structure of the problem has to be considered: Is it unconstrained, bound constrained, or linearly constrained? The NLPC solver automatically identifies the structure and chooses an appropriate variant of the algorithm for the problem.

Next, it is important to consider the type of derivatives of the objective function and the constraints that are needed, and whether these are analytically tractable or not. This section provides some guidelines for making the choice. For an optimization problem, computing the gradient takes more computer time than computing the function value, and computing the Hessian matrix sometimes takes much more computer time and memory than computing the gradient, especially when there are many decision variables. Optimization techniques that do not use the Hessian usually require more iterations than techniques that do use the Hessian. The former tend to be slower and less reliable. However, the techniques that use the Hessian can be prohibitively slow for larger problems.

The following guidelines can be used in choosing an algorithm for a particular problem.

- Without nonlinear constraints:
  - **Smaller Problems: TRUREG or NEWTYP**
    if $n \leq 1000$ and the Hessian matrix is not expensive to compute. Sometimes NEWTYP can be faster than TRUREG, but TRUREG is generally more stable.
  - **Larger Problems: CONGRA**
    if $n > 1000$, the objective function and the gradient can be computed much more quickly than the Hessian, and too much memory is needed to store the Hessian. CONGRA in general needs more iterations than NEWTYP or TRUREG, but each iteration tends to be much faster. Since CONGRA needs less memory, many larger problems can be solved more efficiently by CONGRA.
- With nonlinear constraints:

– **QUANEW** (experimental)
QUANEW is the optimization method in the NLPC solver that solves problems with nonlinear constraints.

Trust Region Method (TRUREG)

The TRUREG method uses the gradient $\nabla f(x^k)$ and the Hessian matrix $\nabla^2 f(x^k)$, and it thus requires that the objective function $f(x)$ have continuous first- and second-order partial derivatives inside the feasible region.

The trust region method iteratively optimizes a quadratic approximation to the nonlinear objective function within a hyperelliptic trust region with radius Δ that constrains the step length corresponding to the quality of the quadratic approximation. The trust region method is implemented using the techniques described in Dennis, Gay, and Welsch (1981), Gay (1983), and Moré and Sorensen (1983).

The trust region method performs well for small to medium-sized problems and does not require many function, gradient, and Hessian calls. If the evaluation of the Hessian matrix is computationally expensive in larger problems, the conjugate gradient algorithm might be more appropriate.

Newton-Type Method with Line Search (NEWTYP)

The NEWTYP technique uses the gradient $\nabla f(x^k)$ and the Hessian matrix $\nabla^2 f(x^k)$, and it thus requires that the objective function have continuous first- and second-order partial derivatives inside the feasible region. If second-order partial derivatives are computed efficiently and precisely, the NEWTYP method can perform well for medium to large problems.

This algorithm uses a pure Newton step when the Hessian is positive definite and when the Newton step reduces the value of the objective function successfully. Otherwise, a combination of ridging and line search is done to compute successful steps. If the Hessian is not positive definite, a multiple of the identity matrix is added to the Hessian matrix to make it positive definite (Eskow and Schnabel 1991).

In each iteration, a line search is done along the search direction to find an approximate optimum of the objective function. The line-search method uses quadratic interpolation and cubic extrapolation.

Conjugate Gradient Method (CONGRA)

Second-order derivatives are not used by CONGRA. The CONGRA algorithm can be expensive in function and gradient calls but needs only $O(n)$ memory for unconstrained optimization. In general, many iterations are needed to obtain a precise solution by using CONGRA, but each iteration is computationally inexpensive. The update formula for generating the conjugate directions uses the automatic restart update method of Powell (1977) and Beale (1972).

The CONGRA method should be used for optimization problems with large n. For the unconstrained or bound constrained case, CONGRA needs only $O(n)$ bytes of working memory, whereas all other optimization methods require $O(n^2)$ bytes of working memory. During n successive iterations, uninterrupted by restarts or changes in the working set, the conjugate gradient algorithm computes a cycle of n conjugate

search directions. In each iteration, a line search is done along the search direction to find an approximate optimum of the objective function value. The line-search method uses quadratic interpolation and cubic extrapolation to obtain a step length α satisfying the Goldstein conditions (Fletcher 1987). Only one of the Goldstein conditions needs to be satisfied if the feasible region defines an upper limit for the step length.

Quasi-Newton Method (QUANEW) (Experimental)

The quasi-Newton method uses the gradient to approximate the Hessian. It works well for medium to moderately large optimization problems where the objective function and the gradient are much faster to compute than the Hessian, but in general it requires more iterations than the TRUREG and NEWTYP techniques, which compute the exact Hessian.

The specific algorithms implemented in the QUANEW technique depend on whether or not there are nonlinear constraints.

Unconstrained or Linearly Constrained Problems

If there are no nonlinear constraints, QUANEW updates the Cholesky factor of the approximate Hessian. In each iteration, a line search is done along the search direction to find an approximate optimum. The line-search method uses quadratic interpolation and cubic extrapolation to obtain a step length satisfying the Goldstein conditions.

Nonlinearly Constrained Problems

The algorithm implemented in the quasi-Newton method is an efficient modification of Powell's (1978, 1982) *Variable Metric Constrained WatchDog* (VMCWD) algorithm. A similar but older algorithm (VF02AD) is part of the Harwell library. Both VMCWD and VF02AD use Fletcher's VE02AD algorithm (part of the Harwell library) for strictly convex quadratic programming. The implementation in the NLPC solver uses a quadratic programming algorithm that updates the approximation of the Cholesky factor when the active set changes. The QUANEW method is not a feasible-point algorithm, and the value of the objective function might not decrease (minimization) or increase (maximization) monotonically. Instead, the algorithm tries to reduce the value of a *merit function*, a linear combination of the objective function and constraint violations.

Conditions of Optimality

To facilitate discussion of the optimality conditions, we rewrite the general form of nonlinear optimization problems from the section "Overview" on page 897 by grouping the equality constraints and inequality constraints. We also rewrite all the general nonlinear inequality constraints and bound constraints in one form as "$\geq$" inequality constraints. Thus we have the following formulation:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \\ & c_i(x) \geq 0, \quad i \in \mathcal{I} \end{aligned}$$

where $\mathcal{E}$ is the set of indices of the equality constraints, $\mathcal{I}$ is the set of indices of the inequality constraints, and $m = |\mathcal{E}| + |\mathcal{I}|$.

A point x is *feasible* if it satisfies all the constraints $c_i(x) = 0, i \in \mathcal{E}$ and $c_i(x) \geq 0, i \in \mathcal{I}$. The feasible region $\mathcal{F}$ consists of all the feasible points. In unconstrained cases, the feasible region $\mathcal{F}$ is the entire $\mathbb{R}^n$ space.

A feasible point x^* is a *local solution* of the problem if there exists a neighborhood $\mathcal{N}$ of x^* such that

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

Further, a feasible point x^* is a *strict local solution* if strict inequality holds in the preceding case; i.e.,

$$f(x) > f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

A feasible point x^* is a *global solution* of the problem if no point in $\mathcal{F}$ has a smaller function value than $f(x^*)$; i.e.,

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{F}$$

All the algorithms in the NLPC solver find a local solution of an optimization problem.

Unconstrained Optimization

The following conditions hold true for unconstrained optimization problems:

- **First-order necessary conditions**: If x^* is a local solution and $f(x)$ is continuously differentiable in some neighborhood of x^*, then

$$\nabla f(x^*) = 0$$

- **Second-order necessary conditions**: If x^* is a local solution and $f(x)$ is twice continuously differentiable in some neighborhood of x^*, then $\nabla^2 f(x^*)$ is positive semidefinite.
- **Second-order sufficient conditions**: If $f(x)$ is twice continuously differentiable in some neighborhood of x^*, $\nabla f(x^*) = 0$, and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local solution.

Constrained Optimization

For constrained optimization problems, the *Lagrangian function* is defined as follows:

$$L(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

where $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$, are called *Lagrange multipliers*. $\nabla_x L(x, \lambda)$ is used to denote the gradient of the Lagrangian function with respect to x, and $\nabla_x^2 L(x, \lambda)$ is used to denote the Hessian of the Lagrangian function with respect to x. The active set at a feasible point x is defined as

$$\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : c_i(x) = 0\}$$

We also need the following definition before we can state the first-order and second-order necessary conditions:

- **Linear independence constraint qualification and regular point**: A point x is said to satisfy the *linear independence constraint qualification* if the gradients of active constraints

$$\nabla c_i(x), \quad i \in \mathcal{A}(x)$$

are linearly independent. Further, we refer to such a point x as a *regular point*.

We now state the theorems that are essential in the analysis and design of algorithms for constrained optimization:

- **First-order necessary conditions**: Suppose that x^* is a local minimum and also a regular point. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are continuously differentiable, there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ such that the following conditions hold:

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*) = 0 \\ c_i(x^*) &= 0, \quad i \in \mathcal{E} \\ c_i(x^*) &\geq 0, \quad i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \quad i \in \mathcal{I} \\ \lambda_i^* c_i(x^*) &= 0, \quad i \in \mathcal{I} \end{aligned}$$

The preceding conditions are often known as the *Karush-Kuhn-Tucker conditions*, or *KKT conditions* for short. Also, the first set of equations are referred to as the *stationarity condition*, and the last set of equations are referred to as the *complementarity condition*.

- **Second-order necessary conditions**: Suppose x^* is a local minimum and also a regular point. Let λ^* be the Lagrange multipliers that satisfy the KKT conditions. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are twice continuously differentiable, the following conditions hold:

$$z^{\mathrm{T}} \nabla_x^2 L(x^*, \lambda^*) z \geq 0$$

for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^{\mathrm{T}} z = 0, \quad i \in \mathcal{A}(x*)$$

- **Second-order sufficient conditions**: Suppose there exist a point x^* and some Lagrange multipliers λ^* such that the KKT conditions are satisfied. If the conditions

$$z^{\mathrm{T}} \nabla_x^2 L(x^*, \lambda^*) z > 0$$

for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^{\mathrm{T}} z = 0, \quad i \in \mathcal{A}(x^*)$$

hold true, then x^* is a strict local solution.

Note that the set of all such z's forms the null space of the matrix $\left[\nabla c_i(x^*)^{\mathrm{T}}\right]_{i \in \mathcal{A}(x*)}$. Hence we can search for strict local solutions by numerically checking the Hessian of the Lagrangian function projected onto the null space. For a rigorous treatment of the optimality conditions, see Fletcher (1987) and Nocedal and Wright (1999).

The optimization algorithms in the NLPC solver apply an iterative process that results in a sequence of points, $x^0, ..., x^k, ...$, that converge to a local solution x^* satisfying the first-order conditions. At the solution the NLPC solver performs tests to confirm that the second-order conditions are also satisfied.

Optimality Control

The ABSOPTTOL= and RELOPTTOL= options are based on satisfying the first-order necessary conditions. In particular, the stationarity condition requires the gradient of the objective function or Lagrange function to be zero at the optimal point x^*, i.e.,

$$\nabla f(x^*) = 0$$

in unconstrained cases and

$$\nabla_x L(x^*, \lambda^*) = 0$$

in constrained cases.

Since the computation is carried out in finite-precision arithmetic, rounding errors prevent the algorithms from exactly satisfying the preceding condition. Instead, we terminate the algorithms at some small threshold values for the preceding gradients. These threshold values can be measured in an absolute or relative sense. We define ξ_1^k, the *absolute optimality error* at x^k, as

$$\xi_1^k = \|\nabla f(x^k)\|_\infty$$

for unconstrained problems, or

$$\xi_1^k = \|\nabla_x L(x^k, \lambda^k)\|_\infty$$

for constrained problems. We define ξ_2^k, the *relative optimality error* at x^k, as

$$\xi_2^k = \frac{\|\nabla f(x^k)\|_\infty}{\max\{1, |f(x^k)|\}}$$

for unconstrained problems, or

$$\xi_2^k = \frac{\|\nabla_x L(x^k, \lambda^k)\|_\infty}{\max\{1, \|\nabla f(x^k)\|_\infty\}}$$

for constrained problems.

An optimization algorithm terminates at x^k if both termination criteria as specified by the ABSOPTTOL=δ_1 and RELOPTTOL=δ_2 options are satisfied—that is, if $\xi_1^k \leq \delta_1$

and $\xi_2^k \leq \delta_2$. The default value of ABSOPTTOL=1.0E−3 effectively prevents an algorithm from terminating at some point x^k where the absolute optimality error, as measured by the maximum magnitude of the gradient elements at x^k, is large but the relative optimality error is small.

If you set a very small value for the ABSOPTTOL= or RELOPTTOL= option, depending on the problem, the termination criteria might not be able to be satisfied. This is especially true when a very small value of the ABSOPTTOL= option is used, and it often occurs when the magnitudes of the objective value or the gradient elements are very large. It can also occur when finite-difference approximations of derivatives are used.

In addition, the complementarity condition must be satisfied at the optimal point x^*. This condition is checked at the end of the optimization by using the same criteria characterized by the ABSOPTTOL= or RELOPTTOL= option. See the section "Conditions of Optimality" on page 915 for more information about the optimality conditions.

Infeasibility

Like any iterative algorithm, an optimization algorithm is carried out in finite-precision arithmetic and is subject to numerical rounding errors. Thus, when an algorithm terminates, the constraints might not be satisfied exactly. Instead we consider a constraint to be satisfied if the violation is within some prescribed tolerance. Such a violation can be measured in an absolute or relative sense.

For an optimization problem of the general form described in the section "Overview" on page 897, we rewrite the constraints (including bound constraints) in the form

$$\begin{aligned} c_i(x) &= b_i, \quad i \in \mathcal{E} \\ c_i(x) &\geq b_i, \quad i \in \mathcal{I} \end{aligned}$$

where $\mathcal{E}$ and $\mathcal{I}$ denote the sets of equality and inequality constraints, respectively. We define the *absolute infeasibility* ν_1^k at x^k to be the maximum constraint violation in absolute measure as follows:

$$\nu_1^k = \max_{i \in \mathcal{E}, j \in \mathcal{I}} \{|c_i(x^k) - b_i|, b_j - c_j(x^k)\}$$

We define the *relative infeasibility* ν_2^k at x^k to be the maximum constraint violation in relative measure as follows:

$$\nu_2^k = \max_{i \in \mathcal{E}, j \in \mathcal{I}} \left\{ \frac{|c_i(x^k) - b_i|}{\max\{1, |b_i|\}}, \frac{b_j - c_j(x^k)}{\max\{1, |b_j|\}} \right\}$$

For feasibility control, we choose ϵ to be the tolerance for the relative infeasibility. For a solution x^k to be considered feasible, we require that the following hold for the relative infeasibility:

$$\nu_2^k \leq \epsilon$$

In the NLPC solver, we set $\epsilon = 1.0\text{E}{-6}$.

Associated with the infeasibility is the *conditional optimality*. A solution x^k is considered to be conditionally optimal if x^k satisfies the optimality criteria described in the section "Optimality Control" on page 918 and if the following is true:

$$\epsilon < \nu_2^k < 1.0\text{E}-3$$

That is, the default tolerance for the relative infeasibility is not satisfied, but the relative infeasibility is not too large. This is useful for problems whose constraints are not well scaled.

Feasible Starting Point

You can specify a starting point for the optimization. If the specified point is infeasible to linear and/or bound constraints, two schemes are used to obtain a feasible starting point (feasible to linear and bound constraints only), depending on the type of problem. They are as follows.

- When only bound constraints are specified:
  - If the variable x_i, $i = 1, \ldots, n$, violates a two-sided bound constraint $l_i \leq x_i \leq u_i$, the variable is given a new value inside the feasible interval, as follows:

$$x_i = \begin{cases} l_i, & \text{if } u_i = l_i \\ l_i + \frac{1}{2}(u_i - l_i), & \text{if } u_i - l_i < 4 \\ l_i + \frac{1}{10}(u_i - l_i), & \text{if } u_i - l_i \geq 4 \end{cases}$$

  - If the variable x_i, $i = 1, \ldots, n$, violates a one-sided bound constraint $l_i \leq x_i$ or $x_i \leq u_i$, the variable is given a new value near the violated bound, as follows:

$$x_i = \begin{cases} l_i + \max\{1, \frac{1}{10}l_i\}, & \text{if } x_i < l_i \\ u_i - \max\{1, \frac{1}{10}u_i\}, & \text{if } x_i > u_i \end{cases}$$

- When general linear constraints are specified, the scheme to find a feasible starting point involves two algorithms that 1) find a feasible point independent of the starting point or 2) find a feasible point closest to the starting point. Both algorithms are active set methods.

Line-Search Method

At each iteration k, the conjugate gradient (CONGRA), Newton-type (NEWTYP) and quasi-Newton (QUANEW) optimization techniques use iterative line-search algorithms. These algorithms try to optimize a quadratic or cubic approximation of some merit function along the search direction s^k by computing an approximately optimal step length α^k that is used as follows:

$$x^{k+1} = x^k + \alpha^k s^k, \quad \alpha^k > 0$$

A line-search algorithm is an iterative process that optimizes a nonlinear function of one variable α within each iteration k of the main optimization algorithm, which itself tries to optimize a quadratic approximation of the nonlinear objective function $f(x)$. Since the outside iteration process is based only on the approximation of the objective function, the inside iteration of the line-search algorithm does not have to be perfect. Usually the appropriate choice of α is one that significantly reduces (in the case of minimization) the objective function value. Criteria often used for termination of line-search algorithms are the Goldstein conditions; see Fletcher (1987).

The line-search method in the NLPC solver is implemented as described in Fletcher (1987).

Computational Problems

First-Iteration Overflows

If you provide bad initial values for the decision variables, the computation of the value of the objective function (and its derivatives) can lead to arithmetic overflows in the first iteration. The line-search algorithms that work with cubic extrapolation are especially sensitive to arithmetic overflows. Hence the starting point x^0 must be a point at which all the functions involved in your problem can be evaluated. In the event of arithmetic overflow, consider the following corrective actions:

- Provide a new set of initial values.
- Scale the variables or functions.
- If possible, use bound or linear constraints to avoid regions where overflows can happen.

Problems in Evaluating the Objective or Constraint Functions

During the optimization process, the algorithm might iterate to a point x^k where the objective function or nonlinear constraint functions and their derivatives cannot be evaluated. If you can identify the problematic region, you can prevent the algorithm from stepping into it by adding some bound or linear constraints to the problem, provided that adding these constraints does not alter the characteristics of the problem. As a result, the optimization algorithm reduces the step length and stays closer to the points at which the functions and their derivatives can be evaluated successfully.

Numerical difficulty is also often encountered when evaluation of the objective function or nonlinear constraints results in undesirably large function values. In this case, a possible remedy is to scale the objective functions or nonlinear constraints so that undesirably large function values do not occur.

Precision of Solution

In some applications the NLPC solver can return solutions that are not precise enough. Usually this means that the algorithm terminated too early at a point too far from the optimal point. The termination criteria define the size of the termination

region around the optimal point. Any point inside this region can be accepted for terminating the optimization process. The default values of the termination criteria are set to satisfy a reasonable compromise between the computational effort (CPU time) and the precision of the solutions for the most common applications. However, there can be circumstances in which the default values of the termination criteria specify a region that is either too large or too small. If the termination region is too large, it can contain points with low precision. In such cases where the messages in the SAS log usually give some hints, you can often obtain a solution with higher precision simply by specifying a smaller value for the termination criterion that was satisfied during the previous run.

If the termination region is too small, the optimization process can take longer to find a point inside such a region or might not even find such a point due to rounding errors in function values and derivatives. This can happen when finite-difference approximations of derivatives are used or when the RELOPTTOL= or ABSOPTTOL= termination criteria are too small with respect to rounding errors in the gradient. In such cases, try specifying larger values for the RELOPTTOL= or ABSOPTTOL= option.

Iteration Log

The iteration log consists of one line of output containing the most important information for each iteration. By default the iteration log is not displayed, but the user can use the PRINTFREQ= option to display it.

The following columns are common to all four optimization techniques. The words in parentheses indicate the column headings in long form; the words in angle brackets indicate the column headings in short form.

- iteration number (Iter)
- number of function calls (Function Calls) <Func Calls>
- value of the objective function (Objective Value) <Obj Value>
- relative optimality error as described in the section "Optimality Control" on page 918 (Optimality Error) <Optim Error>

In addition to the preceding common columns, there are columns specific to an optimization technique, as follows.

- For the conjugate gradient (CONGRA) method, the following three additional columns are displayed:
  - step size (Step Size)
  - slope of the search direction, i.e., inner product of the gradients of the objective function and the search direction (Slope of Search Direction) <Slope Search Direc>
  - number of iteration restarts (Restarts) <Rest>

- For the Newton-type (NEWTYP) method, the following two additional columns are displayed:
  - step size (Step Size)
  - slope of the search direction (Slope of Search Direction) <Slope Search Direc>
- For the trust region (TRUREG) method, the following additional column is displayed:
  - trust region radius (Trust Region Radius)
- For the quasi-Newton (QUANEW) method, the columns displayed depend on whether the problem has nonlinear constraints or not. Without nonlinear constraints, the same two additional columns as in the Newton-type (NEWTYP) method are displayed. If there are nonlinear constraints, the following three additional columns are displayed:
  - relative infeasibility as described in the section "Infeasibility" on page 919 (Infeasibility) <Infeas>
  - step size (Step Size)
  - predicted function reduction using the quadratic model (Predicted Function Reduction) <Pred Func Reduc>

By default, the SAS list output is accumulated to a full page before it is displayed in the output window, so you do not see the iteration log immediately as the iteration proceeds. In order to view the iteration log immediately after every iteration, you need to set the AUTO SCROLL option to 1 in the SAS session. You can do this in either of the following two ways:

- Select a list output window. Then enter the command **ascroll 1** in the command line window.
- In the output window, select **Tools ▶ Options ▶ Preferences**. In the Preferences window, click the **Advanced** tab and select **Scroll lines** from the **Output** group. Type 1 in the list box, and click **OK** to close the window.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put _OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the NLPC solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally

SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit
FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

ABS_OPTIMALITY_ERROR
indicates the absolute optimality error at the solution. See the section "Optimality Control" on page 918 for details.

REL_OPTIMALITY_ERROR
indicates the relative optimality error at the solution. See the section "Optimality Control" on page 918 for details.

ABS_INFEASIBILITY
indicates the absolute infeasibility at the solution. See the section "Infeasibility" on page 919 for details.

REL_INFEASIBILITY
indicates the relative infeasibility at the solution. See the section "Infeasibility" on page 919 for details.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Examples

Example 10.1. Least-Squares Problem

Although the current release of the NLPC solver does not implement techniques specialized for least-squares problems, this example illustrates how the NLPC solver can solve least-squares problems by using general nonlinear optimization techniques. The following Bard function (see Moré, Garbow, and Hillstrom 1981) is a least-squares problem with 3 parameters and 15 residual functions f_k:

$$f(x) = \frac{1}{2}\sum_{k=1}^{15} f_k^2(x), \quad x = (x_1, x_2, x_3)$$

where

$$f_k(x) = y_k - \left(x_1 + \frac{u_k}{v_k x_2 + w_k x_3}\right)$$

with $u_k = k$, $v_k = 16 - k$, $w_k = \min\{u_k, v_k\}$, and

$$y = (.14, .18, .22, .25, .29, .32, .35, .39, .37, .58, .73, .96, 1.34, 2.10, 4.39)$$

The minimum function value $f(x^*) = 4.107\text{E}{-3}$ is at the point $x^* = (0.08, 1.13, 2.34)$. The starting point $x^0 = (1, 1, 1)$ is used.

You can use the following SAS code to formulate and solve this least-squares problem:

```
proc optmodel;
   set S = 1..15;
   number u{k in S} = k;
   number v{k in S} = 16 - k;
   number w{k in S} = min(u[k], v[k]);
   number y{S} = [ .14 .18 .22 .25 .29 .32 .35 .39 .37 .58
                   .73 .96 1.34 2.10 4.39 ];
   var x{1..3} init 1;

   min f = 0.5*sum{k in S} ( y[k] -
             ( x[1] + u[k]/(v[k]*x[2] + w[k]*x[3]) )
           )^2;

   solve with nlpc / printfreq=1;
   print x;
quit;
```

A problem summary is displayed in Output 10.1.1. Since there is no explicit optimization technique specified (using the TECH= option), the default algorithm of the trust region method is used. Output 10.1.2 displays the iteration log. The solution summary and the solution are shown in Output 10.1.3.

Output 10.1.1. Least-Squares Problem Solved with TRUREG: Problem Summary

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Nonlinear
Number of Variables	3
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	3
Fixed	0
Number of Constraints	0

Output 10.1.2. Least-Squares Problem Solved with TRUREG: Iteration Log

Iteration Log: Trust Region Method

Iter	Function Calls	Objective Value	Optimality Error	Trust Region Radius
0	0	20.8408	1.2445	.
1	2	2.8333	2.7031	1.000
2	3	0.6302	2.3180	0.989
3	4	0.1077	0.7732	0.998
4	5	0.009011	0.1396	1.007
5	6	0.004134	0.0125	1.042
6	7	0.004107	0.0000805	0.199
7	8	0.004107	4.3523E-8	0.0207

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.

Output 10.1.3. Least-Squares Problem Solved with TRUREG: Solution

Solution Summary

Solver	NLPC/Trust Region
Objective Function	f
Solution Status	Optimal
Objective Value	0.0041074
Iterations	7
Absolute Optimality Error	4.3523352E-8
Relative Optimality Error	4.3523352E-8

[1]	x
1	0.082411
2	1.133036
3	2.343696

Alternatively, you can specify the Newton-type method with line search by using the following statement:

```
solve with nlpc / tech=newtyp printfreq=1;
```

You get the output for the NEWTYP method as shown in Output 10.1.4 and Output 10.1.5.

Output 10.1.4. Least-Squares Problem Solved with NEWTYP: Iteration Log

Iteration Log: Newton-Type Method with Line Search

Iter	Function Calls	Objective Value	Optimality Error	Step Size	Slope of Search Direction
0	0	20.8408	1.2445	.	.
1	2	2.2590	2.6880	1.000	-35.485
2	3	0.6235	2.3158	1.000	-2.577
3	4	0.1116	0.7488	1.000	-0.831
4	5	0.0115	0.1703	1.000	-0.172
5	6	0.004188	0.0167	1.000	-0.0136
6	7	0.004109	0.000427	1.000	-0.0002
7	8	0.004109	0.0000637	1.000	-578E-9
8	9	0.004108	0.0000352	1.000	-95E-8
9	10	0.004107	8.7568E-6	1.000	-52E-8
10	11	0.004107	1.1411E-6	1.000	-33E-9
11	12	0.004107	8.2006E-8	1.000	-63E-11

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.

Output 10.1.5. Least-Squares Problem Solved with NEWTYP: Solution

Solution Summary

Solver	NLPC/Newton-Type
Objective Function	f
Solution Status	Optimal
Objective Value	0.0041074
Iterations	11
Absolute Optimality Error	8.2005644E-8
Relative Optimality Error	8.2005644E-8

[1]	x
1	0.082411
2	1.133059
3	2.343674

You can also select the conjugate gradient method as follows:

```
solve with nlpc / tech=congra printfreq=2;
```

Note that the PRINTFREQ= option was used to reduce the number of rows in the iteration log. As Output 10.1.6 shows, only every other iteration is displayed. Output 10.1.7 gives a summary of the solution and prints the solution.

Output 10.1.6. Least-Squares Problem Solved with CONGRA: Iteration Log

Iteration Log: Conjugate Gradient Method

Iter	Function Calls	Objective Value	Optimality Error	Step Size	Slope of Search Direction	Restarts
0	0	20.8408	1.2445	.	.	0
2	5	0.7199	1.1503	0.108	−11.045	1
4	11	0.0119	0.4362	0.150	−0.0010	2
6	15	0.004903	0.008169	35.849	−0.0001	3
8	20	0.004788	0.0819	22.628	−0.0001	4
10	25	0.004233	0.0436	0.679	−0.0034	6
12	29	0.004201	0.0191	0.345	−0.0001	7
14	33	0.004166	0.0161	20.000	−129E−7	8
16	37	0.004160	0.003508	0.112	−205E−8	9
18	41	0.004108	0.001920	9.125	−108E−7	10
20	45	0.004108	0.000637	2.000	−178E−9	12
22	49	0.004107	0.000581	0.291	−245E−9	13
24	53	0.004107	2.5337E−6	2.000	−62E−13	14
25	55	0.004107	7.4262E−8	0.142	−15E−12	14

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E−6) are satisfied.

Output 10.1.7. Least-Squares Problem Solved with CONGRA: Solution

Solution Summary

Solver	NLPC/Conjugate Gradient
Objective Function	f
Solution Status	Optimal
Objective Value	0.0041074
Iterations	25
Absolute Optimality Error	7.4261895E−8
Relative Optimality Error	7.4261895E−8

[1]	x
1	0.082411
2	1.133038
3	2.343693

Although the number of iterations required for each optimization technique to converge varies, all three techniques produce the identical solution, given the same starting point.

Example 10.2. Maximum Likelihood Weibull Model

This model is obtained from Lawless (1982, pp. 190–194). Suppose you want to find the maximum likelihood estimates for the three-parameter Weibull model. The observed likelihood function is

$$L(\mu,\alpha,\beta) = \frac{\beta^r}{\alpha^r}\left[\prod_{i\in D}\left(\frac{t_i-\mu}{\alpha}\right)^{\beta-1}\right]\prod_{i=1}^{n}\exp\left[-\left(\frac{t_i-\mu}{\alpha}\right)^{\beta}\right]$$

where n is the number of individuals involved in the experiment, D is the set of individuals whose lifetimes are observed, and $r = |D|$. Then the log-likelihood function is

$$l(\mu,\alpha,\beta) = r\log\beta - r\beta\log\alpha + (\beta-1)\sum_{i\in D}\log(t_i-\mu) - \sum_{i=1}^{n}\left(\frac{t_i-\mu}{\alpha}\right)^{\beta}$$

Note that for $\beta < 1$, the logarithmic terms become infinite as $\mu \uparrow \min_{i\in D}(t_i)$. That is, $l(\mu,\alpha,\beta)$ is unbounded. Thus our interest is restricted to β values greater than or equal to 1. Further, for the logarithmic terms to be defined, it is required that $\alpha > 0$ and $\mu < \min_{i\in D}(t_i)$.

The following data from Pike (1966) represent the number of days it took the rats painted with the carcinogen DMBA to develop carcinomas:

143, 164, 188, 188, 190, 192, 206, 209, 213, 216, 220, 227,
230, 234, 246, 265, 304, 216*, 244*

Note that the last two observations have an asterisk next to them. This indicates that those two rats did not develop carcinomas. These are referred to as *censored* data.

For these data, a local maximum of $l(\mu,\alpha,\beta)$ occurs at $\mu^* = 122$, $\alpha^* = 108.4$, and $\beta^* = 2.712$. You can create the input data set by using the following SAS code:

```
data pike;
   input days cens @@;
   datalines;
143  0  164  0  188  0  188  0
190  0  192  0  206  0  209  0
213  0  216  0  220  0  227  0
230  0  234  0  246  0  265  0
304  0  216  1  244  1
;
```

The first $r = 17$ observations in the data set correspond to the rats that develop carcinomas; $\min_{i\in D}(t_i) = 143$ in this case. Using the preceding data, you can formulate the Weibull model as follows:

```
proc optmodel;
   set S;  /* set of rats in the experiment */
   set D;  /* set of rats that develop carcinomas */
   number r = card(D);
   number t{S};
   number cens{S};
   var alpha >= 0 init 1;
   var beta  >= 1 init 1;
   var mu >= 0 <= min{i in D} t[i] init 10;
   max logl = r*log(beta) - r*beta*log(alpha)
              + (beta - 1)*sum{i in D} log(t[i] - mu)
              - sum{i in S} ((t[i] - mu)/alpha)^beta;

   read data pike into S=[_n_] t=days cens;
   D = {i in S : cens[i] = 0};

   solve with nlpc / printfreq=1;
   print mu alpha beta;
quit;
```

Assume a starting point at $\alpha^0 = 1$, $\beta^0 = 1$, and $\mu^0 = 10$. The data for all the rats in the experiment are read using the READ statement. The following statement subsets D to the set of rats that develop carcinomas:

```
D = {i in S : cens[i] = 0};
```

The NLPC solver invokes the default optimization technique (the trust region method) to solve this problem.

Output 10.2.1 through Output 10.2.3 show the problem summary, iteration log, solution summary, and solution. As you can see, the solution obtained by the trust region method matches closely with the local maximum given in Lawless (1982, p. 193).

Output 10.2.1. Maximum Likelihood Weibull Model: Problem Summary

Problem Summary

Objective Sense	Maximization
Objective Function	logl
Objective Type	Nonlinear
Number of Variables	3
Bounded Above	0
Bounded Below	2
Bounded Below and Above	1
Free	0
Fixed	0
Number of Constraints	0

Output 10.2.2. Maximum Likelihood Weibull Model: Iteration Log

Iteration Log: Trust Region Method

Iter	Function Calls	Objective Value	Optimality Error	Trust Region Radius
0	0	−3905	0.1874	.
1	2	−2606	0.1357	1.000
2	3	−1747	0.0990	1.015
3	4	−1235	0.0754	0.954
4	5	−959.2572	0.0620	2.060
5	6	−670.6203	0.0472	1.910
6	7	−520.6900	0.0390	3.942
7	8	−371.3259	0.0305	3.850
8	9	−292.0677	0.0256	3.987
9	10	−219.7420	0.0214	7.985
10	11	−179.8333	0.0192	8.222
11	13	−144.7603	0.0196	18.466
12	14	−125.6711	0.0248	19.148
13	18	−106.4779	1.0000	11.525
14	22	−88.1397	1.0000	15.904
15	23	−87.4005	0.4041	15.904
16	24	−87.3426	0.0115	4.288
17	25	−87.3270	0.006032	4.355
18	26	−87.3243	0.002287	4.355
19	27	−87.3242	0.0000486	2.666
20	28	−87.3242	1.4815E−8	0.359

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E−6) are satisfied.

Output 10.2.3. Maximum Likelihood Weibull Model: Solution

Solution Summary

Solver	NLPC/Trust Region
Objective Function	logl
Solution Status	Optimal
Objective Value	−87.324
Iterations	20
Absolute Optimality Error	1.4815454E−8
Relative Optimality Error	1.4815454E−8
Absolute Infeasibility	0
Relative Infeasibility	0

mu	alpha	beta
122.03	108.38	2.7115

Example 10.3. Simple Pooling Problem

Consider Example 4.7 in the PROC NLP documentation. In this simple pooling problem, two liquid chemicals, X and Y, are produced by the pooling and blending of three input liquid chemicals, A, B, and C. There are three group of constraints: 1) the mass balance constraints, which are linear constraints; 2) the sulfur concentration constraints, which are nonlinear constraints; and 3) the bound constraints. The objective is to maximize the profit.

You can formulate this problem in PROC OPTMODEL as follows:

```
proc optmodel;
   num costa = 6, costb = 16, costc = 10,
       costx = 9, costy = 15;
   var amountx init 1 >= 0 <= 100,
       amounty init 1 >= 0 <= 200;
   var amounta init 1 >= 0,
       amountb init 1 >= 0,
       amountc init 1 >= 0;
   var pooltox init 1 >= 0,
       pooltoy init 1 >= 0;
   var ctox init 1 >= 0,
       ctoy init 1 >= 0;
   var pools init 1 >=1 <= 3;

   max f = costx*amountx + costy*amounty
           - costa*amounta - costb*amountb - costc*amountc;
   con mb1: amounta + amountb = pooltox + pooltoy,
       mb2: pooltox + ctox    = amountx,
       mb3: pooltoy + ctoy    = amounty,
       mb4: ctox    + ctoy    = amountc;
   con sc1: 2.5*amountx - pools*pooltox - 2*ctox >= 0,
       sc2: 1.5*amounty - pools*pooltoy - 2*ctoy >= 0,
       sc3: 3*amounta + amountb - pools*(amounta + amountb) = 0;

   solve with nlpc / tech=quanew printfreq=1;
   print amountx amounty amounta amountb amountc;
quit;
```

The quantities amounta, amountb, amountc, amountx, and amounty are the amounts to be determined for the liquid chemicals A, B, C, X, and Y, respectively. pooltox, pooltoy, ctox, ctoy, and pools are intermediate decision variables, whose values are not printed in the solution. The constraints **mb1**–**mb4** are the mass balance constraints, and the constraints **sc1**–**sc3** are the sulfur concentration constraints. For more information about converting PROC NLP models into PROC OPTMODEL models, see the section "Rewriting NLP Models for PROC OPTMODEL" on page 788.

To solve the optimization problem, the experimental quasi-Newton method (QUANEW) is invoked. The problem summary is shown in Output 10.3.1. Output 10.3.2 displays the iteration log.

Output 10.3.1. Simple Pooling Problem: Problem Summary

Problem Summary

Objective Sense	Maximization
Objective Function	f
Objective Type	Linear
Number of Variables	10
Bounded Above	0
Bounded Below	7
Bounded Below and Above	3
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	0
Linear EQ (=)	4
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	1
Nonlinear GE (>=)	2
Nonlinear Range	0

Output 10.3.2. Simple Pooling Problem: Iteration Log

Iteration Log: Quasi-Newton Method with BFGS Update

Iter	Function Calls	Objective Value	Infeasibility	Optimality Error	Step Size	Predicted Function Reduction
0	0	3.8182	0.6818	0.0527	.	.
1	4	-2.6893	0.1731	0.2383	1.000	3.0428
2	5	-1.9376	0.0709	0.2202	1.000	4.6944
3	6	1.2603	0.0541	0.0566	1.000	1.5883
4	7	2.4293	0.008492	0.4107	1.000	1.1871
5	8	3.2903	2.107E-10	0.1666	1.000	2.5714
6	9	5.0987	3.161E-10	0.3283	1.000	4.9682
7	10	10.0669	2.107E-10	2.6895	1.000	63.9047
8	12	28.3332	0.4785	8.3364	0.286	115.6
9	13	103.3932	2.0648	31.3704	1.000	112.7
10	14	159.4439	7.105E-15	6.9418	1.000	76.2665
11	15	235.7103	0.4503	18.6325	1.000	92.5610
12	16	322.9693	0.0401	1.0835	1.000	149.4
13	17	397.9146	0.0823	0.5481	1.000	0.8634
14	18	397.9584	0.0000169	1.2472	1.000	0.2766
15	19	398.2348	0.000734	0.8181	1.000	2.4087
16	20	400.0000	0.0301	0.0263	1.000	0.3617
17	21	400.0000	7.216E-7	5.2031E-8	1.000	9.708E-6

NOTE: Optimality criteria (ABSOPTTOL=0.001, RELOPTTOL=1E-6) are satisfied.

Output 10.3.3 displays the solution summary and the solution. The optimal solution shows that to obtain a maximum profit of 400, 200 units of Y should be produced by blending 100 units of B and 100 units of C. Liquid A should not be used for the blending. Further, liquid X should not be produced at all.

Output 10.3.3. Simple Pooling Problem: Solution

Solution Summary

Solver	NLPC/Quasi-Newton
Objective Function	f
Solution Status	Optimal
Objective Value	400
Iterations	17
Absolute Optimality Error	8.3249025E-7
Relative Optimality Error	5.2030641E-8
Absolute Infeasibility	7.2159716E-7
Relative Infeasibility	7.2159716E-7

amountx	amounty	amounta	amountb	amountc
-8.8566E-08	200	-1.0537E-10	100	100

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Chapter 11
The Unconstrained Nonlinear Programming Solver

Chapter Contents

Chapter 11
The Unconstrained Nonlinear Programming Solver

Overview

The unconstrained nonlinear programming solver (NLPU) is a component of the OPTMODEL procedure, and it can be used for solving general unconstrained nonlinear programming (NLP) problems.

Mathematically, an unconstrained nonlinear programming problem can be stated as follows:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

where $f(x)$: $\mathbb{R}^n \mapsto \mathbb{R}$ is the nonlinear objective function and $x \in \mathbb{R}^n$ is the decision variable.

For purely unconstrained optimization, PROC OPTMODEL implements the following algorithms:

- Fletcher-Reeves nonlinear conjugate gradient algorithm for convex unconstrained optimization
- Polak-Ribière nonlinear conjugate gradient algorithm for convex unconstrained optimization
- Limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm for general nonconvex unconstrained optimization (default)

If the objective function is *convex*, then the optimal solution is a global optimum; otherwise it is a local optimum. In other words, the optimal solution obtained by the NLPU solver depends on the starting point. An example of a nonlinear function with multiple local optima is displayed in Figure 11.1.

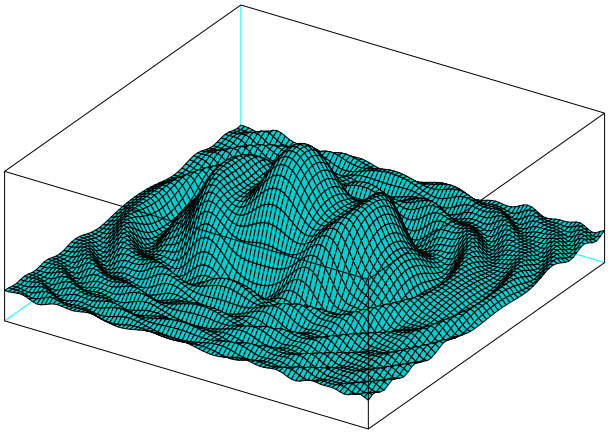


Figure 11.1. An Example of Multiple Local Optimal Points

The function displayed in Figure 11.1 is

$$f(x,y) = \text{sinc}\left(x^2+y^2\right) + \text{sinc}\left((x-2)^2+y^2\right)$$

where sinc(·), also called the "sampling function," is a function that arises frequently in signal processing. The function is defined as

$$\text{sinc}(x) \equiv \begin{cases} 1 & \text{for } x = 0, \\ \dfrac{\sin x}{x} & \text{otherwise} \end{cases}$$

Getting Started

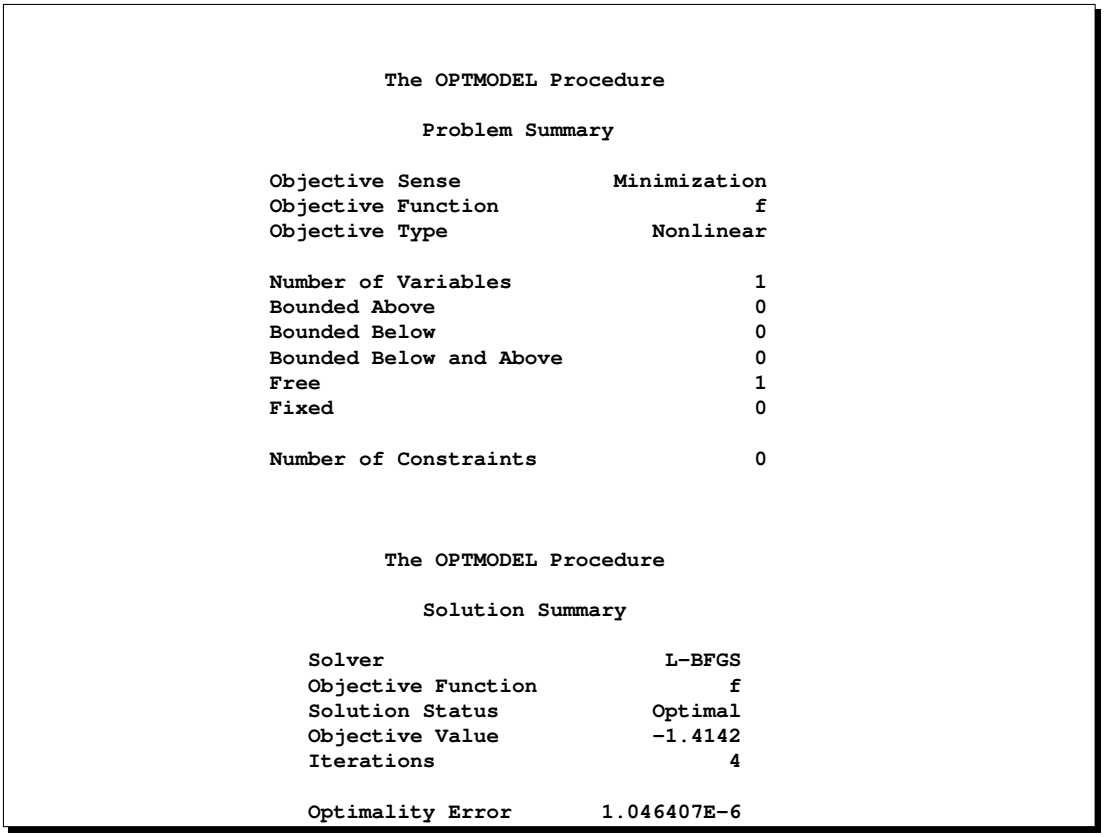
Consider a simple nonlinear problem as follows:

$$\min f = \sin(x) + \cos(x)$$

You can formulate and solve the problem by using PROC OPTMODEL as follows:

```
proc optmodel;
  var x;

  minimize f = sin(x) + cos(x);

  solve with nlpu / tech = lbfgs;

quit;
```

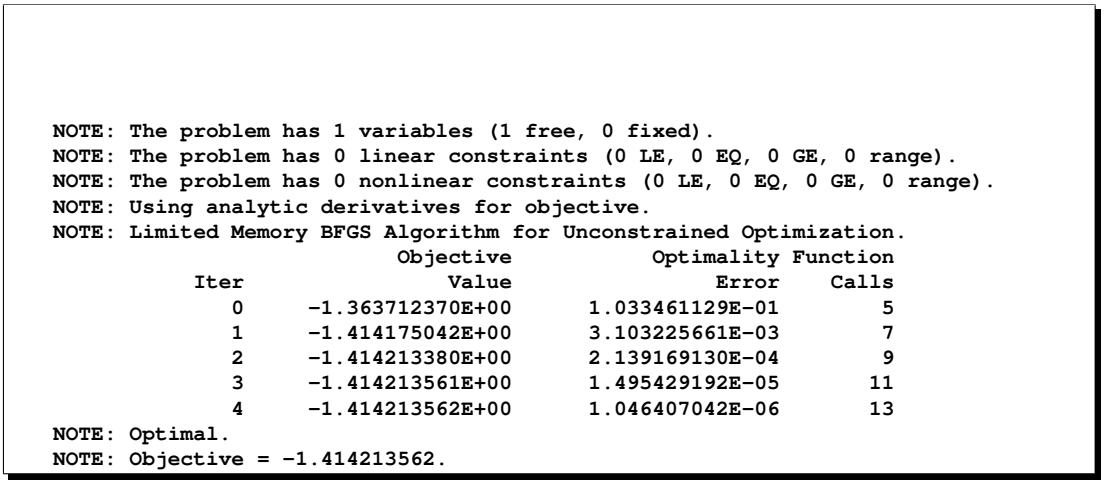
A problem summary and a solution summary are displayed in Figure 11.2.



```
                    The OPTMODEL Procedure

                       Problem Summary

              Objective Sense            Minimization
              Objective Function                    f
              Objective Type                Nonlinear

              Number of Variables                   1
              Bounded Above                         0
              Bounded Below                         0
              Bounded Below and Above               0
              Free                                  1
              Fixed                                 0

              Number of Constraints                 0



                    The OPTMODEL Procedure

                       Solution Summary

                 Solver                        L-BFGS
                 Objective Function                 f
                 Solution Status              Optimal
                 Objective Value              -1.4142
                 Iterations                         4

                 Optimality Error       1.046407E-6
```

Figure 11.2. Optimal Solution of the Example Problem

The iteration log is shown in Figure 11.3.



```
NOTE: The problem has 1 variables (1 free, 0 fixed).
NOTE: The problem has 0 linear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: Using analytic derivatives for objective.
NOTE: Limited Memory BFGS Algorithm for Unconstrained Optimization.
                          Objective           Optimality Function
           Iter               Value                Error    Calls
              0     -1.363712370E+00      1.033461129E-01        5
              1     -1.414175042E+00      3.103225661E-03        7
              2     -1.414213380E+00      2.139169130E-04        9
              3     -1.414213561E+00      1.495429192E-05       11
              4     -1.414213562E+00      1.046407042E-06       13
NOTE: Optimal.
NOTE: Objective = -1.414213562.
```

Figure 11.3. Iteration Log

Syntax

To invoke the NLPU solver, use the following statement:

SOLVE WITH NLPU / *options* **;**

Functional Summary

Table 11.1 outlines the options that can be used with the SOLVE WITH NLPU statement.

Table 11.1. Options for the NLPU Solver

Description	Option
Optimization Option:	
optimization technique	TECH=
Solver Options:	
unboundedness detection parameter	OBJLIMIT=
number of Hessian corrections	LBFGSCORR=
Armijo parameter for line search	LSARMIJO=
Wolfe parameter for line search	LSWOLFE=
Termination Criterion Options:	
termination based on relative gradient norm	OPTTOL=
maximum number of line-search iterations	LSMAXITER=
maximum number of iterations	MAXITER=
upper limit (in seconds) on the CPU time for optimization	MAXTIME=
Printed Output Option:	
display iteration log	PRINTFREQ=

NLPU Solver Options

The following options are supported by the NLPU solver.

LBFGSCORR=*n*

specifies the number of Hessian corrections for the L-BFGS algorithms. The default value is 7 corrections.

LSARMIJO=c_1

specifies the Armijo parameter $c_1 \in (0, 1)$ for the line-search technique. The default value is 2.0E–3. See the section "Line-Search Algorithm" on page 945 for details.

Note: The value of the LSARMIJO= option must be less than the value of the LSWOLFE= option.

LSWOLFE=c_2
specifies the Wolfe parameter $c_2 \in (0, 1)$ for the line-search technique. The default value is 0.7. See the section "Line-Search Algorithm" on page 945 for details.

Note: The value of the LSWOLFE= option must be greater than the value of the LSARMIJO= option.

LSMAXITER=*N*
specifies the maximum number of line-search iterations within one L-BFGS or conjugate gradient method. It can also be viewed as the maximum number of objective function evaluations. The default value is 200 iterations.

MAXITER=*N*
specifies that the optimization process stop after a maximum of N iterations. The default value is 100,000 iterations.

MAXTIME=r
specifies an upper limit of r seconds of real time for the optimization process. If you do not specify this option, the procedure does not stop based on the amount of time elapsed. Note that the time specified by the MAXTIME= option is checked only at the end of each iteration. The optimization terminates when the actual running time is greater than or equal to r.

OPTTOL=ε
specifies the desired tolerance of the relative gradient norm, which is defined as follows:

$$\frac{\|\nabla f(x)\|_2}{\max\left(\|x\|_2, 1\right)}, \quad x \in \mathbb{R}^n$$

The solver terminates when it finds a point where the relative gradient norm of the objective function is less than ε. See the section "Conditions of Optimality" on page 944 for details. The default value of this option is 1.0E–6.

OBJLIMIT=*M*
specifies an upper limit on the magnitude of the objective value. For a minimization problem, the algorithm terminates when the objective value becomes less than $-M$; for a maximization problem, the algorithm stops when the objective value exceeds M. When this happens, it implies that either the problem is unbounded or the algorithm diverges. If optimization were allowed to continue, numerical difficulty might be encountered. The default value is $M = 1.0\text{E}{+}20$.

PRINTFREQ=*j*
specifies that the printing of the solution progress to the iteration log should occur after every j iterations. The print frequency, j, is an integer greater than or equal to zero. The value $j = 0$ disables the printing of the progress of the solution. Note that the first and last iterations are also displayed. The default value for this option is 1.

TECH=*keyword*
TECHNIQUE=*keyword*
SOLVER=*keyword*
specifies the optimization technique. Valid keywords are as follows:

- FLETREEV
  uses the Fletcher-Reeves nonlinear conjugate gradient algorithm. See Fletcher (1987) for details.
- LBFGS
  uses the limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm. This is the default solver. See the section "Broyden-Fletcher-Goldfarb-Shanno (BFGS) Algorithm" on page 946 for details.
- POLRIB
  uses the Polak-Ribière nonlinear conjugate gradient algorithm. See Fletcher (1987) for details.

Details

Conditions of Optimality

Before beginning the discussion, we present the following notation for easy reference:

n	dimension of x, i.e., the number of decision variables
x	iterate, i.e., the vector of n decision variables
$f(x)$	objective function
$\nabla f(x)$	gradient of the objective function
$\nabla^2 f(x)$	Hessian matrix of the objective function

Denote the feasible region as $\mathcal{F}$. In unconstrained problems, any point $x \in \mathbb{R}^n$ is a feasible point. Therefore, the set $\mathcal{F}$ is the entire $\mathbb{R}^n$ space.

A point x^* is a *local solution* of the problem if there exists a neighborhood $\mathcal{N}$ of x^* such that

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

Further, a point x^* is a *strict local solution* if strict inequality holds in the preceding case, i.e.,

$$f(x) > f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

A point $x^* \in \mathbb{R}^n$ is a global solution of the problem if no point in $\mathcal{F}$ has a smaller function value than $f(x^*)$, i.e.,

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{F}$$

All the algorithms in the NLPU solver find a local minimum of an optimization problem.

Unconstrained Optimization

The following conditions hold for unconstrained optimization problems:

- **First-order necessary conditions**: If x^* is a local solution and $f(x)$ is continuously differentiable in some neighborhood of x^*, then

$$\nabla f(x^*) = 0$$

- **Second-order necessary conditions**: If x^* is a local solution and $f(x)$ is twice continuously differentiable in some neighborhood of x^*, then $\nabla^2 f(x^*)$ is positive semidefinite.
- **Second-order sufficient conditions**: If $f(x)$ is twice continuously differentiable in some neighborhood of x^*, and $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local solution.

Line-Search Algorithm

The NLPU solver implements an inexact line-search algorithm. Given a search direction $d \in \mathbb{R}^n$ (produced, e.g., by a quasi-Newton method), each iteration of the line search selects an appropriate step length α, which would in some sense approximate α^*, an optimal solution to the following problem:

$$\alpha^* = \arg \min_{\alpha \geq 0} \; f(x + \alpha d)$$

Let us define $\phi(\alpha)$ as

$$\phi(\alpha) \equiv f(x + \alpha d)$$

During subsequent line-search iterations, objective function values $\phi(\alpha_{i-1})$, $\phi(\alpha_i)$ and their gradients $\phi'(\alpha_{i-1})$, $\phi'(\alpha_i)$ are used to construct a cubic polynomial interpolation, whose minimizer α_{i+1} over $[\alpha_{i-1}, \alpha_i]$ gives the next iterate step length.

An early (economical) line-search termination criterion is given by strong Wolfe conditions:

$$\begin{aligned} f(x + \alpha d) &\leq f(x) + c_1 \alpha \langle f'(x), d \rangle \\ |\langle f'(x + \alpha d), d \rangle| &\leq c_2 |\langle f'(x), d \rangle| \end{aligned}$$

where c_1 is a sufficient decrease condition constant, known as *Armijo's* constant, and c_2 is a strong curvature condition constant, known as *Wolfe's* constant. If $f(\cdot)$ is bounded below, and d is a descent direction at x (such that $\langle f'(x), d \rangle < 0$), then there is always a step length α^*, which satisfies strong Wolfe conditions indicated previously.

Broyden-Fletcher-Goldfarb-Shanno (BFGS) Algorithm

The NLPU solver implements large-scale limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithms (recursive and matrix forms). The matrix form is used for bound-constrained optimization, while the recursive loop is used for unconstrained optimization.

Recursive Hessian matrix update:

$$
\begin{aligned}
H_k = {} & (V_{k-1}^{\mathrm{T}} \dots V_{k-m}^{\mathrm{T}}) H_k^0 (V_{k-m} \dots V_{k-1}) + \\
& \rho_{k-m} (V_{k-1}^{\mathrm{T}} \dots V_{k-m+1}^{\mathrm{T}}) s_{k-m} s_{k-m}^{\mathrm{T}} (V_{k-m+1} \dots V_{k-1}) + \\
& \rho_{k-m+1} (V_{k-1}^{\mathrm{T}} \dots V_{k-m+2}^{\mathrm{T}}) s_{k-m+1} s_{k-m+1}^{\mathrm{T}} (V_{k-m+2} \dots V_{k-1}) + \\
& \dots \\
& \rho_{k-1} s_{k-1} s_{k-1}^{\mathrm{T}}
\end{aligned}
$$

Compact matrix update:

$$
\begin{aligned}
H_k = {} & \gamma_k I + \begin{bmatrix} S_k & \gamma_k Y_k \end{bmatrix} \times \\
& \times \begin{bmatrix} R_k^{-T} (D_k + \gamma_k Y_k^{\mathrm{T}} Y_k) R_k^{-1} & -R^{-T} \\ -R^{-1} & 0 \end{bmatrix} \begin{bmatrix} S_k^{\mathrm{T}} \\ \gamma_k Y_k^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put _OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the NLPU solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit
FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

OPTIMALITY_ERROR
relative gradient norm at the solution.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Examples

Example 11.1. Solving a Highly Nonlinear Problem

Consider the following example of minimizing a nonlinear function of three variables, x, y, and z:

$$\min_{x,y,z} \quad x^2 + e^{\frac{y}{10}+10} + \sin(y\,z)$$

You can use the following SAS code to formulate and solve the problem in PROC OPTMODEL:

```
proc optmodel;
  var x, y, z;

  minimize obj = x^2 + exp(y/10 + 10) + sin(z*y);

  solve with nlpu / tech = fletreev  maxiter   = 100
                    opttol = 1e-7;
  print x y z;

quit;
```

The optimal solution is displayed in Output 11.1.1.

Output 11.1.1. Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	3
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	3
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	Fletcher-Reeves
Objective Function	obj
Solution Status	Optimal
Objective Value	-1
Iterations	6
Optimality Error	2.714258E-10

	x	y	z
0		-2943.7	0.090182

The following iteration log displays information about the type of algorithm used, and the objective value, relative gradient norm, and number of function evaluations at each iteration.

Output 11.1.2. Iteration Log

```
NOTE: The problem has 3 variables (3 free, 0 fixed).
NOTE: The problem has 0 linear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: Using analytic derivatives for objective.
NOTE: Fletcher-Reeves CG algorithm for unconstrained optimization.
                          Objective           Optimality Function
           Iter               Value                 Error    Calls
              0     3.209635865E-124      9.999999423E-01        5
              1     -7.997986244E-01      6.002683723E-01       16
              2     -9.900967320E-01      1.403868196E-01       30
              3     -9.997379974E-01      2.288966059E-02       41
              4     -9.999998987E-01      4.500075251E-04       48
              5     -9.999999991E-01      4.279638786E-05       52
              6     -1.000000000E+00      2.714258433E-10       55
NOTE: Optimal.
NOTE: Objective = -1.
```

Example 11.2. Solving the Accumulated Rosenbrock Function

Suppose you want to solve the accumulated Rosenbrock function comprising 1000 variables. For ease of understanding the formulation, define $\mathcal{I}$ to be an ordered set of indices of the decision variables and $\mathcal{J}$ to be an ordered set of all indices in $\mathcal{I}$ except the last one. In other words,

$$\begin{aligned} \mathcal{I} &\equiv \{1,\ 2,\ \ldots,\ 1000\} \\ \mathcal{J} &\equiv \{1,\ 2,\ \ldots,\ 999\} \end{aligned}$$

The mathematical formulation of the problem is as follows:

$$\min \quad \sum_{j \in \mathcal{J}} (1 - x_j)^2 + 100(x_{j+1} - x_j^2)^2$$

You can use the following SAS code to formulate and solve the problem by using PROC OPTMODEL:

```
proc optmodel;

  set setI = {1 .. 1000};          /* declare index sets */
  set setJ = setI diff {1000};
  set elementsToPrint = {1 .. 10};

  var x {setI};

  minimize z = sum{j in setJ} ((1 - x[j])^2 +
               100*(x[j + 1] - x[j]^2)^2);
```

```
  solve with nlpu / printfreq = 0;
  print {k in elementsToPrint} x[k];

quit;
```

The Rosenbrock function has a unique global minimizer $(1, \ldots, 1)$. The optimal solution is displayed in Output 11.2.1.

Output 11.2.1. Optimal Solution to the Accumulated Rosenbrock Function

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	z
Objective Type	Nonlinear
Number of Variables	1000
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	1000
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	L-BFGS
Objective Function	z
Solution Status	Optimal
Objective Value	5.1404E-11
Iterations	43
Optimality Error	5.2161487E-6

	[1]
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1

References

Bazaraa, M. S., Sherali, H. D., and Shetty, C. M. (1993), *Nonlinear Programming: Theory and Algorithms*, New York: John Wiley & Sons.

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Chapter 12
The Quadratic Programming Solver (Experimental)

Chapter Contents

Chapter 12
The Quadratic Programming Solver (Experimental)

Overview

The OPTMODEL procedure provides a framework for specifying and solving quadratic programs.

Mathematically, a quadratic programming (QP) problem can be stated as follows:

$$\begin{aligned} \min \quad & \tfrac{1}{2}\,\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

where

$\mathbf{Q} \in \mathbb{R}^{n \times n}$	is the quadratic (also known as Hessian) matrix
$\mathbf{A} \in \mathbb{R}^{m \times n}$	is the constraints matrix
$\mathbf{x} \in \mathbb{R}^{n}$	is the vector of decision variables
$\mathbf{c} \in \mathbb{R}^{n}$	is the vector of linear objective function coefficients
$\mathbf{b} \in \mathbb{R}^{m}$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l} \in \mathbb{R}^{n}$	is the vector of lower bounds on the decision variables
$\mathbf{u} \in \mathbb{R}^{n}$	is the vector of upper bounds on the decision variables

The quadratic matrix $\mathbf{Q}$ is assumed to be symmetric; i.e.,

$$q_{ij} = q_{ji}, \quad \forall i, j = 1, \ldots, n$$

Indeed, it is easy to show that even if $\mathbf{Q} \neq \mathbf{Q}^{\mathrm{T}}$, then the simple modification

$$\tilde{\mathbf{Q}} = \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^{\mathrm{T}})$$

produces an equivalent formulation $\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \mathbf{x}^{\mathrm{T}}\tilde{\mathbf{Q}}\mathbf{x}$; hence symmetry is assumed. When specifying a quadratic matrix it suffices to list only lower triangular coefficients.

In addition to being symmetric, $\mathbf{Q}$ is also required to be positive semidefinite:

$$\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^{n}$$

for minimization type of models; it is required to be negative semidefinite for maximization type of models. Convexity can come as a result of a matrix-matrix multiplication

$$\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$$

or as a consequence of physical laws, etc. See Figure 12.1 for examples of convex, concave, and nonconvex objective functions.

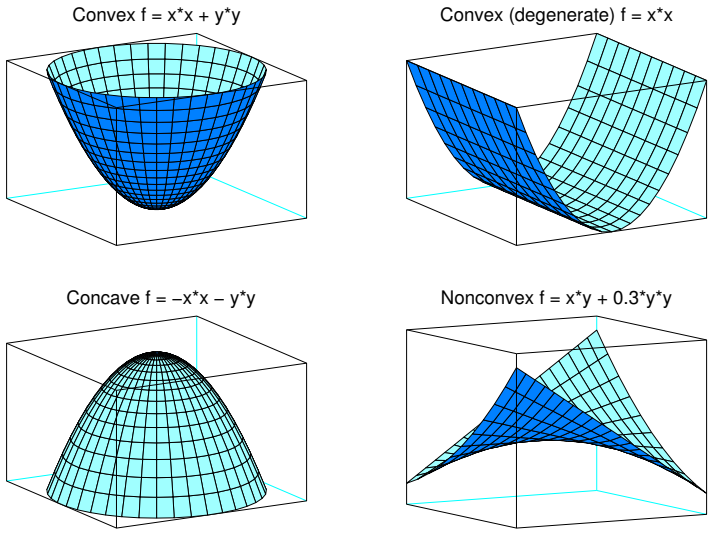

Figure 12.1. Examples of Convex, Concave, and Nonconvex Objective Functions

The order of constraints is insignificant. Some or all components of $\mathbf{l}$ or $\mathbf{u}$ (lower/upper bounds) can be omitted.

Getting Started

Consider a small illustrative example. Suppose you want to minimize a two-variable quadratic function $f(x_1, x_2)$ on the nonnegative quadrant, subject to two constraints:

$$\begin{array}{rrcrcrcrcl} \min & 2x_1 & + & 3x_2 & + & x_1^2 & + & 10x_2^2 & + & 2.5x_1x_2 \\ \text{subject to} & x_1 & - & x_2 & \leq & 1 & & & & \\ & x_1 & + & 2x_2 & \geq & 100 & & & & \\ & x_1 & & & \geq & 0 & & & & \\ & & & x_2 & \geq & 0 & & & & \end{array}$$

To use the OPTMODEL procedure, it is not necessary to fit this problem into the general QP formulation mentioned in the section "Overview" on page 955 and to compute the corresponding parameters. However, since these parameters are closely related to the QPS-format data set, which is used by the OPTQP procedure, we compute these parameters for illustrative purposes as follows. The linear objective function coefficients, vector of right-hand sides, and lower and upper bounds are identified

immediately as

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 100 \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}$$

Let us carefully construct the quadratic matrix $\mathbf{Q}$. Observe that you can use symmetry to separate the main-diagonal and off-diagonal elements:

$$\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \frac{1}{2}\sum_{i,j=1}^{n} x_i\, q_{ij}\, x_j = \frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2 + \sum_{i>j} x_i\, q_{ij}\, x_j$$

The first expression

$$\frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2$$

sums the main-diagonal elements. Thus in this case you have

$$q_{11} = 2, \quad q_{22} = 20$$

Notice that the main-diagonal values are doubled in order to accommodate the 1/2 factor. Now the second term

$$\sum_{i>j} x_i\, q_{ij}\, x_j$$

sums the off-diagonal elements in the strict lower triangular part of the matrix. The only off-diagonal ($x_i\, x_j,\ i \neq j$) term in the objective function is $2.5\, x_1\, x_2$, so you have

$$q_{21} = 2.5$$

Notice that you do not need to specify the upper triangular part of the quadratic matrix.

Finally, the matrix of constraints is as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

The following OPTMODEL program formulates the preceding problem in a manner that is very close to the mathematical specification of the given problem.

```
/* getting started */
proc optmodel;
   var x1 >= 0; /* declare nonnegative variable x1 */
   var x2 >= 0; /* declare nonnegative variable x2 */

   /* objective: quadratic function f(x1, x2) */
   minimize f =
       /* the linear objective function coefficients */
       2 * x1 + 3 * x2 +

       /* quadratic <x, Qx> */
       x1 * x1 + 2.5 * x1 * x2 + 10 * x2 * x2;

   /* subject to the following constraints */
   con r1: x1 - x2 <= 1;
   con r2: x1 + 2 * x2 >= 100;

   /* specify iterative interior point algorithm (QP)
    * in the SOLVE statement */
   solve with qp;

   /* print the optimal solution */
   print x1 x2;
   save qps qpsdata;
quit;
```

The “with qp” clause in the SOLVE statement invokes the QP solver to solve the problem. The output is shown in Figure 12.2.

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Quadratic
Number of Variables	2
Bounded Above	0
Bounded Below	2
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	1
Linear EQ (=)	0
Linear GE (>=)	1
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	QP
Objective Function	f
Solution Status	Optimal
Objective Value	15018
Iterations	10
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	9.9575984E-8
Complementarity	0.0014952645

x1	x2
34	33

Figure 12.2. Summaries and Optimal Solution

In this example, the SAVE QPS statement is used to save the QP problem in the QPS-format data set qpsdata, shown in Figure 12.3. The data set is consistent with the parameters of general quadratic programming previously computed. Also, the data set can be used as input to the OPTQP procedure.

Obs	FIELD1	FIELD2	FIELD3	FIELD4	FIELD5	FIELD6
1	NAME		qpsdata	.		.
2	ROWS			.		.
3	N	f		.		.
4	L	r1		.		.
5	G	r2		.		.
6	COLUMNS			.		.
7		x1	f	2.0	r1	1
8		x1	r2	1.0		.
9		x2	f	3.0	r1	-1
10		x2	r2	2.0		.
11	RHS			.		.
12		.RHS.	r1	1.0		.
13		.RHS.	r2	100.0		.
14	QSECTION			.		.
15		x1	x1	2.0		.
16		x1	x2	2.5		.
17		x2	x2	20.0		.
18	ENDATA			.		.

Figure 12.3. QPS-Format Data Set

Syntax

The following statement is available in the OPTMODEL procedure:

SOLVE WITH QP < / *options* > **;**

Functional Summary

Table 12.1 summarizes the list of options available for the SOLVE WITH QP statement, classified by function.

Table 12.1. Options for the QP Solver

Description	Option
Control Options:	
maximum number of iterations	MAXITER=
upper limit on real time used to solve the problem	MAXTIME=
type of presolve	PRESOLVER=
frequency of printing solution progress	PRINTFREQ=
Interior Point Algorithm Options:	
stopping criterion based on duality gap	STOP_DG=
stopping criterion based on dual infeasibility	STOP_DI=
stopping criterion based on primal infeasibility	STOP_PI=

QP Solver Options

This section describes the options recognized by the QP solver. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the QP solver is explicitly specified using a WITH clause.

Control Options

MAXITER=k
specifies the maximum number of iterations. The value k can be any integer greater than or equal to one. If you do not specify this option, the procedure does not stop based on the number of iterations performed.

MAXTIME=k
specifies an upper limit of k seconds of real time for the optimization process. If you do not specify this option, the procedure does not stop based on the amount of time elapsed.

PRESOLVER=*option*
PRESOL=*option*
specifies one of the following presolve options:

Option	Description
NONE (0)	Disable presolver.
AUTOMATIC (-1)	Apply presolver by using default setting.
BASIC (1)	Apply basic presolver.
MODERATE (2)	Apply moderate presolver.
AGGRESSIVE (3)	Apply aggressive presolver.

You can also specify the option by integers from -1 to 3. The integer value for each option is indicated in parentheses. The default option is AUTOMATIC.

PRINTFREQ=k
specifies that the printing of the solution progress to the iteration log should occur after every k iterations. The print frequency, k, is an integer greater than or equal to zero.

The value $k = 0$ disables the printing of the progress of the solution. The default value of this option is 1.

Interior Point Algorithm Options

STOP_DG=δ
specifies the desired relative duality gap, $\delta \in$ [1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 962 for details.

STOP_DI=β

specifies the maximum allowed relative dual constraints violation, $\beta \in$ [1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 962 for details.

STOP_PI=α

specifies the maximum allowed relative bound and primal constraints violation, $\alpha \in$ [1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 962 for details.

Interior Point Algorithm: Overview

The QP solver implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following QP formulation (the primal):

$$\begin{aligned} \min \quad & \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The corresponding dual is as follows:

$$\begin{aligned} \max \quad & -\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & -\mathbf{Q}\mathbf{x} + \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of dual slack variables.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker) conditions, which can be stated as follows:

$$\begin{aligned} \mathbf{A}\mathbf{x} - \mathbf{s} &= \mathbf{b} \quad (\text{Primal Feasibility}) \\ -\mathbf{Q}\mathbf{x} + \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} &= \mathbf{c} \quad (\text{Dual Feasibility}) \\ \mathbf{W}\mathbf{X}\mathbf{e} &= \mathbf{0} \quad (\text{Complementarity}) \\ \mathbf{S}\mathbf{Y}\mathbf{e} &= \mathbf{0} \quad (\text{Complementarity}) \\ \mathbf{x},\, \mathbf{y},\, \mathbf{w},\, \mathbf{s} &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal *slack* variables.

Note: Slack variables (the s vector) are automatically introduced by the solver when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the solver to handle slack variables much more efficiently.

The letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{S}$ denote matrices with corresponding x, y, w, and s on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations representing the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original QP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{Q} - \mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively; Θ and Ξ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. The QP solver uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point algorithm is that it takes full advantage of the sparsity in the constraint and quadratic matrices, thereby enabling it to efficiently solve large-scale quadratic programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore it is of interest to observe the following four measures:

- Relative primal infeasibility measure α:

$$\alpha = \frac{\|\mathbf{Ax} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- Relative dual infeasibility measure β:

$$\beta = \frac{\|\mathbf{Qx} + \mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- Relative duality gap δ:

$$\delta = \frac{|\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- Absolute complementarity γ:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

where $\|v\|_2$ is the Euclidean norm of the vector v. These measures are displayed in the iteration log.

Iteration Log

The following information is displayed in the iteration log:

Iter	indicates the iteration number
Complement	indicates the (absolute) complementarity
Duality Gap	indicates the (relative) duality gap
Primal Infeas	indicates the (relative) primal infeasibility measure
Bound Infeas	indicates the (relative) bound infeasibility measure
Dual Infeas	indicates the (relative) dual infeasibility measure

If the sequence of solutions converges to an optimal solution of the problem, you should see all columns in the iteration log converge to zero or very close to zero. If they do not, it can be the result of insufficient iterations being performed to reach optimality. In this case, you might need to increase the value specified in the option MAXITER= or MAXTIME=. If the complementarity and/or the duality gap do not converge, the problem might be infeasible or unbounded. If the infeasibility columns do not converge, the problem might be infeasible.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put _OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the QP solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax

DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit
FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY
indicates the (relative) infeasibility of the primal constraints at the solution. See the section "Interior Point Algorithm: Overview" on page 962 for details.

DUAL_INFEASIBILITY
indicates the (relative) infeasibility of the dual constraints at the solution. See the section "Interior Point Algorithm: Overview" on page 962 for details.

BOUND_INFEASIBILITY
indicates the (relative) violation of the optimal solution over the lower and upper bounds. See the section "Interior Point Algorithm: Overview" on page 962 for details.

DUALITY_GAP
indicates the (relative) duality gap. See the section "Interior Point Algorithm: Overview" on page 962 for details.

COMPLEMENTARITY
indicates the (absolute) complementarity at the solution. See the section "Interior Point Algorithm: Overview" on page 962 for details.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Examples

This section presents examples that illustrate the use of the OPTMODEL procedure to solve quadratic programming problems. Example 12.1 illustrates how to model a linear least-squares problem and solve it by using PROC OPTMODEL. Example 12.2 and Example 12.3 show in detail how to model the portfolio optimization/selection problem.

Example 12.1. Linear Least-Squares Problem

The linear least-squares problem arises in the context of determining a solution to an over-determined set of linear equations. In practice, these could arise in data fitting and estimation problems. An over-determined system of linear equations can be defined as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $m > n$. Since this system usually does not have a solution, we need to be satisfied with some sort of approximate solution. The most widely used approximation is the least-squares solution, which minimizes $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.

This problem is called a least-squares problem for the following reason. Let $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ be defined as previously. Let $k_i(x)$ be the kth component of the vector $\mathbf{A}\mathbf{x} - \mathbf{b}$:

$$k_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i, \ i = 1, 2, \ldots, m$$

By definition of the Euclidean norm, the objective function can be expressed as follows:

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \sum_{i=1}^{m} k_i(x)^2$$

Therefore the function we minimize is the sum of squares of m terms $k_i(x)$; hence the term least-squares. The following example is an illustration of the *linear* least-squares problem; i.e., each of the terms k_i is a linear function of x.

Consider the following least-squares problem defined by

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This translates to the following set of linear equations:

$$4x_1 = 1, \quad -x_1 + x_2 = 0, \quad 3x_1 + 2x_2 = 1$$

The corresponding least-squares problem is

$$\text{minimize} \quad (4x_1 - 1)^2 + (-x_1 + x_2)^2 + (3x_1 + 2x_2 - 1)^2$$

The preceding objective function can be expanded to

$$\text{minimize} \quad 26x_1^2 + 5x_2^2 + 10x_1x_2 - 14x_1 - 4x_2 + 2$$

In addition, we impose the following constraint so that the equation $3x_1 + 2x_2 = 1$ is satisfied within a tolerance of 0.1:

$$0.9 \leq 3x_1 + 2x_2 \leq 1.1$$

You can use the following SAS code to solve the least-squares problem:

```
/* example 1: linear least-squares problem */
proc optmodel;
   var x1; /* declare free (no explicit bounds) variable x1 */
   var x2; /* declare free (no explicit bounds) variable x2 */
   /* declare slack variable for ranged constraint */
   var w >= 0 <= 0.2;

   /* objective function: minimize is the sum of squares */
   minimize f = 26 * x1 * x1 + 5 * x2 * x2 + 10 * x1 * x2
       - 14 * x1 - 4 * x2 + 2;

   /* subject to the following constraint */
   con L: 3 * x1 + 2 * x2 - w = 0.9;

   solve with qp;

   /* print the optimal solution */
   print x1 x2;
quit;
```

The output is shown in Output 12.1.1.

Output 12.1.1. Summaries and Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Quadratic
Number of Variables	3
Bounded Above	0
Bounded Below	0
Bounded Below and Above	1
Free	2
Fixed	0
Number of Constraints	1
Linear LE (<=)	0
Linear EQ (=)	1
Linear GE (>=)	0
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	QP
Objective Function	f
Solution Status	Optimal
Objective Value	0.0095238
Iterations	11
Primal Infeasibility	1.449279E-11
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	4.3804698E-7
Complementarity	1.3805642E-6

x1	x2
0.23809	0.1619

Example 12.2. Portfolio Optimization

Consider a portfolio optimization example. The two competing goals of investment are (1) long-term growth of capital and (2) low risk. A good portfolio grows steadily without wild fluctuations in value. The Markowitz model is an optimization model for balancing the return and risk of a portfolio. The decision variables are the amounts invested in each asset. The objective is to minimize the variance of the portfolio's total return, subject to the constraints that (1) the expected growth of the portfolio reaches at least some target level and (2) you do not invest more capital than you have.

Let $x_1, \ldots, x_n$ be the amount invested in each asset, $\mathcal{B}$ be the amount of capital

you have, $\mathbf{R}$ be the random vector of asset returns over some period, and $\mathbf{r}$ be the expected value of $\mathbf{R}$. Let G be the minimum growth you hope to obtain, and $\mathcal{C}$ be the covariance matrix of $\mathbf{R}$. The objective function is $\text{Var}\left(\sum_{i=1}^{n} x_i R_i\right)$, which can be equivalently denoted as $\mathbf{x}^{\mathrm{T}}\mathcal{C}\mathbf{x}$.

Assume, for example, $n = 4$. Let $\mathcal{B} = 10{,}000$, $G = 1000$, $\mathbf{r} = [0.05, -0.2, 0.15, 0.30]$, and

$$\mathcal{C} = \begin{bmatrix} 0.08 & -0.05 & -0.05 & -0.05 \\ -0.05 & 0.16 & -0.02 & -0.02 \\ -0.05 & -0.02 & 0.35 & 0.06 \\ -0.05 & -0.02 & 0.06 & 0.35 \end{bmatrix}$$

The QP formulation can be written as

$$\begin{aligned} \min \quad & 0.08x_1^2 - 0.1x_1x_2 - 0.1x_1x_3 - 0.1x_1x_4 + \\ & 0.16x_2^2 - 0.04x_2x_3 - 0.02x_2x_4 + 0.35x_3^2 + \\ & 0.12x_3x_4 + 0.35x_4^2 \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & \\ (\text{budget}) \quad & x_1 + x_2 + x_3 + x_4 \le 10000 \\ (\text{growth}) \quad & 0.05x_1 - 0.2x_2 + 0.15x_3 + 0.30x_4 \ge 1000 \\ & x_1,\, x_2,\, x_3,\, x_4 \ge 0 \end{aligned}$$

Use the following SAS code to solve the problem:

```
/* example 2: portfolio optimization */
proc optmodel;
   /* let x1, x2, x3, x4 be the amount invested in each asset */
   var x{1..4} >= 0;

   num coeff{1..4, 1..4} = [0.08 -.05 -.05 -.05
                            -.05 0.16 -.02 -.02
                            -.05 -.02 0.35 0.06
                            -.05 -.02 0.06 0.35];
   num r{1..4}=[0.05 -.20 0.15 0.30];

   /* minimize the variance of the portfolio's total return */
   minimize f = sum{i in 1..4, j in 1..4}coeff[i,j]*x[i]*x[j];

   /* subject to the following constraints */
   con BUDGET: sum{i in 1..4}x[i] <= 10000;
   con GROWTH: sum{i in 1..4}r[i]*x[i] >= 1000;

   solve with qp;

   /* print the optimal solution */
   print x;
```

The summaries and the optimal solution are shown in Output 12.2.1.

Output 12.2.1. Portfolio Optimization

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Quadratic
Number of Variables	4
Bounded Above	0
Bounded Below	4
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	1
Linear EQ (=)	0
Linear GE (>=)	1
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	QP
Objective Function	f
Solution Status	Optimal
Objective Value	2232314
Iterations	7
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	4.4734294E-7
Complementarity	0.9932470284

[1]	x
1	3.4529E+03
2	7.0389E-04
3	1.0688E+03
4	2.2235E+03

Thus, the minimum variance portfolio that earns an expected return of at least 10% is $x_1 = 3452$, $x_2 = 0$, $x_3 = 1068$, $x_4 = 2223$. Asset 2 gets nothing because its expected return is -20% and its covariance with the other assets is not sufficiently negative for it to bring any diversification benefits. What if we drop the nonnegativity assumption?

Financially, that means you are allowed to short-sell—i.e., sell low-mean-return assets and use the proceeds to invest in high-mean-return assets. In other words, you put a negative portfolio weight in low-mean assets and "more than 100%" in high-mean

assets.

To solve the portfolio optimization problem with the short-sale option, continue to submit the following SAS code:

```
   /* example 2: portfolio optimization with short-sale option */
   /* dropping nonnegativity assumption */
   for {i in 1..4} x[i].lb=-x[i].ub;

   solve with qp;

   /* print the optimal solution */
   print x;
quit;
```

You can see in the optimal solution displayed in Output 12.2.2 that the decision variable x_2, denoting Asset 2, is equal to -1563.61, which means short sale of that asset.

Output 12.2.2. Portfolio Optimization with Short-Sale Option

```
                The OPTMODEL Procedure

                   Solution Summary

Solver                              QP
Objective Function                   f
Solution Status                Optimal
Objective Value                1907123
Iterations                           6

Primal Infeasibility                 0
Dual Infeasibility                   0
Bound Infeasibility                  0
Duality Gap                6.1120033E-7
Complementarity            1.1690054718


                 [1]           x

                  1       1684.35
                  2      -1563.61
                  3        682.51
                  4       1668.95
```

Example 12.3. Portfolio Selection with Transactions

Consider a portfolio selection problem with a slight modification. You are now required to take into account the current position and transaction costs associated with buying and selling assets. The objective is to find the minimum variance portfolio. In order to understand the scenario better, consider the following data.

You are given three assets. The current holding of the three assets is denoted by the vector **c** = [200, 300, 500], the amount of asset bought and sold is denoted by b_i and s_i, respectively, and the net investment in each asset is denoted by x_i and is defined by the following relation:

$$x_i - b_i + s_i = c_i, \quad i = 1, 2, 3$$

Let us say that you pay a transaction fee of 0.01 every time you buy or sell. Let the covariance matrix $\mathcal{C}$ be defined as

$$\mathcal{C} = \begin{bmatrix} 0.027489 & -0.00874 & -0.00015 \\ -0.00874 & 0.109449 & -0.00012 \\ -0.00015 & -0.00012 & 0.000766 \end{bmatrix}$$

Assume that you hope to obtain at least 12% growth. Let **r** = [1.109048, 1.169048, 1.074286] be the vector of expected return on the three assets, and let $\mathcal{B}$=1000 be the available funds. Mathematically, this problem can be written in the following manner:

$$\begin{aligned} \min \quad & 0.027489x_1^2 - 0.01748x_1x_2 - 0.0003x_1x_3 + 0.109449x_2^2 \\ & -0.00024x_2x_3 + 0.000766x_3^2 \end{aligned}$$

subject to

$$\begin{array}{lrcll} \text{(return)} & \sum_{i=1}^{3} r_i x_i & \geq & 1.12\mathcal{B} & \\ \text{(budget)} & \sum_{i=1}^{3} x_i + \sum_{i=1}^{3} 0.01(b_i + s_i) & = & \mathcal{B} & \\ \text{(balance)} & x_i - b_i + s_i & = & c_i, & i = 1, 2, 3 \\ & x_i,\ b_i,\ s_i & \geq & 0, & i = 1, 2, 3 \end{array}$$

The problem can be solved by the following SAS code:

```
/* example 3: portfolio selection with transactions */
proc optmodel;
   /* let x1, x2, x3 be the amount invested in each asset */
   var x{1..3} >= 0;
   /* let b1, b2, b3 be the amount of asset bought */
```

```
   var b{1..3} >= 0;
   /* let s1, s2, s3 be the amount of asset sold */
   var s{1..3} >= 0;

   /* current holdings */
   num c{1..3}=[ 200 300 500];
   /* covariance matrix */
   num coeff{1..3, 1..3} = [0.027489  -.008740  -.000150
                            -.008740  0.109449  -.000120
                            -.000150  -.000120  0.000766];
   /* returns */
   num r{1..3}=[1.109048 1.169048 1.074286];

   /* minimize the variance of the portfolio's total return */
   minimize f = sum{i in 1..3, j in 1..3}coeff[i,j]*x[i]*x[j];

   /* subject to the following constraints */
   con BUDGET: sum{i in 1..3}(x[i]+.01*b[i]+.01*s[i]) <= 1000;
   con RETURN: sum{i in 1..3}r[i]*x[i] >= 1120;
   con BALANC{i in 1..3}: x[i]-b[i]+s[i]=c[i];

   solve with qp;

   /* print the optimal solution */
   print x;
quit;
```

The output is displayed in Output 12.3.1.

Output 12.3.1. Portfolio Selection with Transactions

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	f
Objective Type	Quadratic
Number of Variables	9
Bounded Above	0
Bounded Below	9
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	5
Linear LE (<=)	1
Linear EQ (=)	3
Linear GE (>=)	1
Linear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	QP
Objective Function	f
Solution Status	Optimal
Objective Value	19561
Iterations	17
Primal Infeasibility	4.516565E-15
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	5.8692587E-8
Complementarity	0.0011480356

[1]	x
1	397.58
2	406.12
3	190.17

References

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Chapter 13
The Sequential Quadratic Programming Solver

Chapter Contents

Chapter 13
The Sequential Quadratic Programming Solver

Overview

The sequential quadratic programming (SQP) solver is a component of the OPTMODEL procedure, and it can be used for solving general nonlinear programming (NLP) problems.

The general form of nonlinear optimization problems can be mathematically described as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c(x) \; \{\leq \mid = \mid \geq\} \; 0 \\ & l \leq x \leq u \end{aligned}$$

where f: $\mathbb{R}^n \mapsto \mathbb{R}$ is the nonlinear objective function, c: $\mathbb{R}^n \mapsto \mathbb{R}^m$ is the set of general nonlinear equality and inequality constraints, and l and u are lower and upper bounds, respectively, on the decision variable x.

The SQP solver solves NLP problems by using an iterative procedure. An improved estimate of the solution is obtained at the end of each iteration by taking a step along a certain search direction. This direction is computed by solving a quadratic programming (QP) subproblem (Fletcher 1987).

It has been found that SQP-based algorithms are very efficient for solving NLP problems. Such a view is based on their strong convergence properties, supported by good numerical results from test experiments (Conn, Gould, and Toint 1994).

The SQP solver basically adapts the ideas from Bartholomew-Biggs's work (Bartholomew-Biggs 1988). It uses an *augmented Lagrangian penalty function* as a merit function. The solver solves a QP subproblem to obtain a search direction, which is an approximation of the Newton direction to the minimum of the merit function. A *line-search algorithm* is then implemented along the resulting search direction. In the SQP solver, the line search is terminated when Wolfe-Powell conditions are satisfied, thereby ensuring global convergence to a local minimum (Fletcher 1987).

The SQP solver has several features that make it different from the other NLP solvers in the SAS/OR suite (see Chapter 10, "The NLPC Nonlinear Optimization Solver," and Chapter 11, "The Unconstrained Nonlinear Programming Solver"). To name a few:

- It can handle general nonlinear constraints.

- It exploits the sparsity (of the Jacobian) in NLP problems. This enables the SQP solver to solve fairly large NLP problems.
- It finds the negative curvature of the NLP problem at a stationary point of the problem, thereby giving the solver a better chance of moving away from a stationary point.
- It tries to find a good estimate of Lagrange multipliers based on a good starting point for the decision variables.

In general, the SQP solver approaches the solution of a NLP problem through a sequence of points that might not be in the feasible region. Therefore it requires that all the functions in the NLP problem be defined in the entire space, not just at the points that satisfy the constraints.

Another requirement is that all the functions in NLP problems be smooth. You do not need to supply the derivatives of these functions; the SQP solver calculates them, if needed.

Getting Started

Consider a simple example of minimizing the following nonlinear programming problem:

$$\begin{aligned} \min \quad & (1 - x_1)^2 \\ \text{subject to} \quad & x_2 - {x_1}^2 = 0 \end{aligned}$$

Assume the starting point $\mathbf{x}^0 = (\,-1.2, 1.0\,)$. You can use the following SAS code to solve the problem:

```
proc optmodel;
  var x {1..2};
  minimize obj = (1-x[1])^2;

  con cons1: x[2] - x[1]^2 = 0;

  /*starting point */
  x[1] = -1.2;
  x[2] =  1;

  solve with SQP/ printfreq = 5;
  print x;

quit;
```

The summary of the solver output along with the optimal solution, $(\,1.0, 1.0\,)$, is displayed in Figure 13.1.

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Quadratic
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	2
Fixed	0
Number of Constraints	1
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	1
Nonlinear GE (>=)	0
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	2.7360E-16
Iterations	18
Infeasibility	9.2771104E-7
Optimality Error	1.5590633E-7
Complementarity	0

[1]	x
1	1
2	1

Figure 13.1. Optimal Solution

The iteration log in Figure 13.2 displays information about the objective function value, the norm of the violated constraints, the norm of the gradient of the Lagrangian function, and the norm of the violated complementarity condition at each iteration.

```
NOTE: The problem has 2 variables (2 free, 0 fixed).
NOTE: The problem has 0 linear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1 nonlinear constraints (0 LE, 1 EQ, 0 GE, 0 range).
NOTE: Using analytic derivatives for objective.
NOTE: Using analytic derivatives for nonlinear constraints.
NOTE: The SQP solver is called.
                  Objective                      Optimality
        Iter          Value   Infeasibility          Error   Complementarity
           0       4.840000        0.440000       0.289283          0.000000
           5       0.710665        1.221065       0.810306          0.000000
          10       0.006302        0.166703       0.136975          0.000000
          15       0.000003        0.004772       0.003551          0.000000
          18       0.000000        0.000001       0.000000          0.000000
NOTE: Converged.
NOTE: Objective = 2.736033E-16
```

Figure 13.2. Iteration Log

Syntax

The following PROC OPTMODEL statement is available for the SQP solver:

SOLVE WITH SQP < / *options* > **;**

Functional Summary

Table 13.1 summarizes the options that can be used with the SOLVE WITH SQP statement.

Table 13.1. Options for the SQP Solver

Description	Option
Solver Options:	
check second-order optimality conditions	HESCHECK / NOHESCHECK
maximum number of iterations	MAXITER=
upper limit on the real time for optimization	MAXTIME=
reduction rate of penalty parameters in the Lagrangian function	PENALTY=
convergence tolerance for both stationary and complementary conditions	OPTTOL=
convergence tolerance for feasibility	FEASTOL=
Output Option:	
print objective function value, norm of violated constraints, and norm of Lagrangian function gradient at each iteration	PRINTFREQ=

SQP Solver Options

The following options are supported by the SQP solver.

HESCHECK|NOHESCHECK
specifies that the solver check (or not check, in the case of NOHESCHECK) the second-order optimality of the solution found—i.e., the nonnegativity of the projected Hessian of the Lagrangian function at the solution. The option NOHESCHECK is used by default.

MAXITER=N
specifies that the optimization process stop after maximum number of N iterations. The value of this option is an integer greater than or equal to zero. An iteration in the SQP solver includes obtaining a search direction and finding a step size along that direction. The default value is 10,000 iterations.

MAXTIME=r
specifies an upper limit of r seconds of real time for the optimization process. The default value is 7200 seconds. Note that the time specified by the MAXTIME= option is checked only at the end of each iteration. The optimization terminates when the actual running time is greater than or equal to r.

PENALTY=γ
specifies the rate at which the penalty parameters used in the Lagrangian function are reduced. In other words, the real number $\gamma \in (0, 1)$ controls the rate at which the solver converges. A smaller number often results in the solver taking fewer iterations to converge to a solution. However, if the number is set too small, it is known to cause numerical difficulties for the solver. The default value for the PENALTY= option is 0.75.

PRINTFREQ=*j*
specifies that the printing of the solution progress to the iteration log should occur after every j iterations. The print frequency, j, is an integer greater than or equal to zero. The value $j = 0$ disables the printing of the progress of the solution. Note that the first and last iterations are also displayed. The default value for this option is 1.

Note: The PRINT statement in the OPTMODEL procedure can be used to print the solution of the decision variables to the output.

OPTTOL=ϵ
specifies the convergence tolerance for both stationary and complementary conditions. The value of this option is a positive real number. The default value is 1E–5. See the section "Solver's Criteria" on page 986 for details.

FEASTOL=ϵ
specifies the convergence tolerance for feasibility. The value of this option is a positive real number. The default value is 1E–6. See the section "Solver's Criteria" on page 986 for details.

Details

Conditions of Optimality

To facilitate discussion of the optimality conditions, we present the notation to be used for easy reference:

m	number of general nonlinear constraints, which include the linear constraints but not the bound constraints
n	dimension of x, i.e., the number of decision variables
x	iterate, i.e., the vector of n decision variables
$f(x)$	objective function
$\nabla f(x)$	gradient of the objective function
$\nabla^2 f(x)$	Hessian matrix of the objective function
λ	Lagrange multiplier vector, $\lambda \in \mathbb{R}^m$
$L(x, \lambda)$	Lagrangian function of constrained problems
$\nabla_x L(x, \lambda)$	gradient of the Lagrangian function with respect to x

For x and λ, the superscript k is used to denote the value at the kth iteration, such as x^k, and the superscript $*$ is used to denote their corresponding optimal values, such as λ^*.

We rewrite the general form of nonlinear optimization problems in the section "Overview" on page 979 by grouping the equality constraints and inequality constraints. We also rewrite all the general nonlinear inequality constraints and bound constraints in one form as "$\geq$" inequality constraints. Thus we have the following:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \\ & c_i(x) \geq 0, \quad i \in \mathcal{I} \end{aligned}$$

where $\mathcal{E}$ is the set of indices of the equality constraints, $\mathcal{I}$ is the set of indices of the inequality constraints, and $m = |\mathcal{E}| + |\mathcal{I}|$.

A point x is feasible if it satisfies all the constraints $c_i(x) = 0, i \in \mathcal{E}$, and $c_i(x) \geq 0, i \in \mathcal{I}$. The feasible region $\mathcal{F}$ consists of all the feasible points. In unconstrained cases, the feasible region $\mathcal{F}$ is the entire $\mathbb{R}^n$ space.

A feasible point x^* is a *local solution* of the problem if there exists a neighborhood $\mathcal{N}$ of x^* such that

$$f(x) \geq f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

Further, a feasible point x^* is a *strict local solution* if strict inequality holds in the preceding case, i.e.,

$$f(x) > f(x^*) \quad \text{for all } x \in \mathcal{N} \cap \mathcal{F}$$

A feasible point x^* is a *global solution* of the problem if no point in $\mathcal{F}$ has a smaller function value than $f(x^*)$, i.e.,

$$f(x) \geq f(x^*) \text{ for all } x \in \mathcal{F}$$

The SQP solver finds a local minimum of an optimization problem.

Unconstrained Optimization

The following conditions hold for unconstrained optimization problems:

- **First-order necessary conditions**: If x^* is a local solution and $f(x)$ is continuously differentiable in some neighborhood of x^*, then

$$\nabla f(x^*) = 0$$

- **Second-order necessary conditions**: If x^* is a local solution and $f(x)$ is twice continuously differentiable in some neighborhood of x^*, then $\nabla^2 f(x^*)$ is positive semidefinite.
- **Second-order sufficient conditions**: If $f(x)$ is twice continuously differentiable in some neighborhood of x^*, and if $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict local solution.

Constrained Optimization

For constrained optimization problems, the *Lagrangian function* is defined as follows:

$$L(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

where $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$, are called *Lagrange multipliers*. $\nabla_x L(x, \lambda)$ is used to denote the gradient of the Lagrangian function with respect to x, and $\nabla_x^2 L(x, \lambda)$ is used to denote the Hessian of the Lagrangian function with respect to x. The *active set* at a feasible point x is defined as

$$\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : c_i(x) = 0\}$$

We also need the following definition before we can state the first-order and second-order necessary conditions:

- **Linear independence constraint qualification and regular point**: A point x is said to satisfy the linear independence constraint qualification if the gradients of active constraints

$$\nabla c_i(x), \quad i \in \mathcal{A}(x)$$

  are linearly independent. We refer to such a point x as a *regular point*.

We now state the theorems that are essential in the analysis and design of algorithms for constrained optimization:

- **First-order necessary conditions**: Suppose x^* is a local minimum and also a regular point. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are continuously differentiable, there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ such that the following conditions hold:

$$\nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*) = 0$$

$$\begin{aligned} c_i(x^*) &= 0, \quad i \in \mathcal{E} \\ c_i(x^*) &\geq 0, \quad i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \quad i \in \mathcal{I} \\ \lambda_i^* c_i(x^*) &= 0, \quad i \in \mathcal{I} \end{aligned}$$

  The preceding conditions are often known as the *Karush-Kuhn-Tucker conditions*, or KKT conditions for short.

- **Second-order necessary conditions**: Suppose x^* is a local minimum and also a regular point. Let λ^* be the Lagrange multipliers that satisfy the KKT conditions. If $f(x)$ and $c_i(x), i \in \mathcal{E} \cup \mathcal{I}$, are twice continuously differentiable, the following conditions hold:

$$z^{\mathrm{T}} \nabla_x^2 L(x^*, \lambda^*) z \geq 0$$

  for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^{\mathrm{T}} z = 0, \quad i \in \mathcal{A}(x*)$$

- **Second-order sufficient conditions**: Suppose there exist a point x^* and some Lagrange multipliers λ^* such that the KKT conditions are satisfied. If the conditions

$$z^{\mathrm{T}} \nabla_x^2 L(x^*, \lambda^*) z > 0$$

  hold for all $z \in \mathbb{R}^n$ that satisfy

$$\nabla c_i(x^*)^{\mathrm{T}} z = 0, \quad i \in \mathcal{A}(x^*)$$

  then x^* is a strict local solution.

  Note that the set of all such z's forms the null space of matrix $\left[\nabla c_i(x^*)^{\mathrm{T}}\right]_{i \in \mathcal{A}(x*)}$. Hence we can numerically check the Hessian of the Lagrangian function projected onto the null space. For a rigorous treatment of the optimality conditions, see Fletcher (1987) and Nocedal and Wright (1999).

Solver's Criteria

The SQP solver declares a solution to be optimal when all the following criteria are satisfied:

- The relative norm of the gradient of the Lagrangian function is less than OPTTOL, and the absolute norm of the gradient of the Lagrangian function is less than 1.0E–2.

When the problem is a constrained optimization problem, the relative norm of the gradient of the Lagrangian function is defined as

$$\frac{\|\nabla_x L(x,\lambda)\|}{1+\|\nabla f(x)\|}$$

When the problem is an unconstrained optimization problem, it is defined as

$$\frac{\| \nabla f(x) \|}{1+|f(x)|}$$

- The absolute value of each equality constraint is less than FEASTOL.
- The value of each "$\geq$" inequality is greater than the right-hand-side constant minus FEASTOL.
- The value of each "$\leq$" inequality is less than the right-hand-side constant plus FEASTOL.
- The value of each Lagrange multiplier for inequalities is greater than $-$OPTTOL.
- The minimum of both the values of an inequality and its corresponding Lagrange multiplier is less than OPTTOL.

The SQP solver provides the HESCHECK option to verify that the second-order conditions are also satisfied.

Solution Techniques

Some of the solution techniques used by the SQP solver are outlined as follows:

- The SQP solver solves QP subproblems to find a search direction. This subproblem involves some linear equality and inequality constraints, and an *active set* method is used to solve the subproblem.
- An augmented Lagrangian function is employed as a merit function for the line search. Global convergence is ensured when the Wolfe-Powell conditions are satisfied.
- A *quasi-Newton* update formula is used to approximate the inverse of the Hessian of the Lagrangian function.
- The solver also provides an option, HESCHECK, to verify the second-order necessary condition for a local optimal solution (see Example 13.2). If the second-order necessary conditions are not satisfied for a given solution, and the HESCHECK option is specified, then the solver finds a search direction based on the second-order information to improve the solution. This feature ensures that the solver does not terminate at a saddle point, which is not optimal.
- The SQP solver has a built-in procedure to find a good estimate of the Lagrange multipliers, given the starting points for the decision variables.
- The SQP solver incorporates some strategies to scale an NLP problem so that the resulting scaled problem is easier to solve than the original one.

Solver Termination

The SQP solver terminates with one of the following messages:

Converged	The SQP solver has found a local minimum within the convergence tolerance specified.
Number of iterations taken is more than MAXITER	The SQP solver has reached the limit on the maximum number of iterations, but has not satisfied the convergence criteria.
Time exceeded MAXTIME	The SQP solver has spent more time than the prespecified maximum real time for the optimization process, but has not satisfied the convergence criteria.
Line search cannot improve further	The SQP solver cannot make any progress in the line search. One possibility is that a solution has been found, but the convergence tolerance is set too small.
Functions not defined in the area where the algorithm enters	The SQP solver has entered an area where a function in the NLP problem is not defined.
Problem size too big	The NLP problem is too large for the SQP solver to solve. This could be due to a limitation in the memory allocated to the procedure.
Function or gradient cannot be evaluated at starting point	The functions or gradients of the function involved in the NLP problem cannot be evaluated at the starting point.
Problem not solved	The NLP problem has not been solved. There could be several reasons why this happens. For instance, maybe a QP subproblem could not be solved within a preset number of iterations.
Second-order optimality is not satisfied	SQP has located a point satisfying the first-order condition but not the second-order condition.
Problem may be unbounded	SQP has located a feasible point, but its objective function value is extremely low. SQP can improve the objective function even further as well.
Problem may be infeasible	SQP cannot locate a feasible point.

Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying %put

&_OROPTMODEL_; and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the SQP solver is called are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	solver terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
SEMANTIC_ERROR	evaluation error, such as an invalid operand type
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
BAD_PROBLEM_TYPE	problem type is unsupported by solver
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	solver reached its execution time limit
FUNCTION_CALL_LIMIT_REACHED	solver reached its limit on function evaluations
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

INFEASIBILITY
indicates the norm of the feasibility error at the solution.

FIRST_ORDER_ERROR
indicates the relative norm of the gradient of the Lagrangian function at the solution.

COMPLEMENTARITY
indicates the norm of the complementarity error at the solution.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Examples

Example 13.1. Solving a Highly Nonlinear Problem

Consider the following example of minimizing a highly nonlinear problem:

$$\begin{array}{rll} \min & \sum_{j=1}^{10} e^{x_j}(c_j + x_j - \log \sum_{k=1}^{10} e^{x_k}) & \\ \text{subject to} & e^{x_1} + 2e^{x_2} + 2e^{x_3} + e^{x_6} + e^{x_{10}} & = \quad 8 \\ & e^{x_4} + 2e^{x_5} + e^{x_6} + e^{x_7} & = \quad 1 \\ & e^{x_3} + e^{x_7} + e^{x_8} + 2e^{x_9} + e^{x_{10}} & = \quad 1 \end{array}$$

Assume the starting point $\mathbf{x}^0 = (\ -2.3, ..., -2.3, ..., -2.3\ )$. You can use the following SAS code to solve the problem:

```
proc optmodel;
   var x {1..10} >= -100 <= 100   /* variable bounds */
      init -2.3;   /* starting point */

   number c {1..10} = [-6.089 -17.164 -34.054 -5.914 -24.721
      -14.986 -24.100 -10.708 -26.662 -22.179];

   number a{1..3,1..10}=[1 2 2 0 0 1 0 0 0 1
                         0 0 0 1 2 1 1 0 0 0
                         0 0 1 0 0 0 1 1 2 1];

   number b{1..3}=[8 1 1];

   minimize obj =
      sum{j in 1..10}exp(x[j])*(c[j]+x[j]
         -log(sum {k in 1..10}exp(x[k])));

   con cons{i in 1..3}:
      sum{j in 1..10}a[i,j]*exp(x[j])=b[i];

   solve with sqp / printfreq = 0;
quit;
```

The output is displayed in Output 13.1.1.

Output 13.1.1. Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	10
Bounded Above	0
Bounded Below	0
Bounded Below and Above	10
Free	0
Fixed	0
Number of Constraints	3
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	3
Nonlinear GE (>=)	0
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	-102.08
Iterations	33
Infeasibility	8.9721776E-6
Optimality Error	8.2864441E-6
Complementarity	0

Example 13.2. Using the HESCHECK Option

The following example illustrates how the HESCHECK option could be useful:

$$\begin{aligned} \min \quad & {x_1}^2 + {x_2}^2 \\ \text{subject to} \quad & x_1 + {x_2}^2 \geq 1 \\ & 0 \leq x_i \leq 8, \qquad i = 1, 2 \end{aligned}$$

Use the following SAS code to solve the problem:

```
proc optmodel;
   var x {1..2} <= 8 >= 0   /* variable bounds */
      init 0;   /* starting point */

   minimize obj = x[1]^2 + x[2]^2;

   con cons:
      x[1] + x[2]^2 >= 1;

   solve with sqp / printfreq = 5;
   print x;
quit;
```

When $\mathbf{x}^0 = (0, 0)$ is chosen as the starting point, the SQP solver converges to (1, 0), as displayed in Output 13.2.1. It can be easily verified that (1, 0) is a stationary point and not an optimal solution.

Output 13.2.1. Stationary Point

```
                    The OPTMODEL Procedure

                       Problem Summary

Objective Sense            Minimization
Objective Function                  obj
Objective Type                Quadratic

Number of Variables                   2
Bounded Above                         0
Bounded Below                         0
Bounded Below and Above               2
Free                                  0
Fixed                                 0

Number of Constraints                 1
Linear LE (<=)                        0
Linear EQ (=)                         0
Linear GE (>=)                        0
Linear Range                          0
Nonlinear LE (<=)                     0
Nonlinear EQ (=)                      0
Nonlinear GE (>=)                     1
Nonlinear Range                       0


                    The OPTMODEL Procedure

                       Solution Summary

Solver                              SQP
Objective Function                  obj
Solution Status                 Optimal
Objective Value                 0.99999
Iterations                           12

Infeasibility              5.8182019E-6
Optimality Error           4.1308973E-6
Complementarity            5.8182019E-6


                  [1]             x

                   1         0.99999
                   2         0.00000
```

To resolve this issue, you can use the HESCHECK option in the SOLVE statement as follows:

```
proc optmodel;
...
   solve with sqp / printfreq = 1 hescheck;
...
quit;
```

For the same starting point $\mathbf{x}^0 = (0, 0)$, the SQP solver now converges to the optimal solution, $\left(0.5, \sqrt{2}/2\right)$, as displayed in Output 13.2.2.

Output 13.2.2. Optimal Solution Using the HESCHECK Option

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Quadratic
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	2
Free	0
Fixed	0
Number of Constraints	1
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	0
Nonlinear GE (>=)	1
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	0.75
Iterations	25
Infeasibility	0
Optimality Error	4.2661052E-7
Complementarity	2.8807365E-6

[1]	x
1	0.50000
2	0.70711

Example 13.3. Choosing a Good Starting Point

To solve a constrained nonlinear optimization problem, both the optimal primal and optimal dual variables have to be computed. Therefore, a good estimate of both the primal and dual variables is essential for fast convergence. An important feature of the SQP solver is that it computes a good estimate of the dual variables if an estimate of the primal variables is known.

The following example illustrates how a good initial starting point helps in securing the desired optimum:

$$\begin{aligned} \min \quad & e^{x_1}(4\,x_1^2 + 2\,x_2^2 + 4\,x_1\,x_2 + 2\,x_2 + 1) \\ \text{subject to} \quad & x_1\,x_2 - x_1 - x_2 \leq -1.5 \\ & x_1\,x_2 \geq -10 \end{aligned}$$

Assume the starting point $\mathbf{x}^0 = (1, -1)$. The SAS code is as follows:

```
proc optmodel;
   var x {1..2} <= 8 >= -10;   /* variable bounds */

   minimize obj =
      exp(x[1])*(4*x[1]^2 + 2*x[2]^2 + 4*x[1]*x[2] + 2*x[2] + 1);

   con cons1:
      x[1]*x[2] -x[1] - x[2] <= -1.5;
   con cons2:
      x[1]*x[2] >= -10;

   x[1] = 1; x[2] = -1; /* starting point */
   solve with sqp / printfreq = 5 ;
   print x;
```

The solution obtained by the SQP solver is displayed in Output 13.3.1.

Output 13.3.1. Solution Obtained with the Starting Point (0, 0)

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	2
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	1
Nonlinear EQ (=)	0
Nonlinear GE (>=)	1
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	3.0608
Iterations	23
Infeasibility	4.4461568E-6
Optimality Error	7.0276172E-7
Complementarity	4.4461568E-6

[1]	x
1	1.1825
2	-1.7398

Now assume a starting point of (–1, 1). Continue to submit the following code:

```
/* starting point */
x[1] = -1;
x[2] =  1;

solve with sqp / printfreq = 5 ;
print x;
quit;
```

The corresponding solution is displayed in Output 13.3.2.

Output 13.3.2. Solution Obtained with the Starting Point (–1, 1)

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	2
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	1
Nonlinear EQ (=)	0
Nonlinear GE (>=)	1
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	0.023552
Iterations	19
Infeasibility	0
Optimality Error	2.5795177E-6
Complementarity	0.0000394954

[1]	x
1	-9.5473
2	1.0474

The preceding illustration demonstrates the importance of a good starting point in obtaining the optimal solution. The SQP solver ensures global convergence only to a local optimum. Therefore you need to have sufficient knowledge of your problem in order to be able to get a "good" estimate of the starting point.

Example 13.4. Using the PENALTY= Option

The PENALTY= option plays an important role in ensuring a good convergence rate. Consider the following example:

$$
\begin{aligned}
\min \quad & (x_1 - 10)^3 + (x_2 - 20)^3 \\
\text{subject to} \quad & (x_1 - 5)^2 + (x_2 - 5)^2 \geq 100 \\
& (x_6 - 5)^2 + (x_2 - 5)^2 \geq 0 \\
& (x_1 - 6)^2 + (x_2 - 5)^2 \leq 82.81 \\
& 13 \leq x_1 \leq 16 \\
& 0 \leq x_2 \leq 15
\end{aligned}
$$

Assume the starting point $\mathbf{x}^0 = (\,14.35, 8.6\,)$. You can use the following SAS code to solve the problem:

```
proc optmodel;
   var x1 >=13 <=16, x2 >=0 <=15;

   minimize obj =
      (x1-10)^3 + (x2-20)^3;

   con cons1:
      (x1-5)^2 + (x2-5)^2 - 100 >= 0;
   con cons2:
      (x1-6)^2 + (x2-5)^2 >= 0;
   con cons3:
      82.81 - (x1-6)^2 - (x2-5)^2 >= 0;

   /* starting point */
   x1 = 14.35;
   x2 = 8.6;

   solve with sqp / printfreq = 5;
   print x1 x2;
quit;
```

The optimal solution is displayed in Output 13.4.1.

Output 13.4.1. Optimal Solution

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	2
Bounded Above	0
Bounded Below	0
Bounded Below and Above	2
Free	0
Fixed	0
Number of Constraints	3
Linear LE (<=)	0
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Nonlinear LE (<=)	0
Nonlinear EQ (=)	0
Nonlinear GE (>=)	3
Nonlinear Range	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	-6961.8
Iterations	60
Infeasibility	9.2766298E-6
Optimality Error	8.696722E-10
Complementarity	9.2766298E-6

x1	x2
14.095	0.84295

The SQP solver can converge to the optimal solution faster if you specify PENALTY = 0.1. You can specify the PENALTY= option in the SOLVE statement as follows:

```
proc optmodel;
...
   solve with sqp/ printfreq = 5 penalty=0.1;
...
quit;
```

The iteration log is shown in Output 13.4.2:

Output 13.4.2. Iteration Log

```
NOTE: The problem has 2 variables (0 free, 0 fixed).
NOTE: The problem has 0 linear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 3 nonlinear constraints (0 LE, 0 EQ, 3 GE, 0 range).
NOTE: Using analytic derivatives for objective.
NOTE: Using analytic derivatives for nonlinear constraints.
NOTE: The SQP solver is called.
                    Objective                      Optimality
        Iter            Value   Infeasibility          Error   Complementarity
           0     -1399.231125        0.000000       0.997442          0.000000
           5     -22160.43028       90.422481       0.999577         90.422481
          10     -5574.824558        6.056792       1.432838          6.056792
          15     -7779.950111        5.723764       0.179844          5.723764
          20     -7951.247692        0.968937       0.000002          0.968937
          25     -7515.515769        0.517712       0.021069          0.517712
          30     -7391.884007        0.401901       0.000092          0.401901
          35     -6966.094726        0.003853       0.000146          0.003853
          37     -6961.818928        0.000005       0.000000          0.000005
NOTE: Converged.
NOTE: Objective = -6961.818928
```

The optimal solution is displayed in Output 13.4.3.

Output 13.4.3. Optimal Solution Using the PENALTY= Option

```
                    The OPTMODEL Procedure

                       Problem Summary

Objective Sense                 Minimization
Objective Function                       obj
Objective Type                     Nonlinear

Number of Variables                        2
Bounded Above                              0
Bounded Below                              0
Bounded Below and Above                    2
Free                                       0
Fixed                                      0

Number of Constraints                      3
Linear LE (<=)                             0
Linear EQ (=)                              0
Linear GE (>=)                             0
Linear Range                               0
Nonlinear LE (<=)                          0
Nonlinear EQ (=)                           0
Nonlinear GE (>=)                          3
Nonlinear Range                            0


                    The OPTMODEL Procedure

                      Solution Summary

Solver                                   SQP
Objective Function                       obj
Solution Status                      Optimal
Objective Value                      -6961.8
Iterations                                37

Infeasibility                   4.684243E-6
Optimality Error                2.3364017E-7
Complementarity                 4.684243E-6


                      x1          x2

                  14.095     0.84296
```

You can see from Output 13.4.3 that the SQP solver took less iterations to converge to the optimal solution (see Output 13.4.1).

Example 13.5. Unconstrained NLP Optimization

The SQP solver is designed mainly for constrained optimization problems, but it can be used for solving unconstrained optimization problems as well. Consider the following example:

$$\min \quad \sin x + \cos x$$

Assume the starting point $x^0 = 0$. You can use the following SAS code to solve the problem:

```
proc optmodel;
   var x init 0;   /* starting point */

   minimize obj =
      sin(x) + cos(x);

   solve with sqp/ printfreq = 0;
   print x;
quit;
```

The optimal solution is displayed in Output 13.5.1.

Output 13.5.1. Optimal Solution Using the SQP Solver

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	1
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	1
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	SQP
Objective Function	obj
Solution Status	Optimal
Objective Value	-1.4142
Iterations	4
Infeasibility	0
Optimality Error	8.7523537E-7
Complementarity	0

x
-2.3562

You can also solve the same function by using the NLPU solver for unconstrained NLP problems. The SAS code is as follows:

```
proc optmodel;
   var x init 0;   /* starting point */

   minimize obj =
      sin(x) + cos(x);

   solve;  /* the default solver is NLPU */
   print x;
quit;
```

The optimal solution is displayed in Output 13.5.2.

Output 13.5.2. Optimal Solution Using the NLPU Solver

The OPTMODEL Procedure

Problem Summary

Objective Sense	Minimization
Objective Function	obj
Objective Type	Nonlinear
Number of Variables	1
Bounded Above	0
Bounded Below	0
Bounded Below and Above	0
Free	1
Fixed	0
Number of Constraints	0

The OPTMODEL Procedure

Solution Summary

Solver	L-BFGS
Objective Function	obj
Solution Status	Optimal
Objective Value	-1.4142
Iterations	4
Optimality Error	1.046407E-6

x
-2.3562

The L-BFGS method is the default technique used in the NLPU solver when a nonlinear programming problem with no variable bounds is specified. See Chapter 11, "The Unconstrained Nonlinear Programming Solver," for details.

Example 13.6. Solving Large-Scale NLP Problems

The SQP solver exploits the sparsity of the Jacobian in NLP problems. This enables the SQP solver to solve reasonably large NLP problems. One such problem is the test example "porous1" from Vanderbei. It has 5184 variables and 4900 constraints. It is a known difficult problem to solve.

You can formulate this problem by using PROC OPTMODEL as follows:

```
proc optmodel;
title 'Vanderbei test example - porous1';
/*AMPL Model by Hande Y. Benson Copyright (C) 2001 Princeton
*University All Rights Reserved Permission to use, copy, modify,
*and distribute this software and its documentation for any
```

```
*purpose and without fee is hereby granted, provided that the
*above copyright notice appear in all copies and that the
*copyright notice and this permission notice appear in all
*supporting documentation.
*Source: example 3.2.4 in S. Eisenstat and H. Walker,
*"Choosing the forcing terms in an inexact Newton
*method" Report 6 / 94 / 75, Dept of Maths, Utah State University,
*1994. SIF input: Ph. Toint, July 1994. classification NOR2 - MN -
*V - V */

   number P = 72;
   number D = 50.0;
   number H = 1 / (P - 1);
   var u{i in 1..P,j in 1..P} init 1 - (i - 1) * (j - 1) * H^2;
   minimize f = 0;
   con cons1{i in 2..P - 1, j in 2..P - 2}: ((u[i + 1,j]^2 + u[i -
      1,j]^2  + u[i,j - 1]^2 + u[i,j + 1]^2 - 4 * u[i,j]^2) / H^2 + D *
      (u[i + 1,j]^3 - u[i - 1,j]^3) / (2 * H)) = 0;
   con cons2{i in 2..P - 2, j in P - 1..P - 1}: ((u[i + 1,j]^2 + u[i -
      1,j]^2  + u[i,j - 1]^2 + u[i,j + 1]^2 - 4 * u[i,j]^2) / H^2 + D *
      (u[i + 1,j]^3 - u[i - 1,j]^3) / (2 * H)) = 0;
   con cons3: ((u[P,P - 1]^2 + u[P - 2,P - 1]^2  + u[P - 1,P - 2]^2 +
      u[P - 1,P]^2 - 4 * u[P - 1,P - 1]^2) / H^2 + D * (u[P,P - 1]^3 -
      u[P - 2,P - 1]^3) / (2 * H) + 50) = 0;
   for {j in 1..P} fix u[1,j] = 1.0;
   for {j in 1..P} fix u[P,j] = 0.0;
   for {i in 2..P - 1} fix u[i,P] = 1.0;
   for {i in 2..P - 1} fix u[i,1] = 0.0;
   solve with sqp / printfreq = 5;
quit;
```

The iteration log in Output 13.6.1 displays the optimization progress and the optimal solution.

Output 13.6.1. Iteration Log

```
NOTE: The problem has 5180 variables (4900 free, 280 fixed).
NOTE: The problem has 0 linear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 4900 nonlinear constraints (0 LE, 4900 EQ, 0 GE, 0 range)
NOTE: The OPTMODEL presolver removed 280 variables, 0 linear constraints, and 0
      nonlinear constraints.
NOTE: The OPTMODEL presolved problem has 4900 variables, 0 linear constraints,
      and 4900 nonlinear constraints.
NOTE: Using analytic derivatives for objective.
NOTE: Using analytic derivatives for nonlinear constraints.
NOTE: The SQP solver is called.
                  Objective                      Optimality
        Iter          Value   Infeasibility          Error   Complementarity
           0       0.000000    11642.161333       0.000000          0.000000
           5       0.000000       47.211433     366.464895          0.000000
          10       0.000000        0.028692       0.005785          0.000000
          15       0.000000        0.000258       0.005546          0.000000
          18       0.000000        0.000002       0.000006          0.000000
NOTE: Converged.
NOTE: Objective = 0
```

References

Bartholomew-Biggs, M. C. (1988), "A Global Convergent Version of REQP for Constrained Minimization," *IMA Journal of Numerical Analysis*, 8, 253–271.

Conn, R., Gould, N. I. M., and Toint, P. L. (1994), "Large-Scale Nonlinear Constrained Optimization: A Current Survey," *Algorithms for Continuous Optimization: The State of the Art*, 434, 287–332.

Fletcher, R. (1987), *Practical Methods of Optimization*, Second Edition, Chichester, UK: John Wiley & Sons.

Nocedal, J. and Wright, S. J. (1999), *Numerical Optimization*, New York: Springer-Verlag.

Chapter 14
The MPS-Format SAS Data Set

Chapter Contents

Chapter 14
The MPS-Format SAS Data Set

Overview

The MPS file format is a format commonly used in industry for describing linear programming (LP) and integer programming (IP) problems (Murtagh 1981; IBM 1988). It can be extended to the QPS format (Maros and Meszaros 1999), which describes quadratic programming (QP) problems. MPS-format and QPS-format files are in text format and have specific conventions for the order in which the different pieces of the mathematical model are specified. The MPS-format SAS data set corresponds closely to the format of an MPS-format or QPS-format file and is used to describe linear programming, mixed integer programming, and quadratic programming problems for SAS/OR.

Observations

An MPS-format data set contains six variables: field1, field2, field3, field4, field5, and field6. The variables field4 and field6 are numeric type; the others are character type. Among the character variables, only the value of field1 is case-insensitive and leading blanks are ignored. Values of field2, field3, and field5 are case-sensitive and leading blanks are NOT ignored. Not all variables are used in a particular observation.

Observations in an MPS-format SAS data set are grouped into sections. Each section starts with an *indicator record*, followed by associated *data records*. Indicator records specify the names of sections and the format of the following data records. Data records contain the actual data values for a section.

Order of Sections

Sections of an MPS-format SAS data set must be specified in a **fixed** order.

Sections of linear programming problems are listed in the following order:

- NAME
- ROWS
- COLUMNS
- RHS (optional)
- RANGES (optional)
- BOUNDS (optional)
- ENDATA

Sections of quadratic programming problems are listed in the following order:

- NAME
- ROWS
- COLUMNS
- RHS (optional)
- RANGES (optional)
- BOUNDS (optional)
- QSECTION
- ENDATA

Sections of mixed integer programming problems are listed in the following order:

- NAME
- ROWS
- COLUMNS
- RHS (optional)
- RANGES (optional)
- BOUNDS (optional)
- BRANCH (optional)
- ENDATA

Sections Format

The following subsections describe the format of the records for each section of the MPS-format data set. Note that each section contains two types of records: an indicator record and multiple data records. The following subsections of this documentation describe the two different types of records for each section of the MPS data set.

NAME Section

The NAME section contains only a single record identifying the name of the problem.

Field1	Field2	Field3	Field4	Field5	Field6
NAME	Blank	Input model name	.	Blank	.

ROWS Section

The ROWS section contains the name and type of the rows (linear constraints or objectives). The type of each row is specified by the indicator code in field1 as follows:

- **MIN**: minimization objective
- **MAX**: maximization objective
- **N**: objective
- **G**: $\geq$ constraint
- **L**: $\leq$ constraint
- **E**: $=$ constraint

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
ROWS	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Indicator code	Row name	Blank	.	Blank	.

Notes:

1. At least one objective row should be specified in the ROWS section. It is possible to specify multiple objective rows. However, among all the data records indicating the objective, only the first one is regarded as the objective row, while the rest are ignored. If a type-N row is taken as the objective row, minimization is assumed.
2. Duplicate entries of field2 in the ROWS section are not allowed. In other words, row name is unique. The variable field2 in the ROWS section cannot take a missing value.

COLUMNS Section

The COLUMNS section defines the column (i.e., variable or decision variable) names of the problem. It also specifies the coefficients of the columns for each row.

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
COLUMNS	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Blank	Column name (e.g., col)	Row name (e.g., rowi)	Matrix element in row rowi, column col	Row name (e.g., rowj)	Matrix element in row rowj, column col

Notes:

1. All elements belonging to one column must be grouped together.
2. A missing coefficient value is ignored. A data record with missing values in both field4 and field6 is ignored.
3. Duplicate entries in each pair of column and row are not allowed.
4. When a sequence of data records have an identical value in field2, you can specify the value in the first occurrence and omit the value by giving a missing value in the other records. The value in field2 of the first data record in the section cannot be missing.

Mixed integer programming (MIP) problems require you to specify which variables are constrained to be integers. Integer variables can be specified in the COLUMNS section with the use of special marker records in the following form:

Field1	Field2	Field3	Field4	Field5	Field6
Blank	Marker name	'MARKER' (including the quotation marks)	.	'INTORG' or 'INTEND' (including the quotation marks)	.

A marker record with field5 that contains the value 'INTORG' indicates the start of binary variables. In the marker record that indicates the end of binary variables, field5 must be 'INTEND'. All variables between INTORG and INTEND markers are taken to be binary variables. In BOUNDS sections, you can change a variable from binary to integer by specifying its upper bound to be an integer other than one or specifying its lower bound to be an integer other than zero. An alternative way to specify integer variables is described in the section "BOUNDS Section (Optional)" on page 1015.

Notes:

1. INTORG and INTEND markers must appear in pairs in the COLUMNS section. The marker pairs can appear any number of times.
2. The marker name in field2 should be different from the preceding and following column names.
3. For every column enclosed by INTORG and INTEND markers, by default the upper bound is 1 and the lower bound is 0.

RHS Section (Optional)

The RHS section specifies the right-hand-side value for the rows. Any row unspecified in this section is considered to have an RHS value of 0. Missing the entire RHS section implies that all RHS values are 0.

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
RHS	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Blank	RHS name	Row name (e.g., rowi)	RHS value for row rowi	Row name (e.g., rowj)	RHS value for row rowj

Notes:

1. The rows that have an RHS element defined in this section need not be specified in the same order in which the rows were specified in the ROWS section. However, a row in the RHS section should be defined in the ROWS section.
2. It is possible to specify multiple RHS vectors, which are labeled by different RHS names. Normally, the first RHS vector encountered in the RHS section is used, and all other RHS vectors are discarded. All the elements of the selected RHS vector must be specified before other RHS vectors are introduced. Within a specific RHS vector, for a given row, duplicate assignments of RHS values are not allowed.
3. An RHS value assigned to the objective row is ignored by PROC OPTLP and PROC OPTMILP, while it is taken as a constant term of the objective function by PROC OPTQP.
4. A missing value in field4 or field6 is ignored. A data record with missing values in both field4 and field6 is ignored.

5. When a sequence of data records have an identical value in field2, you can specify the value in the first occurrence and omit the value by giving a missing value in the other records. If the value in field2 of the first data record in the section is missing, it means the name of the first vector is the missing value.

RANGES Section (Optional)

The RANGES section specifies the range of the RHS value for the constraint rows. With range specification, a row can be constrained from above and below.

For a constraint row c, if b is the RHS value and r is the range for this row, then the equivalent constraints are given in Table 14.1, depending on the type of row and the sign of r.

Table 14.1. Range Effect

Type of Row	Sign of r	Equivalent Constraints
G	$\pm$	$b \leq c \leq b + \|r\|$
L	$\pm$	$b - \|r\| \leq c \leq b$
E	$+$	$b \leq c \leq b + r$
E	$-$	$b + r \leq c \leq b$

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
RANGES	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Blank	Range name	Row name (e.g., rowi)	Range for RHS of row rowi	Row name (e.g., rowj)	Range for RHS of row rowj

Notes:

1. Range assignment for an objective row (i.e., **MAX**, **MIN**, or **N** row) is not allowed.
2. The rows that have a range element defined in this section need not be specified in the same order in which the rows were specified in the ROWS or RHS section. However, a row in the RANGES section should be defined in the ROWS section.

3. It is possible to specify multiple range vectors, which are labeled by different range names. Normally, the first range vector encountered in the RANGES section is used, and all other range vectors are discarded. All the elements in a range vector must be specified before other range vectors are introduced. Within the specific range vector, for a given range, duplicate assignments of range values are not allowed.
4. A missing value in field4 or field6 is ignored. A data record with missing values in both field4 and field6 is ignored.
5. When a sequence of data records have an identical value in field2, you can specify the value in the first occurrence and omit the value by giving a missing value in the other records. If the value in field2 of the first data record in the section is missing, it means the name of the first vector is the missing value.

BOUNDS Section (Optional)

The BOUNDS section specifies bounds for the columns.

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
BOUNDS	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Bound type	Bound name	Column name	Bound for the column	Blank	Blank

Notes:

1. For every column, by default, the upper bound is $+\infty$ and the lower bound is 0.
2. General bound types include **LO**, **UP**, **FX**, **FR**, **MI**, and **PL**. Suppose the bound for a column identified in field3 is specified as b in field4. Table 14.2 explains the effects of different bound types.

Table 14.2. Bound Type Rules

Bound Type	Ignore b	Resultant Lower Bound	Resultant Upper Bound
LO	No	b	*unspecified*
UP	No	*unspecified*	b
FX	No	b	b
FR	Yes	$-\infty$	$+\infty$
MI	Yes	$-\infty$	*unspecified*
PL	Yes	*unspecified*	$+\infty$

Mixed integer programming problems can specify integer variables in the BOUNDS section. Table 14.3 shows bound types defined for MIP.

Table 14.3. Bound Type Rules

Bound Type	Ignore b	Variable Type	Value
BV	Yes	binary	0 or 1
LI	No	integer	$[b, \infty)$
UI	No	integer	$(-\infty, b]$

3. The columns that have bounds need not be specified in the same order in which the columns were specified in the COLUMNS section. However, all columns in the BOUNDS section should be defined in the COLUMNS section.
4. It is possible to specify multiple bound vectors, which are labeled by different bound names. Normally, the first bound vector encountered in the BOUNDS section is used, and all other bound vectors are discarded. All the elements of the selected bound vector must be specified before other bound vectors are introduced.
5. Within a particular BOUNDS vector, for a given column, if a bound (lower or upper) is explicitly specified by the bound type rules listed in Table 14.2, any other specification is considered to be an error.
6. If the value in field1 is **LO**, **UP**, **FX**, **LI**, or **UI**, then a data record with a missing value in field4 is ignored.
7. When a sequence of data records have an identical value in field2, you can specify the value in the first occurrence and omit the value by giving a missing value in the other records. If the value in field2 of the first data record in the section is missing, it means the name of the first vector is the missing value.

BRANCH Section (Optional)

Sometimes the user wants to specify branching priorities or directions for integer variables to improve performance. Variables with higher priorities are branched on before variables with lower priorities. The branch direction is used to decide which branch to take first at each node.

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
BRANCH	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Branch direction	Blank	First column name	First column priority	Second column name	Second column priority

Notes:

1. Valid directions include **UP** (up branch), **DN** (down branch), **RD** (rounding) and **CB** (closest bound). If field1 is blank, the solver automatically decides the direction.
2. If field4 is missing, then the name defined in field3 is ignored. Similarly, if field6 is missing, then the name defined in field5 is ignored.
3. The priority value in field4 and field6 must be nonnegative. Zero is the lowest priority and is also the default.

QSECTION Section

The QSECTION section is needed only to describe quadratic programming problems. It specifies the coefficients of the quadratic terms in the objective function.

- Indicator record:

Field1	Field2	Field3	Field4	Field5	Field6
QSECTION or QUADOBJ	Blank	Blank	.	Blank	.

- Data record:

Field1	Field2	Field3	Field4	Field5	Field6
Blank	Column name	Column name	Coefficient in objective function	Blank	.

Notes:

1. The QSECTION section is required for PROC OPTQP and should not appear for PROC OPTLP. For PROC OPTQP, there should be at least one valid data record in the QSECTION section. For PROC OPTLP, an error is reported when the submitted data set contains a QSECTION section.
2. The variables field2 and field3 contain the names of the columns that form a quadratic term in the objective function. They must have been defined in the COLUMNS section. They need not refer to the same column. Zero entries should not be specified.
3. Duplicate entries of a quadratic term are not allowed. This means the combination of (field2, field3) must be unique, where (k, j) and (j, k) are considered to be the same combination.
4. If field4 of one data record is missing or takes a value of zero, then this data record is ignored.

ENDATA Section

The ENDATA section simply declares the end of all records. It contains only one indicator record, where field1 takes the value ENDATA and the values of the remaining variables are blank or missing.

Details

Converting an MPS/QPS-Format File: %MPS2SASD

As described in the section "Overview" on page 1009, the MPS or QPS format is a standard file format for describing linear, integer, and quadratic programming problems. The MPS/QPS format is defined for plain text files, whereas in the SAS System it is more convenient to read data from SAS data sets. Therefore, a facility is required to convert MPS/QPS-format text files to MPS-format SAS data sets. The SAS macro %MPS2SASD serves this purpose.

In the MPS/QPS-format text file, a record refers to a single line of text that is divided into six fields. MPS/QPS files can be read and printed in both *fixed* and *free* format. In fixed MPS/QPS format, each field of a data record must occur in specific columns:

Field	Field 1	Field 2	Field 3	Field 4	Field 5	Field 6
Columns	2–3	5–12	15–22	25–36	40–47	50–61

In free format, fields of a record are separated by a space. Both fixed and free format have limitations. If users need to use row names or column names longer than 8 characters, then there is not enough space to hold them in fixed format. If users use a space as a part of a row name or column name, such as "ROW NAME", then free-format MPS format interprets this symbol as two fields, "ROW" and "NAME".

You can insert a comment record, denoted by an asterisk (*) in column 1, anywhere in an MPS/QPS file. Also, if a dollar sign ($) is the first character in field 3 or field 5 of any record, the information from that point to the end of the record is treated as a comment. All comments are ignored by the %MPS2SASD macro, described as follows.

%MPS2SASD Macro Parameters

%MPS2SASD (MPSFILE='infilename', OUTDATA=mpsdata, MAXLEN=n, FORMAT=FIXED/FREE);

MPSFILE=*'infilename'*
specifies the path and name of the input MPS-format file. The input file is a plain text file, normally with either an ".mps" extension for linear programming problems or a ".qps" extension for quadratic programming problems. This parameter is required; there is no default value.

OUTDATA=*mpsdata*
specifies the name of the output MPS-format SAS data set. This parameter is optional; the default value is mpsdata.

MAXLEN=*n*
specifies length of the variables field2, field3, and field5 in the output MPS-format SAS data set. This parameter is optional; the default value is 8.

FORMAT=*FIXED/FREE*

specifies the format of the input MPS file. Valid values can be either FIXED or FREE. This parameter is optional; the default value is the one, if any, specified by the flat file and FIXED otherwise.

Length of Variables

In an MPS-format SAS data set, normally the variables field2, field3, and field5 hold the names of the rows and columns. The length of these character variables is limited to the maximum size of a SAS character variable. This enables you to use sufficiently long names for the rows and columns in your model.

In a SAS data set generated by the %MPS2SASD macro, the length of the variables field2, field3, and field5 is fixed to be 8 ASCII characters by default. This length fits the fixed-format MPS/QPS file well since field 2, field 3, and field 5 are fixed at 8 characters. However, the free-format MPS/QPS files might have longer row names or longer column names. The %MPS2SASD macro provides a parameter **MAXLEN = n**. Using this parameter, you can set the variables field2, field3, and field5 to have a length of n characters in the output SAS data set.

The parameter MAXLEN works only when the given MPS file is in free format. For a fixed-format MPS file, this parameter is ignored and the length of field2, field3, and field5 is 8 characters by default.

Examples

Example 14.1. MPS-Format Data Set for a Product Mix Problem

Consider a simple product mix problem where a furniture company tries to find an optimal product mix of four items: a desk (x_1), a chair (x_2), a cabinet (x_3), and a bookcase (x_4). Each item is processed in a stamping department (STAMP), an assembly department (ASSEMB), and a finishing department (FINISH). The time each item requires in each department is given in the input data. Because of resource limitations, each department has an upper limit on the time available for processing. Furthermore, because of labor constraints, the assembly department must work at least 300 hours. Finally, marketing tells you not to make more than 75 chairs, to make at least 50 bookcases, and to find the range over which the selling price of a bookcase can vary without changing the optimal product mix.

This problem can be expressed as the following linear program:

$$
\begin{array}{lrcrl}
\max & 95x_1 + 41x_2 + 84x_3 + 76x_4 & & & \\
\text{subject to} & 3x_1 + 1.5x_2 + 2x_3 + 2x_4 & \leq & 800 & (\text{STAMP}) \\
& 10x_1 + 6x_2 + 8x_3 + 7x_4 & \leq & 1200 & (\text{ASSEMB}) \\
& 10x_1 + 6x_2 + 8x_3 + 7x_4 & \geq & 300 & (\text{ASSEMB}) \\
& 10x_1 + 8x_2 + 8x_3 + 7x_4 & \leq & 800 & (\text{FINISH}) \\
& x_2 & \leq & 75 & \\
& x_4 & \geq & 50 & \\
& x_i & \geq & 0 & i = 1, 2, 3
\end{array}
$$

The following DATA step saves the problem specification as an MPS-format SAS data set:

```
data prodmix;
   infile datalines;
   input field1 $ field2 $ field3$ field4 field5 $ field6;
   datalines;
NAME          .          PROD_MIX        .          .          .
ROWS          .             .            .          .          .
N           PROFIT          .            .          .          .
L           STAMP           .            .          .          .
L           ASSEMB          .            .          .          .
L           FINISH          .            .          .          .
N           CHNROW          .            .          .          .
N           PRICE           .            .          .          .
COLUMNS       .             .            .          .          .
.           DESK          STAMP          3.0      ASSEMB        10
.           DESK          FINISH        10.0      PROFIT       -95
.           DESK          PRICE        175.0         .          .
.           CHAIR         STAMP          1.5      ASSEMB         6
.           CHAIR         FINISH         8.0      PROFIT       -41
.           CHAIR         PRICE         95.0         .          .
.           CABINET       STAMP          2.0      ASSEMB         8
.           CABINET       FINISH         8.0      PROFIT       -84
.           CABINET       PRICE        145.0         .          .
.           BOOKCSE       STAMP          2.0      ASSEMB         7
.           BOOKCSE       FINISH         7.0      PROFIT       -76
.           BOOKCSE       PRICE        130.0      CHNROW         1
RHS           .             .            .          .          .
.           TIME          STAMP        800.0      ASSEMB      1200
.           TIME          FINISH       800.0         .          .
RANGES        .             .            .          .          .
.           T1            ASSEMB       900.0         .          .
BOUNDS        .             .            .          .          .
UP          BND           CHAIR         75.0         .          .
LO          BND           BOOKCSE       50.0         .          .
```

```
ENDATA          .               .               .          .               .
;
```

Example 14.2. Fixed-MPS-Format File

The following file, example_fix.mps, contains the data from Example 14.1 in the form of a fixed-MPS-format file:

```
* THIS IS AN EXAMPLE FOR FIXED MPS FORMAT.
NAME          PROD_MIX
ROWS
 N  PROFIT
 L  STAMP
 L  ASSEMB
 L  FINISH
 N  CHNROW
 N  PRICE
COLUMNS
    DESK      STAMP          3.00000   ASSEMB        10.00000
    DESK      FINISH        10.00000   PROFIT       -95.00000
    DESK      PRICE        175.00000
    CHAIR     STAMP          1.50000   ASSEMB         6.00000
    CHAIR     FINISH         8.00000   PROFIT       -41.00000
    CHAIR     PRICE         95.00000
    CABINET   STAMP          2.00000   ASSEMB         8.00000
    CABINET   FINISH         8.00000   PROFIT       -84.00000
    CABINET   PRICE        145.00000
    BOOKCSE   STAMP          2.00000   ASSEMB         7.00000
    BOOKCSE   FINISH         7.00000   PROFIT       -76.00000
    BOOKCSE   PRICE        130.00000   CHNROW         1.00000
RHS
    TIME      STAMP        800.00000   ASSEMB       1200.0000
    TIME      FINISH       800.00000
RANGES
    T1        ASSEMB       900.00000
BOUNDS
 UP BND       CHAIR         75.00000
 LO BND       BOOKCSE       50.00000
ENDATA
```

Example 14.3. Free-MPS-Format File

In free format, fields in data records other than the first record have no predefined positions. They can be written anywhere except column 1, with each field separated from the next by one or more blanks (a tab cannot be used as a field separator). However, the fields must appear in the same sequence as in the fixed format. The following file, example_free.mps, is an example. It describes the same problem as in Example 14.2.

```
* THIS IS AN EXAMPLE FOR FREE MPS FORMAT.
NAME          PROD_MIX  FREE
ROWS
 N  PROFIT
    L  STAMP
    L  ASSEMB
    L  FINISH
 N  CHNROW
 N  PRICE
COLUMNS
 DESK       STAMP    3.00000   ASSEMB      10.00000
 DESK       FINISH   10.00000    PROFIT   -95.00000
 DESK       PRICE    175.00000
    CHAIR   STAMP    1.50000   ASSEMB       6.00000
    CHAIR   FINISH   8.00000     PROFIT   -41.00000
    CHAIR   PRICE    95.00000
 CABINET    STAMP    2.00000   ASSEMB       8.00000
 CABINET    FINISH   8.00000     PROFIT   -84.00000
 CABINET    PRICE    145.00000
    BOOKCSE STAMP    2.00000   ASSEMB       7.00000
    BOOKCSE FINISH   7.00000     PROFIT   -76.00000
    BOOKCSE PRICE    130.00000   CHNROW     1.00000
RHS
    TIME STAMP   800.00000  ASSEMB 1200.0000
    TIME FINISH  800.00000
RANGES
    T1   ASSEMB   900.00000
BOUNDS
 UP BND     CHAIR 75.00000
 LO BND   BOOKCSE 50.00000
ENDATA
```

Example 14.4. Using the %MPS2SASD Macro

We illustrate the use of the %MPS2SASD macro in this example, assuming the files example_fix.mps and example_free.mps are in your current SAS working directory.

The MPS2SASD macro function has one required parameter, MPSFILE= 'infile-name', which specifies the path and name of the MPS/QPS-format file. With this single parameter, the macro reads the file, converts the records, and saves the conversion to the default MPS-format SAS data set MPSDATA.

Running the following statements converts the fixed-format MPS file shown in Example 14.2 to the MPS-format SAS data set MPSDATA:

```
%mps2sasd(mpsfile='example_fix.mps');
proc print data=mpsdata;
run;
```

Output 14.4.1 displays the MPS-format SAS data set MPSDATA.

Output 14.4.1. The MPS-Format SAS Data Set MPSDATA

Obs	field1	field2	field3	field4	field5	field6
1	NAME		PROD_MIX	.		.
2	ROWS			.		.
3	N	PROFIT		.		.
4	L	STAMP		.		.
5	L	ASSEMB		.		.
6	L	FINISH		.		.
7	N	CHNROW		.		.
8	N	PRICE		.		.
9	COLUMNS			.		.
10		DESK	STAMP	3.0	ASSEMB	10
11		DESK	FINISH	10.0	PROFIT	-95
12		DESK	PRICE	175.0		.
13		CHAIR	STAMP	1.5	ASSEMB	6
14		CHAIR	FINISH	8.0	PROFIT	-41
15		CHAIR	PRICE	95.0		.
16		CABINET	STAMP	2.0	ASSEMB	8
17		CABINET	FINISH	8.0	PROFIT	-84
18		CABINET	PRICE	145.0		.
19		BOOKCSE	STAMP	2.0	ASSEMB	7
20		BOOKCSE	FINISH	7.0	PROFIT	-76
21		BOOKCSE	PRICE	130.0	CHNROW	1
22	RHS			.		.
23		TIME	STAMP	800.0	ASSEMB	1200
24		TIME	FINISH	800.0		.
25	RANGES			.		.
26		T1	ASSEMB	900.0		.
27	BOUNDS			.		.
28	UP	BND	CHAIR	75.0		.
29	LO	BND	BOOKCSE	50.0		.
30	ENDATA			.		.

Running the following statement converts the free-format MPS file shown in Example 14.3 to the MPS-format SAS data set MPSDATA:

```
%mps2sasd(mpsfile='example_free.mps');
```

The data set is identical to the one shown in Output 14.4.1.

In the following statement, when the free-format MPS file is converted, the length of the variables field2, field3, and field5 in the SAS data set MPSDATA is explicitly set to 10 characters:

```
%mps2sasd(mpsfile='example_free.mps', maxlen=10, format=free);
```

If you want to save the converted data to a SAS data set other than the default data set MPSDATA, you can use the parameter OUTDATA= mpsdata. The following statement reads data from the file example_fix.mps and writes the converted data to the data set PRODMIX:

```
%mps2sasd(mpsfile='example_fix.mps', outdata=PRODMIX);
```

References

IBM (1988), *Mathematical Programming System Extended/370 (MPSX/370) Version 2 Program Reference Manual*, volume SH19-6553-0, IBM.

Maros, I. and Meszaros, C. (1999), "A Repository of Convex Quadratic Programming Problems," *Optimization Methods and Software*, 11–12, 671–681.

Murtagh, B. A. (1981), *Advanced Linear Programming, Computation and Practice*, New York: McGraw-Hill.

Chapter 15
The OPTLP Procedure

Chapter Contents

Chapter 15
The OPTLP Procedure

Overview: OPTLP Procedure

The OPTLP procedure provides three methods of solving linear programs (LPs). A linear program has the following formulation:

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \; \{\geq, =, \leq\} \; \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

where

$\mathbf{x}$	$\in$	$\mathbb{R}^n$	is the vector of decision variables
$\mathbf{A}$	$\in$	$\mathbb{R}^{m \times n}$	is the matrix of constraints
$\mathbf{c}$	$\in$	$\mathbb{R}^n$	is the vector of objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{R}^m$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l}$	$\in$	$\mathbb{R}^n$	is the vector of lower bounds on variables
$\mathbf{u}$	$\in$	$\mathbb{R}^n$	is the vector of upper bounds on variables

The following LP solvers are available in the OPTLP procedure:

- primal simplex solver
- dual simplex solver
- interior point solver (experimental)

The simplex solvers implement the two-phase simplex method. In phase I, the solver tries to find a feasible solution. If no feasible solution is found, the LP is infeasible; otherwise, the solver enters phase II to solve the original LP. The interior point solver implements a primal-dual predictor-corrector interior point algorithm.

PROC OPTLP requires a linear program to be specified using a SAS data set that adheres to the MPS format, a widely accepted format in the optimization community. For details about the MPS format see Chapter 14, "The MPS-Format SAS Data Set."

You can use the SAS macro %LP2MPSD to convert typical PROC LP format data sets into MPS-format SAS data sets. The macro is available online at the SAS Customer Support Center.

Getting Started: OPTLP Procedure

The following example illustrates how you can use the OPTLP procedure to solve linear programs. Suppose you want to solve the following problem:

$$
\begin{array}{rrcrcrcrl}
\min & 2x_1 & - & 3x_2 & - & 4x_3 & & & \\
\text{subject to} & & - & 2x_2 & - & 3x_3 & \geq & -5 & (\text{R1}) \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 4 & (\text{R2}) \\
& x_1 & + & 2x_2 & + & 3x_3 & \leq & 7 & (\text{R3}) \\
& & & x_1, & & x_2, \; x_3 & \geq & 0 &
\end{array}
$$

The corresponding MPS-format SAS data set is as follows:

```
data example;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME          .       EXAMPLE     .    .      .
ROWS          .         .         .    .      .
N             COST      .         .    .      .
G             R1        .         .    .      .
L             R2        .         .    .      .
L             R3        .         .    .      .
COLUMNS       .         .         .    .      .
.             X1      COST        2    R2     1
.             X1      R3          1    .      .
.             X2      COST       -3    R1    -2
.             X2      R2          1    R3     2
.             X3      COST       -4    R1    -3
.             X3      R2          2    R3     3
RHS           .       .          .    .      .
.             RHS     R1         -5    R2     4
.             RHS     R3          7    .      .
ENDATA        .       .          .    .      .
;
```

You can also create this data set from an MPS-format flat file (examp.mps) by using the following SAS macro:

```
%mps2sasd(mpsfile = "examp.mps", outdata = example);
```

Note: The SAS macro %MPS2SASD is provided in SAS/OR software. See the section "Converting an MPS/QPS-Format File: %MPS2SASD" on page 1019 for details.

You can use the following statement to call the OPTLP procedure:

```
proc optlp data=example
  objsense=min
  presolver = automatic
  solver    = primal_spx
  primalout = expout
  dualout   = exdout;
run;
```

Note: The "N" designation for "COST" in the rows section of the data set example also specifies a minimization problem. See the section "ROWS Section" on page 1011 for details.

The optimal primal and dual solutions are stored in the data sets expout and exdout, respectively, and are displayed in Figure 15.1.

The OPTLP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient
1	COST	RHS	X1	N	2
2	COST	RHS	X2	N	-3
3	COST	RHS	X3	N	-4

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	0	1.7977E308	0.0	L	2.0
2	0	1.7977E308	2.5	B	0.0
3	0	1.7977E308	0.0	L	0.5

The OPTLP Procedure
Dual Solution

Obs	Objective Function ID	RHS ID	Constraint Name	Constraint Type	Constraint RHS	Constraint Lower Bound
1	COST	RHS	R1	G	-5	.
2	COST	RHS	R2	L	4	.
3	COST	RHS	R3	L	7	.

Obs	Constraint Upper Bound	Dual Variable Value	Constraint Status	Constraint Activity
1	.	1.5	U	-5.0
2	.	0.0	B	2.5
3	.	0.0	B	5.0

Figure 15.1. Primal and Dual Solution Output

For details about the type and status codes displayed for variables and constraints, see the section "Data Input and Output" on page 1039.

Syntax: OPTLP Procedure

The following statement is available in the OPTLP procedure:

PROC OPTLP < *options* > **;**

Functional Summary

Table 15.1 summarizes the list of options available for the OPTLP procedure, classified by function.

Table 15.1. Options for the OPTLP Procedure

Description	Option
Data Set Options:	
input data set	DATA=
dual input data set for warm start	DUALIN=
dual solution output data set	DUALOUT=
objective sense (maximization or minimization)	OBJSENSE=
primal input data set for warm start	PRIMALIN=
primal solution output data set	PRIMALOUT=
save output data sets only if optimal	SAVE_ONLY_IF_OPTIMAL
Solver Option:	
type of solver	SOLVER=
Presolve Option:	
type of presolve	PRESOLVER=
Control Options:	
feasibility tolerance	FEASTOL=
maximum number of iterations	MAXITER=
upper limit on time used to solve the problem	MAXTIME=
optimality tolerance	OPTTOL=
frequency of printing solution progress	PRINTFREQ=
enable/disable printing summary	PRINTLEVEL=
use CPU/real time	TIMETYPE=
Simplex Algorithm Options:	
type of initial basis	BASIS=
type of pricing strategy	PRICETYPE=
queue size for determining entering variable	QUEUESIZE=

Table 15.1. (continued)

Description	Option
enable or disable scaling of the problem	SCALE=
Interior Point Algorithm Options:	
stopping criterion based on duality gap	STOP_DG=
stopping criterion based on dual infeasibility	STOP_DI=
stopping criterion based on primal infeasibility	STOP_PI=

PROC OPTLP Statement

PROC OPTLP < *options* > **;**

You can specify the following options in the PROC OPTLP statement.

Data Set Options

DATA=*SAS-data-set*
specifies the input data set corresponding to the LP model. If this option is not specified, PROC OPTLP will use the most recently created SAS data set. See Chapter 14, "The MPS-Format SAS Data Set," for more details about the input data set.

DUALIN=*SAS-data-set*
DIN=*SAS-data-set*
specifies the input data set corresponding to the dual solution that is required for warm starting the simplex solvers. See the section "Data Input and Output" on page 1039 for details.

DUALOUT=*SAS-data-set*
DOUT=*SAS-data-set*
specifies the output data set for the dual solution. This data set contains the dual solution information. See the section "Data Input and Output" on page 1039 for details.

OBJSENSE=*option*
specifies whether the LP model is a minimization or a maximization problem. You specify OBJSENSE=MIN for a minimization problem and OBJSENSE=MAX for a maximization problem. Alternatively, you can specify the objective sense in the input data set; see the section "ROWS Section" on page 1011 for details. If for some reason the objective sense is specified differently in these two places, this option supersedes the objective sense specified in the input data set. If the objective sense is not specified anywhere, then PROC OPTLP interprets and solves the linear program as a minimization problem.

PRIMALIN=*SAS-data-set*
PIN=*SAS-data-set*
specifies the input data set corresponding to the primal solution that is required for warm starting the simplex solvers. See the section "Data Input and Output" on page 1039 for details.

PRIMALOUT=*SAS-data-set*
POUT=*SAS-data-set*
specifies the output data set for the primal solution. This data set contains the primal solution information. See the section "Data Input and Output" on page 1039 for details.

SAVE_ONLY_IF_OPTIMAL
specifies that the PRIMALOUT= and DUALOUT= data sets be saved only if the final solution obtained by the solver at termination is optimal. If the PRIMALOUT= and DUALOUT= options are specified, then by default (i.e., omitting the SAVE_ONLY_IF_OPTIMAL option), PROC OPTLP always saves the solutions obtained at termination, regardless of the final status.

Solver Option

SOLVER=*option*
SOL=*option*
specifies one of the following LP solvers:

Option	Description
PRIMAL_SPX (PS)	Use primal simplex solver.
DUAL_SPX (DS)	Use dual simplex solver.
ITERATIVE_INTERIOR (II), experimental	Use interior point solver.

The valid abbreviated value for each option is indicated in parentheses. By default, the dual simplex solver is used.

Presolve Options

PRESOLVER=*option*
PRESOL=*option*
specifies one of the following presolve options:

Option	Description
NONE (0)	Disable presolver.
AUTOMATIC (-1)	Apply presolver by using default setting.
BASIC (1)	Perform basic presolve like removing empty rows, columns, and fixed variables.
MODERATE (2)	Perform basic presolve and apply other inexpensive presolve techniques.
AGGRESSIVE (3)	Perform moderate presolve and apply other aggressive (but expensive) presolve techniques.

You can also specify the option by specifying an integer from -1 to 3. The integer value for each option is indicated in parentheses. The default option is AUTOMATIC (-1). See the section "Presolve" on page 1043 for details.

Control Options

FEASTOL=ϵ

specifies the feasibility tolerance $\epsilon \in$[1E–9, 1E–4] for determining the feasibility of a variable value. The default value is 1E–6.

MAXITER=k

specifies the maximum number of iterations. The value k can be any integer greater than or equal to one. If you do not specify this option, the procedure does not stop based on the number of iterations performed.

MAXTIME=k

specifies an upper limit of k seconds of time for the optimization process. The timer used by this option is determined by the value of the TIMETYPE= option. If you do not specify this option, the procedure does not stop based on the amount of time elapsed.

OPTTOL=ϵ

specifies the optimality tolerance $\epsilon \in$[1E–9, 1E–4] for declaring optimality. The default value is 1E–6.

PRINTFREQ=k

specifies that the printing of the solution progress to the iteration log should occur after every k iterations. The print frequency, k, is an integer greater than or equal to zero.

The value $k = 0$ disables the printing of the progress of the solution.

If the PRINTFREQ= option is not specified, then PROC OPTLP displays the iteration log with a dynamic frequency according to the problem size if one of the simplex solvers is used, or with frequency 1 if the interior point solver is used.

PRINTLEVEL=*0* | *1*

specifies whether a summary of the problem and solution should be printed. If PRINTLEVEL=1, then two ODS (Output Delivery System) tables named "ProblemSummary" and "SolutionSummary" are produced and printed. If PRINTLEVEL=0, then no ODS tables are produced or printed. The default value of this option is 1.

For details about the ODS tables created by PROC OPTLP see the section "ODS Tables" on page 1047.

TIMETYPE=*CPU* | *REAL*

specifies type of the time used in a PROC OPTLP call. Numeric values of time can be specified in the MAXTIME= option or reported in the _OROPTLP_ macro variable. The value of this option determines whether such time is CPU time or real time. The default value of this option is CPU.

Simplex Algorithm Options

BASIS=*option*

specifies the following options for generating an initial basis:

Option	Description
CRASH (0)	Generate an initial basis by using crash techniques (Maros 2003). The procedure creates a triangular basic matrix consisting of both decision variables and slack variables.
SLACK (1)	Generate an initial basis by using all slack variables.
WARMSTART (2)	Start the simplex solvers with a user-specified initial basis. The PRIMALIN= and DUALIN= data sets are required to specify an initial basis.

You can also specify the option by specifying an integer from 0 to 2. The integer value for each option is indicated in parentheses. The default option is CRASH (0).

PRICETYPE=*option*

specifies one of the following pricing strategies for the simplex solvers:

Option	Description
HYBRID (0)	Use a hybrid of Devex and steepest-edge pricing strategies. Available for the primal simplex solver only.
PARTIAL (1)	Use the Dantzig's rule on a queue of decision variables. Optionally, you can specify QUEUESIZE=. Available for the primal simplex solver only.
FULL (2)	Use the Dantzig's rule on all decision variables.
DEVEX (3)	Use Devex pricing strategy.
STEEPESTEDGE (4)	Use steepest-edge pricing strategy.

You can also specify the option by specifying an integer from 0 to 4. The integer value for each option is indicated in parentheses. The default pricing strategy for the primal simplex solver is HYBRID (0) and for the dual simplex solver is STEEPESTEDGE (4). See the section "Pricing Strategies for the Simplex Solvers" on page 1043 for details.

QUEUESIZE=*k*

specifies the queue size $k \in [1, n]$, where n is the number of decision variables. This queue is used for finding an entering variable in the simplex iteration. The default value is chosen adaptively based on the number of decision variables. This option is used only when PRICETYPE=PARTIAL.

SCALE=*option*
specifies one of the following scaling options:

Option	Description
NONE (0)	Disable scaling.
AUTOMATIC (−1)	Automatically apply scaling procedure if necessary.

You can also specify the option by specifying the integer −1 or 0. The integer value for each option is indicated in parentheses. The default option is AUTOMATIC (–1).

Interior Point Algorithm Options

STOP_DG=δ
specifies the desired relative duality gap $\delta \in$[1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 1044 for details.

STOP_DI=β
specifies the maximum allowed relative dual constraints violation $\beta \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 1044 for details.

STOP_PI=α
specifies the maximum allowed relative bound and primal constraints violation $\alpha \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "The Interior Point Algorithm: Overview" on page 1044 for details.

PROC OPTLP Macro Variable

The OPTLP procedure defines a macro variable named _OROPTLP_. This variable contains a character string that indicates the status of the OPTLP procedure upon termination. The various terms of the variable are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	procedure terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	maximum time limit reached
FAILED	solver failed to converge, possibly due to numerical issues

OBJECTIVE
indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY
indicates the (relative) primal infeasibility of the constraints by the solution.

DUAL_INFEASIBILITY
indicates the (relative) dual infeasibility of the constraints by the solution.

BOUND_INFEASIBILITY
indicates the (relative) violation by the solution of the lower and/or upper bounds.

DUALITY_GAP
indicates the (relative) duality gap. This term appears only if the interior point solver is used.

COMPLEMENTARITY
indicates the (absolute) complementarity. This term appears only if the interior point solver is used.

ITERATIONS
indicates the number of iterations taken to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken to solve the problem (seconds).

Note: The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time (default) or real time. The type is determined by the TIMETYPE= option.

Details: OPTLP Procedure

In this section we describe in detail the various data sets pertaining to PROC OPTLP in both regular and warm starting scenarios. We also provide brief overviews of the following:

- the types of presolve techniques available for the solvers
- the various pricing strategies available for the simplex solvers
- the warm starting option for the simplex solvers
- the interior point algorithm

We conclude the section with a description of the iteration log corresponding to each of the solvers.

Data Input and Output

This subsection describes the PRIMALIN= and DUALIN= data sets required to warm start the simplex solvers, and the PRIMALOUT= and DUALOUT= output data sets.

Definitions of Variables in the PRIMALIN= Data Set

The PRIMALIN= data set has two required variables defined as follows:

VAR
specifies the name of the decision variable.

STATUS
specifies the status of the decision variable. It can take one of the following values:

B basic variable

L nonbasic variable at its lower bound

U nonbasic variable at its upper bound

F free variable

A newly added variable in the modified LP model when using the BASIS=WARMSTART option

Note: The PRIMALIN= data set is created from the PRIMALOUT= data set obtained from a previous "normal" run of PROC OPTLP—i.e., using only the DATA= data set as the input.

Definitions of Variables in the DUALIN= Data Set

The DUALIN= data set also has two required variables defined as follows:

ROW
specifies the name of the constraint.

STATUS
specifies the status of the slack variable for a given constraint. It can take one of the following values:

- B basic variable
- L nonbasic variable at its lower bound
- U nonbasic variable at its upper bound
- F free variable
- A newly added variable in the modified LP model when using the BASIS=WARMSTART option

Note: The DUALIN= data set is created from the DUALOUT= data set obtained from a previous "normal" run of PROC OPTLP—i.e., using only the DATA= data set as the input.

Definitions of Variables in the PRIMALOUT= Data Set

The PRIMALOUT= data set contains the primal solution to the LP model; each observation corresponds to a variable of the LP problem. See Example 15.1 for an example of the PRIMALOUT= data set. The variables in the data set have the following names and meanings.

_OBJ_ID_
specifies the name of the objective function. This is particularly useful when there are multiple objective functions, in which case each objective function has a unique name.

Note: PROC OPTLP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_
specifies the name of the variable containing the right-hand-side value of each constraint.

VAR
specifies the name of the decision variable.

TYPE
specifies the type of the decision variable. _TYPE_ can take one of the following values:

- N nonnegative
- D bounded (with both lower and upper bound)
- F free
- X fixed
- O other (with either lower or upper bound)

OBJCOEF
specifies the coefficient of the decision variable in the objective function.

LBOUND
specifies the lower bound on the decision variable.

UBOUND
specifies the upper bound on the decision variable.

VALUE
specifies the value of the decision variable.

STATUS
specifies the status of the decision variable. _STATUS_ can take one of the following values:

B basic variable

L nonbasic variable at its lower bound

U nonbasic variable at its upper bound

F free variable

S superbasic variable (a nonbasic variable with a value between its bounds; interior point solver only)

I LP model infeasible (all decision variables have _STATUS_ equal to I)

_R_COST_
specifies the reduced cost of the decision variable, which is the amount by which the objective function is increased per unit increase in the decision variable. The reduced cost associated with the ith variable is the ith entry of the following vector:

$$\left(\mathbf{c}^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}\right)$$

where $\mathbf{B} \in \mathbb{R}^{m \times m}$ denotes the basis (matrix composed of *basic* columns of the constraints matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$), $\mathbf{c} \in \mathbb{R}^n$ is the vector of objective function coefficients, and $\mathbf{c}_B \in \mathbb{R}^m$ is the vector of objective coefficients of the variables in the basis.

Definitions of Variables in the DUALOUT= Data Set

The DUALOUT= data set contains the dual solution to the LP model; each observation corresponds to a constraint of the LP problem. Information about the objective rows of the LP problems is not included. See Example 15.1 for an example of the DUALOUT= data set. The variables in the data set have the following names and meanings.

_OBJ_ID_
specifies the name of the objective function. This is particularly useful when there are multiple objective functions, in which case each objective function has a unique name.

Note: PROC OPTLP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_
specifies the name of the variable containing the right-hand-side value of each constraint.

ROW
specifies the name of the constraint.

TYPE
specifies the type of the constraint. _TYPE_ can take one of the following values:

- L "less than or equals" constraint
- E equality constraint
- G "greater than or equals" constraint
- R ranged constraint (both "less than or equals" and "greater than or equals")

RHS
specifies the value of the right-hand side of the constraint. It takes a missing value for a ranged constraint.

_L_RHS_
specifies the lower bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

_U_RHS_
specifies the upper bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

VALUE
specifies the value of the dual variable associated with the constraint.

STATUS
specifies the status of the slack variable for the constraint. _STATUS_ can take one of the following values:

- B basic variable
- L nonbasic variable at its lower bound
- U nonbasic variable at its upper bound
- F free variable
- S superbasic variable (a nonbasic variable with a value between its bounds; interior point solver only)
- I LP model infeasible (all decision variables have _STATUS_ equal to I)

ACTIVITY
specifies the left-hand-side value of a constraint. In other words, the value of _ACTIVITY_ for the ith constraint would be equal to $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$, where $\mathbf{a}_i$ refers to the ith row of the constraints matrix and $\mathbf{x}$ denotes the vector of current decision variable values.

Presolve

Presolve in PROC OPTLP uses a variety of techniques to reduce the problem size, improve numerical stability, and detect infeasibility or unboundedness (Andersen and Andersen 1995; Gondzio 1997). During presolve, redundant constraints and variables are identified and removed. Presolve can further reduce the problem size by substituting variables. Variable substitution is a very effective technique, but it might occasionally increase the number of nonzero entries in the constraint matrix.

In most cases, using presolve is very helpful in reducing solution times. You can enable presolve at different levels or disable it by specifying the PRESOLVER= option.

Pricing Strategies for the Simplex Solvers

Several pricing strategies for the simplex solvers are available. Pricing strategies determine which variable enters the basis at each simplex pivot. They can be controlled by specifying the PRICETYPE= option.

The primal simplex solver has the following five pricing strategies:

PARTIAL	uses Dantzig's most violated reduced cost rule (Dantzig 1963). It scans a queue of decision variables and selects the variable with the most violated reduced cost as the entering variable. You can optionally specify the QUEUESIZE= option to control the length of this queue.
FULL	uses Dantzig's most violated reduced cost rule. It compares the reduced costs of all decision variables and selects the variable with the most violated reduced cost as the entering variable.
DEVEX	implements the Devex pricing strategy developed by Harris (1973).
STEEPESTEDGE	uses the steepest-edge pricing strategy developed by Forrest and Goldfarb (1992).
HYBRID	uses a hybrid of the Devex and steepest-edge pricing strategies.

The dual simplex solver has only three pricing strategies available: FULL, DEVEX, and STEEPESTEDGE.

Warm Start for the Simplex Solvers

You can warm start the simplex solvers by specifying the option BASIS=WARMSTART. Additionally you need to specify the PRIMALIN= and DUALIN= data sets. The simplex solvers start with the basis thus provided. If the given basis cannot form a valid basis, the solvers use the basis generated using their *crash* techniques.

After an LP model is solved using the simplex solvers, the BASIS=WARMSTART option enables you to perform sensitivity analysis such as modifying the objective function, changing the right-hand sides of the constraints, adding and/or deleting

constraints and/or decision variables, and combinations of these cases. A faster solution to such a modified LP model can be obtained by starting with the basis in the optimal solution to the original LP model. This can be done by using the BASIS=WARMSTART option, modifying the DATA= input data set, and specifying the PRIMALIN= and DUALIN= data sets. Example 15.4 and Example 15.5 illustrate how to reoptimize an LP problem with a modified objective function and/or a modified right-hand side by using this technique. Example 15.6 shows how to reoptimize an LP problem after adding a new constraint.

CAUTION: Since the presolver uses the objective function and/or right-hand-side information, the basis provided by you might not be valid for the presolved model. It is therefore recommended that you turn the PRESOLVER= option off when using BASIS=WARMSTART.

The Interior Point Algorithm: Overview

The interior point solver (experimental) in PROC OPTLP implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following LP formulation (the primal):

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The corresponding dual is as follows:

$$\begin{aligned} \max \quad & \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of dual slack variables.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker)

conditions, which can be stated as follows:

$$\begin{aligned}
\mathbf{Ax} - \mathbf{s} &= \mathbf{b} \quad (\text{Primal Feasibility}) \\
\mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} &= \mathbf{c} \quad (\text{Dual Feasibility}) \\
\mathbf{WXe} &= \mathbf{0} \quad (\text{Complementarity}) \\
\mathbf{SYe} &= \mathbf{0} \quad (\text{Complementarity}) \\
\mathbf{x},\ \mathbf{y},\ \mathbf{w},\ \mathbf{s} &\geq \mathbf{0}
\end{aligned}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal *slack* variables.

Note: Slack variables (the s vector) are automatically introduced by the solver when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the solver to handle slack variables much more efficiently.

The letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{S}$ denote matrices with corresponding x, y, w, and s on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations representing the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original LP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively; Θ and Ξ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. PROC OPTLP uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point solver is that it takes full advantage of the sparsity in the constraint matrix, thereby enabling it to efficiently solve large-scale linear programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore it is of interest to observe the following four measures:

- Relative primal infeasibility measure α:

$$\alpha = \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- Relative dual infeasibility measure β:

$$\beta = \frac{\|\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- Relative duality gap δ:

$$\delta = \frac{|\mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- Absolute complementarity γ:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

where $\|v\|_2$ is the Euclidean norm of the vector v. These measures are displayed in the iteration log.

Iteration Log for the Simplex Solvers

The simplex solvers implement a two-phase simplex algorithm. When option PRINTFREQ= 1, the following information is printed in the iteration log:

Phase	indicates whether the solver is in phase I or phase II of the simplex method.
Iteration	indicates the iteration number.
Objective Value	indicates the current amount of infeasibility in phase I and the objective value of the current solution in phase II.
Entering Variable	indicates the entering pivot variable. A slack variable entering the basis is indicated by the corresponding row name followed by '(S)'. If the entering nonbasic variable has distinct, finite lower and upper bounds, then a "bound swap" takes place. In other words, if the entering variable is at its upper bound, then it is "flipped" to its lower bound and is indicated in the log as "To lower."
Leaving Variable	indicates the leaving pivot variable. A slack variable leaving the basis is indicated by the corresponding row name followed by '(S)'.

When option PRINTFREQ= is omitted or specified with a value larger than 1, only phase, iteration, and objective value information is printed in the iteration log.

During the solution process, some elements of the LP model might be perturbed to improve performance. After reaching optimality PROC OPTLP solves the original problem by using the optimal basis for the perturbed problem. This can occasionally cause the simplex solver to repeat phase I and phase II in several passes.

Iteration Log for the Interior Point Solver

The interior point solver implements an infeasible primal-dual predictor-corrector interior point algorithm. The following information is displayed in the iteration log:

Iter	indicates the iteration number
Complement	indicates the (absolute) complementarity
Duality Gap	indicates the (relative) duality gap
Primal Infeas	indicates the (relative) primal infeasibility measure
Bound Infeas	indicates the (relative) bound infeasibility measure
Dual Infeas	indicates the (relative) dual infeasibility measure

If the sequence of solutions converges to an optimal solution of the problem, you should see all columns in the iteration log converge to zero or very close to zero. If they do not, it can be the result of insufficient iterations being performed to reach optimality. In this case, you might need to increase the value specified in the option MAXITER= or MAXTIME=. If the complementarity and/or the duality gap do not converge, the problem might be infeasible or unbounded. If the infeasibility columns do not converge, the problem might be infeasible.

ODS Tables

PROC OPTLP creates two ODS (Output Delivery System) tables by default unless the option PRINTLEVEL=0 is specified. One table is a summary of the input LP problem. The other is a brief summary of the solution status. PROC OPTLP assigns a name to each table it creates. You can use these names to reference the table when using the ODS to select tables and create output data sets. For more information about ODS, see*SAS Output Delivery System: User's Guide*.

Table 15.2. ODS Tables Produced by PROC OPTLP

ODS Table Name	Description	Statement	Option
ProblemSummary	Summary of the input LP problem	OPTLP	default
SolutionSummary	Summary of the solution status	OPTLP	default

A typical output of PROC OPTLP is shown in Figure 15.2.

The OPTLP Procedure

Problem Summary

Problem Name	Delfland
Objective Sense	Minimization
Objective Function	OBJFUNCT
RHS	R
Number of Variables	1140
Bounded Above	0
Bounded Below	695
Bounded Above and Below	162
Free	214
Fixed	69
Number of Constraints	709
LE (<=)	295
EQ (=)	414
GE (>=)	0
Range	0
Constraint Coefficients	2749

The OPTLP Procedure

Solution Summary

Solver	Dual simplex
Objective Function	OBJFUNCT
Solution Status	Optimal
Objective Value	21.282493043
Primal Infeasibility	4.547474E-13
Dual Infeasibility	9.2511898E-7
Bound Infeasibility	2.619432E-16
Iterations	295
Presolve Time	0.00
Solution Time	0.02

Figure 15.2. Typical OPTLP Output

You can create output data sets from these tables by using the ODS OUTPUT statement. This can be useful, for example, when you want to create a report to summarize multiple PROC OPTLP runs. The output data sets corresponding to the preceding output are shown in Figure 15.3, where you can also find (at the row following the heading of each data set in display) the variable names that are used in the table definition (template) of each table.

Problem Summary

Obs	Label1	cValue1	nValue1
1	Problem Name	Delfland	.
2	Objective Sense	Minimization	.
3	Objective Function	OBJFUNCT	.
4	RHS	R	.
5			.
6	Number of Variables	1140	1140.000000
7	Bounded Above	0	0
8	Bounded Below	695	695.000000
9	Bounded Above and Below	162	162.000000
10	Free	214	214.000000
11	Fixed	69	69.000000
12			.
13	Number of Constraints	709	709.000000
14	LE (<=)	295	295.000000
15	EQ (=)	414	414.000000
16	GE (>=)	0	0
17	Range	0	0
18			.
19	Constraint Coefficients	2749	2749.000000

Solution Summary

Obs	Label1	cValue1	nValue1
1	Solver	Dual simplex	.
2	Objective Function	OBJFUNCT	.
3	Solution Status	Optimal	.
4	Objective Value	21.282493043	21.282493
5			.
6	Primal Infeasibility	4.547474E-13	4.547474E-13
7	Dual Infeasibility	9.2511898E-7	0.000000925
8	Bound Infeasibility	2.619432E-16	2.619432E-16
9			.
10	Iterations	295	295.000000
11	Presolve Time	0.00	0
12	Solution Time	0.02	0.015625

Figure 15.3. ODS Output Data Sets

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the OPTLP procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to allow the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the *SAS Companion* book for your operating system.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the *SAS Companion* book for your operating system.

Examples: OPTLP Procedure

Example 15.1. Oil Refinery Problem

Consider an oil refinery scenario. A step in refining crude oil into finished oil products involves a distillation process that splits crude into various streams. Suppose there are three types of crude available: Arabian light (a_l), Arabian heavy (a_h), and Brega (br). These crudes are distilled into light naphtha (na_l), intermediate naphtha (na_i), and heating oil (h_o). These in turn are blended into two types of jet fuel. Jet fuel j_1 is made up of 30% intermediate naphtha and 70% heating oil, and jet fuel j_2 is made up of 20% light naphtha and 80% heating oil. What amounts of the three crudes maximize the profit from producing jet fuel (j_1, j_2)? This problem can be formulated as the following linear program:

$$
\begin{array}{rrcrcrcl}
\max & -175\,\mathrm{a\_l} & - & 165\,\mathrm{a\_h} & - & 205\,\mathrm{br} & + & 350\,\mathrm{j\_1} + 350\,\mathrm{j\_2} \\
\text{subject to} & & & & & & & \\
(\mathrm{napha\_l}) & 0.035\,\mathrm{a\_l} & + & 0.03\,\mathrm{a\_h} & + & 0.045\,\mathrm{br} & = & \mathrm{na\_l} \\
(\mathrm{napha\_i}) & 0.1\,\mathrm{a\_l} & + & 0.075\,\mathrm{a\_h} & + & 0.135\,\mathrm{br} & = & \mathrm{na\_i} \\
(\mathrm{htg\_oil}) & 0.39\,\mathrm{a\_l} & + & 0.3\,\mathrm{a\_h} & + & 0.43\,\mathrm{br} & = & \mathrm{h\_o} \\
(\mathrm{blend1}) & & & 0.3\,\mathrm{j\_1} & & & \leq & \mathrm{na\_i} \\
(\mathrm{blend2}) & & & & & 0.2\,\mathrm{j\_2} & \leq & \mathrm{na\_l} \\
(\mathrm{blend3}) & & & 0.7\,\mathrm{j\_1} & + & 0.8\,\mathrm{j\_2} & \leq & \mathrm{h\_o} \\
 & \mathrm{a\_l} & & & & & \leq & 110 \\
 & & & \mathrm{a\_h} & & & \leq & 165 \\
 & & & & & \mathrm{br} & \leq & 80 \\
 & \multicolumn{5}{c}{\mathrm{a\_l, a\_h, br, na\_1, na\_i, h\_o, j\_1, j\_2}} & \geq & 0
\end{array}
$$

The constraints "blend1" and "blend2" ensure that j_1 and j_2 are made with the specified amounts of na_i and na_l, respectively. The constraint "blend3" is actually the reduced form of the following constraints:

$$\begin{aligned} \text{h\_o1} &\geq 0.7\,\text{j\_1} \\ \text{h\_o2} &\geq 0.8\,\text{j\_2} \\ \text{h\_o1} + \text{h\_o2} &\leq \text{h\_o} \end{aligned}$$

where h_o1 and h_o2 are dummy variables.

You can use the following SAS code to create the input data set ex1:

```
data ex1;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME          .          EX1       .     .          .
ROWS          .          .         .     .          .
 N            profit     .         .     .          .
 E            napha_l    .         .     .          .
 E            napha_i    .         .     .          .
 E            htg_oil    .         .     .          .
 L            blend1     .         .     .          .
 L            blend2     .         .     .          .
 L            blend3     .         .     .          .
COLUMNS       .          .         .     .          .
.             a_l        profit    -175  napha_l    .035
.             a_l        napha_i   .100  htg_oil    .390
.             a_h        profit    -165  napha_l    .030
.             a_h        napha_i   .075  htg_oil    .300
.             br         profit    -205  napha_l    .045
.             br         napha_i   .135  htg_oil    .430
.             na_l       napha_l   -1    blend2     -1
.             na_i       napha_i   -1    blend1     -1
.             h_o        htg_oil   -1    blend3     -1
.             j_1        profit    350   blend1     .3
.             j_1        blend3    .7    .          .
.             j_2        profit    350   blend2     .2
.             j_2        blend3    .8    .          .
BOUNDS        .          .         .     .          .
UP            .          a_l       110   .          .
UP            .          a_h       165   .          .
UP            .          br        80    .          .
ENDATA        .          .         .     .          .
;
```

You can use the following call to PROC OPTLP to solve the LP problem:

```
proc optlp data=ex1
   objsense  = max
   solver    = primal_spx
   primalout = ex1pout
   dualout   = ex1dout
   printfreq = 1;
run;
%put &_OROPTLP_;
```

Note that the OBJSENSE=MAX option is used to indicate that the objective function is to be maximized.

The primal and dual solutions are displayed in Output 15.1.1.

Output 15.1.1. Example 1: Primal and Dual Solution Output

The OPTLP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient
1	profit		a_l	D	-175
2	profit		a_h	D	-165
3	profit		br	D	-205
4	profit		na_l	N	0
5	profit		na_i	N	0
6	profit		h_o	N	0
7	profit		j_1	N	350
8	profit		j_2	N	350

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	0	110	110.000	U	10.2083
2	0	165	0.000	L	-22.8125
3	0	80	80.000	U	2.8125
4	0	1.7977E308	7.450	B	0.0000
5	0	1.7977E308	21.800	B	0.0000
6	0	1.7977E308	77.300	B	0.0000
7	0	1.7977E308	72.667	B	0.0000
8	0	1.7977E308	33.042	B	0.0000

The OPTLP Procedure
Dual Solution

Obs	Objective Function ID	RHS ID	Constraint Name	Constraint Type	Constraint RHS	Constraint Lower Bound
1	profit		napha_l	E	0	.
2	profit		napha_i	E	0	.
3	profit		htg_oil	E	0	.
4	profit		blend1	L	0	.
5	profit		blend2	L	0	.
6	profit		blend3	L	0	.

Obs	Constraint Upper Bound	Dual Variable Value	Constraint Status	Constraint Activity
1	.	0.000	L	0.00000
2	.	-145.833	L	0.00000
3	.	-437.500	L	0.00000
4	.	145.833	L	0.00000
5	.	0.000	B	-0.84167
6	.	437.500	L	0.00000

The progress of the solution is printed to the log as follows.

Output 15.1.2. Log: Solution Progress

```
NOTE: The problem EX1 has 8 variables (0 free, 0 fixed).
NOTE: The problem has 6 constraints (3 LE, 3 EQ, 0 GE, 0 range).
NOTE: The problem has 19 constraint coefficients.
WARNING: The objective sense has been changed to maximization.
NOTE: The OPTLP presolver value AUTOMATIC is applied.
NOTE: The OPTLP presolver removed 3 variables and 3 constraints.
NOTE: The OPTLP presolver removed 6 constraint coefficients.
NOTE: The presolved problem has 5 variables, 3 constraints, and 13 constraint
      coefficients.
NOTE: The PRIMAL SIMPLEX solver is called.
NOTE:                    Objective     Entering    Leaving
      Phase Iteration    Value         Variable    Variable
        2           1    1.5411014E-8  j_1         blend1   (S)
        2           2    2.6969274E-8  j_2         blend2   (S)
        2           3    5.2372044E-8  br       (S) blend3
        2           4     1347.916667  blend2   (S) br
NOTE: Optimal.
NOTE: Objective = 1347.9166667.
```

Note that the %put statement immediately after the OPTLP procedure prints value of the macro variable _OROPTLP_ to the log as follows.

Output 15.1.3. Log: Value of the Macro Variable _OROPTLP_

```
STATUS=OK   SOLUTION_STATUS=OPTIMAL   OBJECTIVE=1347.9166667
PRIMAL_INFEASIBILITY=1.836145E-14   DUAL_INFEASIBILITY=0
BOUND_INFEASIBILITY=0   ITERATIONS=4   PRESOLVE_TIME=0.00   SOLUTION_TIME=0.00
```

The value briefly summarizes the status of the OPTLP procedure upon termination.

Example 15.2. Using the Interior Point Solver

You can also solve the oil refinery problem described in Example 15.1 by using the interior point solver. You can create the input data set from an external MPS-format flat file by using the SAS macro %MPS2SASD or SAS DATA step code, both of which are described in the section "Getting Started: OPTLP Procedure" on page 1030. You can use the following SAS code to solve the problem:

```
proc optlp data=ex1
  objsense  = max
  solver    = ii
  primalout = ex1ipout
  dualout   = ex1idout
  printfreq = 1;
run;
```

The optimal solution is displayed in Output 15.2.1.

Output 15.2.1. Interior Point Solver: Primal Solution Output

The OPTLP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient
1	profit		a_l	D	-175
2	profit		a_h	D	-165
3	profit		br	D	-205
4	profit		na_l	N	0
5	profit		na_i	N	0
6	profit		h_o	N	0
7	profit		j_1	N	350
8	profit		j_2	N	350

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	0	110	110.000		.
2	0	165	0.000		.
3	0	80	80.000		.
4	0	1.7977E308	7.450		.
5	0	1.7977E308	21.800		.
6	0	1.7977E308	77.300		.
7	0	1.7977E308	72.667		.
8	0	1.7977E308	33.042		.

The iteration log is displayed in Output 15.2.2.

Output 15.2.2. Log: Solution Progress

```
NOTE: The problem EX1 has 8 variables (0 free, 0 fixed).
NOTE: The problem has 6 constraints (3 LE, 3 EQ, 0 GE, 0 range).
NOTE: The problem has 19 constraint coefficients.
WARNING: The objective sense has been changed to maximization.
NOTE: The OPTLP presolver value AUTOMATIC is applied.
NOTE: The OPTLP presolver removed 3 variables and 3 constraints.
NOTE: The OPTLP presolver removed 6 constraint coefficients.
NOTE: The presolved problem has 5 variables, 3 constraints, and 13 constraint
      coefficients.
NOTE: This is an experimental version of the ITERATIVE INTERIOR solver.
NOTE: The ITERATIVE INTERIOR solver is called.
NOTE:                                        Primal        Bound         Dual
      Iter   Complement  Duality Gap         Infeas       Infeas       Infeas
         0        48146    17.556333       0.285250            0            0
         1  6479.114458    14.641797       0.490805            0            0
         2  1763.666090     0.335357       0.021532            0            0
         3   136.596216     0.041506              0            0            0
         4     8.719046     0.002045    0.000003535            0            0
         5     0.236482     0.000102    0.000000176            0            0
         6     0.011297  0.000005106  8.6973727E-9            0            0
         7     0.000564  0.000000255 4.306989E-10            0            0
NOTE: Optimal.
NOTE: Objective = 1347.9164665.
```

Example 15.3. The Diet Problem

Consider the problem of diet optimization. There are six different foods: bread, milk, cheese, potato, fish, and yogurt. The cost and nutrition values per unit are displayed in Table 15.3.

Table 15.3. Cost and Nutrition Values

	Bread	Milk	Cheese	Potato	Fish	Yogurt
Cost	2.0	3.5	8.0	1.5	11.0	1.0
Protein, g	4.0	8.0	7.0	1.3	8.0	9.2
Fat, g	1.0	5.0	9.0	0.1	7.0	1.0
Carbohydrates, g	15.0	11.7	0.4	22.6	0.0	17.0
Calories	90	120	106	97	130	180

The objective is to find a minimum-cost diet that contains at least 300 calories, not more than 10 grams of protein, not less than 10 grams of carbohydrates, and not less than 8 grams of fat. In addition, the diet should contain at least 0.5 unit of fish and no more than 1 unit of milk.

You can use the following SAS code to create the MPS-format input data set:

```
data ex3;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME          .           EX3       .     .          .
ROWS          .           .         .     .          .
 N            diet        .         .     .          .
 G            calories    .         .     .          .
 L            protein     .         .     .          .
 G            fat         .         .     .          .
 G            carbs       .         .     .          .
COLUMNS       .           .         .     .          .
.             br          diet      2     calories   90
.             br          protein   4     fat        1
.             br          carbs     15    .          .
.             mi          diet      3.5   calories   120
.             mi          protein   8     fat        5
.             mi          carbs     11.7  .          .
.             ch          diet      8     calories   106
.             ch          protein   7     fat        9
.             ch          carbs     .4    .          .
.             po          diet      1.5   calories   97
.             po          protein   1.3   fat        .1
.             po          carbs     22.6  .          .
.             fi          diet      11    calories   130
.             fi          protein   8     fat        7
.             fi          carbs     0     .          .
.             yo          diet      1     calories   180
.             yo          protein   9.2   fat        1
.             yo          carbs     17    .          .
RHS           .           .         .     .          .
.             .           calories  300   protein    10
.             .           fat       8     carbs      10
BOUNDS        .           .         .     .          .
UP            .           mi        1     .          .
LO            .           fi        .5    .          .
ENDATA        .           .         .     .          .
;
```

You can solve the diet problem by using PROC OPTLP as follows:

```
proc optlp data=ex3
  presolver = none
  solver    = ps
  primalout = ex3pout
  dualout   = ex3dout
  printfreq = 1;
run;
```

The solution summary and the optimal primal solution are displayed in Output 15.3.1.

Output 15.3.1. Diet Problem: Solution Summary and Optimal Primal Solution

The OPTLP Procedure

Solution Summary

Solver	Primal simplex
Objective Function	diet
Solution Status	Optimal
Objective Value	12.081337881
Primal Infeasibility	8.881784E-16
Dual Infeasibility	0
Bound Infeasibility	1.816039E-18
Iterations	4
Presolve Time	0.00
Solution Time	0.00

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient
1	diet		br	N	2.0
2	diet		mi	D	3.5
3	diet		ch	N	8.0
4	diet		po	N	1.5
5	diet		fi	O	11.0
6	diet		yo	N	1.0

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	0.0	1.7977E308	0.00000	L	1.19066
2	0.0	1	0.05360	B	0.00000
3	0.0	1.7977E308	0.44950	B	0.00000
4	0.0	1.7977E308	1.86517	B	0.00000
5	0.5	1.7977E308	0.50000	L	5.15641
6	0.0	1.7977E308	-0.00000	L	1.10849

The cost of the optimal diet is 12.08 units.

Example 15.4. Reoptimizing after Modifying the Objective Function

Using the diet problem described in Example 15.3, we now illustrate how to reoptimize an LP problem after modifying the objective function.

Assume that the optimal solution of the diet problem is found and the optimal solutions are stored in the data sets ex3pout and ex3dout.

Suppose the cost of cheese increases from 8 to 10 per unit and the cost of fish decreases from 11 to 7 per serving unit. The COLUMNS section in the input data set ex3 is updated (and the data set is saved as ex4) as follows:

```
COLUMNS      .          .        .      .          .
   ...
.            ch         diet     10      calories  106
   ...
.            fi         diet     7       calories  130
   ...
RHS          .          .        .      .          .
   ...

ENDATA
;
```

You can use the following DATA step to create the data set ex4:

```
data ex4;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME         .          EX4      .      .          .
ROWS         .          .        .      .          .
 N           diet       .        .      .          .
 G           calories   .        .      .          .
 L           protein    .        .      .          .
 G           fat        .        .      .          .
 G           carbs      .        .      .          .
COLUMNS      .          .        .      .          .
.            br         diet     2      calories   90
.            br         protein  4      fat        1
.            br         carbs    15     .          .
.            mi         diet     3.5    calories   120
.            mi         protein  8      fat        5
.            mi         carbs    11.7   .          .
.            ch         diet     10     calories   106
.            ch         protein  7      fat        9
.            ch         carbs    .4     .          .
.            po         diet     1.5    calories   97
.            po         protein  1.3    fat        .1
.            po         carbs    22.6   .          .
.            fi         diet      7     calories   130
```

```
.           fi          protein  8     fat       7
.           fi          carbs    0     .         .
.           yo          diet     1     calories  180
.           yo          protein  9.2   fat       1
.           yo          carbs    17    .         .
RHS         .           .        .     .         .
.           .           calories 300   protein   10
.           .           fat      8     carbs     10
BOUNDS      .           .        .     .         .
UP          .           mi       1     .         .
LO          .           fi       .5    .         .
ENDATA      .           .        .     .         .
;
```

You can use the BASIS=WARMSTART option (and the ex3pout and ex3dout data sets from Example 15.3) in the following call to PROC OPTLP to solve the modified problem:

```
proc optlp data=ex4
   presolver = none
   basis     = warmstart
   primalin  = ex3pout
   dualin    = ex3dout
   solver    = primal_spx
   primalout = ex4pout
   dualout   = ex4dout
   printfreq = 1;
run;
```

The following iteration log indicates that it takes the primal simplex solver no extra iterations to solve the modified problem by using BASIS=WARMSTART, since the optimal solution to the LP problem in Example 15.3 remains optimal after the objective function is changed.

Output 15.4.1. Iteration Log

```
NOTE: The problem EX4 has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 constraint coefficients.
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The PRIMAL SIMPLEX solver is called.
NOTE: Optimal.
NOTE: Objective = 10.980335514.
```

Note that the primal simplex solver is preferred because the primal solution to the original LP is still feasible for the modified problem in this case.

Example 15.5. Reoptimizing after Modifying the Right-Hand Side

You can also modify the right-hand side of your problem and use the BASIS=WARMSTART option to obtain an optimal solution more quickly. Since the dual solution to the original LP is still feasible for the modified problem in this case, the dual simplex solver is preferred. We illustrate this case by using the same diet problem as in Example 15.3. Assume that you now need a diet that supplies at least 150 calories. The RHS section in the input data set ex3 is updated (and the data set is saved as ex5) as follows:

```
   ...
RHS          .           .          .      .          .
.            .           calories   150    protein    10
.            .           fat        8      carbs      10
BOUNDS       .           .          .      .          .
   ...
```

You can use the following DATA step to create the data set ex5:

```
data ex5;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME         .           EX5        .      .          .
ROWS         .           .          .      .          .
 N           diet        .          .      .          .
 G           calories    .          .      .          .
 L           protein     .          .      .          .
 G           fat         .          .      .          .
 G           carbs       .          .      .          .
COLUMNS      .           .          .      .          .
.            br          diet       2      calories   90
.            br          protein    4      fat        1
.            br          carbs      15     .          .
.            mi          diet       3.5    calories   120
.            mi          protein    8      fat        5
.            mi          carbs      11.7   .          .
.            ch          diet       8      calories   106
.            ch          protein    7      fat        9
.            ch          carbs      .4     .          .
.            po          diet       1.5    calories   97
.            po          protein    1.3    fat        .1
.            po          carbs      22.6   .          .
.            fi          diet       11     calories   130
.            fi          protein    8      fat        7
.            fi          carbs      0      .          .
.            yo          diet       1      calories   180
.            yo          protein    9.2    fat        1
.            yo          carbs      17     .          .
RHS          .           .          .      .          .
.            .           calories   150    protein    10
.            .           fat        8      carbs      10
```

```
BOUNDS      .          .        .     .          .
UP          .          mi       1     .          .
LO          .          fi       .5    .          .
ENDATA      .          .        .     .          .
;
```

You can use the BASIS=WARMSTART option in the following call to PROC OPTLP to solve the modified problem:

```
proc optlp data=ex5
  presolver = none
  basis     = warmstart
  primalin  = ex3pout
  dualin    = ex3dout
  solver    = dual_spx
  primalout = ex5pout
  dualout   = ex5dout
  printfreq = 1;
run;
```

Note that the dual simplex solver is preferred because the dual solution to the last solved LP is still feasible for the modified problem in this case.

The following iteration log indicates that it takes the dual simplex solver just one more phase II iteration to solve the modified problem by using BASIS=WARMSTART.

Output 15.5.1. Iteration Log

```
NOTE: The problem EX5 has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 constraint coefficients.
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The DUAL SIMPLEX solver is called.
NOTE:                  Objective     Entering    Leaving
      Phase Iteration  Value         Variable    Variable
        2           1      9.174413  calories(S) carbs    (S)
NOTE: Optimal.
NOTE: Objective = 9.1744131985.
```

Compare this with the following call to PROC OPTLP:

```
proc optlp data=ex5
  presolver = none
  solver    = dual_spx
  printfreq = 1;
run;
```

This call to PROC OPTLP solves the modified problem "from scratch" (without using the BASIS=WARMSTART option) and produces the following iteration log.

Output 15.5.2. Iteration Log

```
NOTE: The problem EX5 has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 constraint coefficients.
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The DUAL SIMPLEX solver is called.
NOTE:                    Objective     Entering    Leaving
      Phase Iteration    Value         Variable    Variable
        2           1       8.650000  mi          fat     (S)
        2           2       8.925676  ch          protein (S)
        2           3       9.174413  po          carbs   (S)
NOTE: Optimal.
NOTE: Objective = 9.1744131985.
```

It is clear that using the BASIS=WARMSTART option saves computation time. For larger or more complex examples, the benefits of using this option are more pronounced.

Example 15.6. Reoptimizing after Adding a New Constraint

Assume that after solving the diet problem in Example 15.3 we need to add a new constraint on sodium intake of no more than 550 mg/day for adults. The updated nutrition data are given in Table 15.4.

Table 15.4. Updated Cost and Nutrition Values

	Bread	Milk	Cheese	Potato	Fish	Yogurt
Cost	2.0	3.5	8.0	1.5	11.0	1.0
Protein, g	4.0	8.0	7.0	1.3	8.0	9.2
Fat, g	1.0	5.0	9.0	0.1	7.0	1.0
Carbohydrates, g	15.0	11.7	0.4	22.6	0.0	17.0
Calories, Cal	90	120	106	97	130	180
sodium, mg	148	122	337	186	56	132

The input data set ex3 is updated (and the data set is saved as ex6) as follows:

```
/* added a new constraint to the diet problem */
data ex6;
input field1 $ field2 $ field3$ field4 field5 $ field6 ;
datalines;
NAME          .           EX6       .      .          .
ROWS          .           .         .      .          .
```

```
 N          diet        .          .      .          .
 G          calories    .          .      .          .
 L          protein     .          .      .          .
 G          fat         .          .      .          .
 G          carbs       .          .      .          .
 L          sodium      .          .      .          .
COLUMNS     .           .          .      .          .
.           br          diet       2      calories   90
.           br          protein    4      fat        1
.           br          carbs      15     sodium     148
.           mi          diet       3.5    calories   120
.           mi          protein    8      fat        5
.           mi          carbs      11.7   sodium     122
.           ch          diet       8      calories   106
.           ch          protein    7      fat        9
.           ch          carbs      .4     sodium     337
.           po          diet       1.5    calories   97
.           po          protein    1.3    fat        .1
.           po          carbs      22.6   sodium     186
.           fi          diet       11     calories   130
.           fi          protein    8      fat        7
.           fi          carbs      0      sodium     56
.           yo          diet       1      calories   180
.           yo          protein    9.2    fat        1
.           yo          carbs      17     sodium     132
RHS         .           .          .      .          .
.           .           calories   300    protein    10
.           .           fat        8      carbs      10
.           .           sodium     550    .          .
BOUNDS      .           .          .      .          .
UP          .           mi         1      .          .
LO          .           fi         .5     .          .
ENDATA      .           .          .      .          .
;
```

For the modified problem we can warm start the simplex solvers to get a solution faster. The dual simplex solver is preferred because a dual feasible solution can be readily constructed from the optimal solution to the diet optimization problem.

Since there is a new constraint in the modified problem, you can use the following SAS code to create a new DUALIN= data set ex6din with this information:

```
data ex6newcon;
_ROW_='sodium  '; _STATUS_='A';
output;
;
/* create a new DUALIN= data set to include the new constraint */
data ex6din;
set ex3dout ex6newcon;
run;
```

Note that this step is optional. In this example, you can still use the data set ex3dout as the DUALIN= data set to solve the modified LP problem by using

the BASIS=WARMSTART option. PROC OPTLP validates the PRIMALIN= and DUALIN= data sets against the input model. Any new variable (or constraint) in the model is added to the PRIMALIN= (or DUALIN=) data set, and its status is assigned to be 'A'. The simplex solvers decide its corresponding status internally. Any variable in the PRIMALIN= and DUALIN= data sets but not in the input model is removed.

The _ROW_ and _STATUS_ columns of the DUALIN= data set ex6din are shown in Output 15.6.1.

Output 15.6.1. DUALIN= Data Set with a Newly Added Constraint

Obs	_ROW_	_STATUS_
1	calories	U
2	protein	L
3	fat	U
4	carbs	B
5	sodium	A

The dual simplex solver is called to solve the modified diet optimization problem more quickly with the following SAS code:

```
proc optlp data=ex6
   objsense=min
   presolver=none
   solver=ds
   primalout=ex6pout
   dualout=ex6dout
   scale=none
   printfreq=1
   basis=warmstart
   primalin=ex3pout
   dualin=ex6din;
run;
```

The optimal primal and dual solutions of the modified problem are displayed in Output 15.6.2.

Output 15.6.2. Primal and Dual Solution Output

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient
1	diet		br	N	2.0
2	diet		mi	D	3.5
3	diet		ch	N	8.0
4	diet		po	N	1.5
5	diet		fi	O	11.0
6	diet		yo	N	1.0

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status	Reduced Cost
1	0.0	1.7977E308	0.00000	L	1.19066
2	0.0	1	0.05360	B	0.00000
3	0.0	1.7977E308	0.44950	B	0.00000
4	0.0	1.7977E308	1.86517	B	0.00000
5	0.5	1.7977E308	0.50000	L	5.15641
6	0.0	1.7977E308	0.00000	L	1.10849

Dual Solution

Obs	Objective Function ID	RHS ID	Constraint Name	Constraint Type	Constraint RHS	Constraint Lower Bound
1	diet		calories	G	300	.
2	diet		protein	L	10	.
3	diet		fat	G	8	.
4	diet		carbs	G	10	.
5	diet		sodium	L	550	.

Obs	Constraint Upper Bound	Dual Variable Value	Constraint Status	Constraint Activity
1	.	0.02179	U	300.000
2	.	-0.55360	L	10.000
3	.	1.06286	U	8.000
4	.	0.00000	B	42.960
5	.	0.00000	B	532.941

The iteration log shown in Output 15.6.3 indicates that it takes the dual simplex solver no more iterations to solve the modified problem by using the BASIS=WARMSTART option, since the optimal solution to the original problem remains optimal after one more constraint is added.

Output 15.6.3. Iteration Log

```
NOTE: The problem EX6 has 6 variables (0 free, 0 fixed).
NOTE: The problem has 5 constraints (2 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 29 constraint coefficients.
NOTE: The OPTLP presolver value NONE is applied.
NOTE: The DUAL SIMPLEX solver is called.
NOTE: Optimal.
NOTE: Objective = 12.081337881.
```

Both this example and Example 15.4 illustrate the situation in which the optimal solution does not change after some perturbation of the parameters of the LP problem. The simplex solver starts from an optimal solution and quickly verifies the optimality. Usually the optimal solution of the slightly perturbed problem can be obtained after performing relatively small number of iterations if starting with the optimal solution of the original problem. In such cases you can expect a dramatic reduction of computation time, for instance, if you want to solve a large LP problem and a slightly perturbed version of this problem by using the BASIS=WARMSTART option rather than solving both problems from scratch.

References

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Chapter 16
The OPTMILP Procedure

Chapter Contents

Chapter 16
The OPTMILP Procedure

Overview: OPTMILP Procedure

The OPTMILP procedure is a solver for general mixed integer linear programs (MILPs).

A standard mixed integer linear program has the following formulation:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \qquad \text{(MILP)} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \\ & \mathbf{x}_i \in \mathbb{Z} \quad \forall i \in \mathcal{S} \end{aligned}$$

where

$\mathbf{x}$	$\in$	$\mathbb{R}^n$	is the vector of structural variables
$\mathbf{A}$	$\in$	$\mathbb{Q}^{m\times n}$	is the matrix of technological coefficients
$\mathbf{c}$	$\in$	$\mathbb{Q}^n$	is the vector of objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{Q}^m$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l}$	$\in$	$\mathbb{Q}^n$	is the vector of lower bounds on variables
$\mathbf{u}$	$\in$	$\mathbb{Q}^n$	is the vector of upper bounds on variables
$\mathcal{S}$			is a nonempty subset of the set $\{1\ldots, n\}$ of indices

The OPTMILP procedure implements an LP-based branch-and-bound algorithm. This divide-and-conquer approach attempts to solve the original problem by solving linear programming relaxations of a sequence of smaller subproblems. The OPTMILP procedure also implements advanced techniques such as presolving, generating cutting planes, and applying primal heuristics to improve the efficiency of the overall algorithm.

The OPTMILP procedure requires a mixed integer linear program to be specified using a SAS data set that adheres to the MPS format, a widely accepted format in the optimization community. Chapter 14 discusses the MPS format in detail. It is also possible to input an incumbent solution in MPS format; see the section "Warm Start" on page 1088 for details.

You can use the SAS macro %LP2MPSD to convert typical PROC LP format data sets into MPS-format SAS data sets. The macro is available online at the SAS Customer Support Center.

The OPTMILP procedure provides various control options and solution strategies. In particular, you can enable, disable, or set levels for the advanced techniques previously mentioned.

The OPTMILP procedure outputs an optimal solution or the best feasible solution found, if any, in SAS data sets. This enables you to generate solution reports and perform additional analyses by using SAS.

Getting Started: OPTMILP Procedure

The following example illustrates the use of the OPTMILP procedure to solve mixed integer linear programs. For more examples, see the section "Examples: OPTMILP Procedure" on page 1099. Suppose you want to solve the following problem:

$$
\begin{array}{lrcrcrcrl}
\min & 2x_1 & - & 3x_2 & - & 4x_3 & & & \\
\text{s.t.} & & - & 2x_2 & - & 3x_3 & \geq & -5 & \text{(R1)} \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 4 & \text{(R2)} \\
& x_1 & + & 2x_2 & + & 3x_3 & \leq & 7 & \text{(R3)} \\
& & & x_1, & x_2, & x_3 & \geq & 0 & \\
& & & x_1, & x_2, & x_3 & \in \mathbb{Z} & &
\end{array}
$$

The corresponding MPS-format SAS data set follows:

```
data ex_mip;
   input field1 $ field2 $ field3 $ field4 field5 $ field6;
   datalines;
NAME          .       EX_MIP       .   .          .
ROWS          .         .          .   .          .
N             COST      .          .   .          .
G             R1        .          .   .          .
L             R2        .          .   .          .
L             R3        .          .   .          .
COLUMNS       .         .          .   .          .
.             MARK00 'MARKER'      .   'INTORG'   .
.             X1     COST          2   R2         1
.             X1     R3            1   .          .
.             X2     COST         -3   R1        -2
.             X2     R2            1   R3         2
.             X3     COST         -4   R1        -3
.             X3     R2            2   R3         3
.             MARK01 'MARKER'      .   'INTEND'   .
RHS           .      .             .   .          .
.             RHS    R1           -5   R2         4
.             RHS    R3            7   .          .
ENDATA        .      .             .   .          .
;
```

You can also create this SAS data set from an MPS-format flat file (ex_mip.mps) by using the following SAS macro:

```
%mps2sasd(mpsfile = "ex_mip.mps", outdata = ex_mip);
```

This problem can be solved by using the following statement to call the OPTMILP procedure:

```
proc optmilp data = ex_mip
   objsense   = min
   primalout  = primal_out
   dualout    = dual_out
   presolver  = automatic
   heuristics = automatic;
run;
```

The DATA= option names the MPS-format SAS data set containing the problem data. The OBJSENSE= option specifies whether to maximize or minimize the objective function. The PRIMALOUT= option names the SAS data set containing the optimal solution or the best feasible solution found by the solver. The DUALOUT= option names the SAS data set containing the constraint activities. The PRESOLVER= and HEURISTICS= options specify the levels for presolving and applying heuristics, respectively. In this example, each option is set to its default value AUTOMATIC, meaning that the solver determines the appropriate levels for presolve and heuristics automatically.

The optimal integer solution and its corresponding constraint activities, stored in the data sets primal_out and dual_out, respectively, are displayed in Figure 16.1 and Figure 16.2.

The OPTMILP Procedure
Primal Integer Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient	Lower Bound	Upper Bound	Variable Value
1	COST	RHS	X1	I	2	0	1.7977E308	0
2	COST	RHS	X2	I	−3	0	1.7977E308	1
3	COST	RHS	X3	I	−4	0	1.7977E308	1

Figure 16.1. Optimal Solution

Constraint Information

Obs	Objective Function ID	RHS ID	Constraint Name	Constraint Type	Constraint RHS
1	COST	RHS	R1	G	-5
2	COST	RHS	R2	L	4
3	COST	RHS	R3	L	7

Obs	Constraint Lower Bound	Constraint Upper Bound	Constraint Activity
1	.	.	-5
2	.	.	3
3	.	.	5

Figure 16.2. Constraint Activities

The solution summary stored in the macro variable _OROPTMILP_ can be viewed by issuing the following statement:

```
%put &_OROPTMILP_;
```

This produces the output shown in Figure 16.3.

```
STATUS=OK   SOLUTION_STATUS=OPTIMAL   OBJECTIVE=-7   RELATIVE_GAP=0   ABSOLUTE_G
AP=0   PRIMAL_INFEASIBILITY=0   BOUND_INFEASIBILITY=0   INTEGER_INFEASIBILITY=0
  NODES=1   ITERATIONS=2   PRESOLVE_TIME=0.00   SOLUTION_TIME=0.00
```

Figure 16.3. Macro Output

See the section "Data Input and Output" on page 1086 for details about the type and status codes displayed for variables and constraints.

Syntax: OPTMILP Procedure

The following statement is available in the OPTMILP procedure:

PROC OPTMILP < *options* > **;**

Functional Summary

Table 16.1 summarizes the options available for the OPTMILP procedure, classified by function.

Table 16.1. Options for the OPTMILP Procedure

Description	Option
Data Set Options	
input data set	DATA=
constraint activities output data set	DUALOUT=
objective sense (maximization or minimization)	OBJSENSE=
primal solution input data set (warm start)	PRIMALIN=
primal solution output data set	PRIMALOUT=
Presolve Option	
type of presolve	PRESOLVER=
Control Options	
stopping criterion based on absolute objective gap	ABSOBJGAP=
cutoff value for node removal	CUTOFF=
emphasize feasibility or optimality	EMPHASIS=
maximum allowed difference between an integer variable's value and an integer	INTTOL=
maximum number of nodes to be processed	MAXNODES=
maximum number of solutions to be found	MAXSOLS=
maximum solution time	MAXTIME=
frequency of printing node log	PRINTFREQ=
toggle ODS output	PRINTLEVEL=
detail of solution progress printed in log	PRINTLEVEL2=
stopping criterion based on relative objective gap	RELOBJGAP=
scale the problem matrix	SCALE=
stopping criterion based on target objective value	TARGET=
use CPU/real time	TIMETYPE=
Heuristics Option	
primal heuristics level	HEURISTICS=
Search Options	
node selection strategy	NODESEL=
use of variable priorities	PRIORITY=
number of simplex iterations performed on each variable in strong branching strategy	STRONGITER=
number of candidates for strong branching	STRONGLEN=
rule for selecting branching variable	VARSEL=

Table 16.1. (continued)

Description	Option
Cut Options	
overall cut level	ALLCUTS=
clique cut level	CUTCLIQUE=
flow cover cut level	CUTFLOWCOVER=
flow path cut level	CUTFLOWPATH=
Gomory cut level	CUTGOMORY=
generalized upper bound (GUB) cover cut level	CUTGUB=
implied bounds cut level	CUTIMPLIED=
knapsack cover cut level	CUTKNAPSACK=
lift-and-project cut level (experimental)	CUTLAP=
mixed integer rounding (MIR) cut level	CUTMIR=
row multiplier factor for cuts	CUTSFACTOR=

PROC OPTMILP Statement

PROC OPTMILP < *options* > **;**

You can specify the following options in the PROC OPTMILP statement.

Data Set Options

DATA=*SAS-data-set*
specifies the input data set corresponding to the MILP model. If this option is not specified, PROC OPTMILP will use the most recently created SAS data set. See Chapter 14, "The MPS-Format SAS Data Set", for more details about the input data set.

DUALOUT=*SAS-data-set*
DOUT=*SAS-data-set*
specifies the output data set containing the constraint activities.

OBJSENSE=*MIN* | *MAX*
specifies whether the MILP model is a minimization or a maximization problem. You can use OBJSENSE=MIN for a minimization problem and OBJSENSE=MAX for a maximization problem. Alternatively, you can specify the objective sense in the input data set. This option supersedes the objective sense specified in the input data set. If the objective sense is not specified anywhere, then PROC OPTMILP interprets and solves the MILP as a minimization problem.

PRIMALIN=*SAS-data-set*
enables you to input an integer feasible solution in a SAS data set. PROC OPTMILP validates both the data set and the solution stored in the data set. If both are valid, then the input solution provides an incumbent solution and a bound for the branch-and-bound algorithm. If either the data set or the solution is not valid, then the PRIMALIN= data are ignored. See the section "Warm Start" on page 1088 for details.

PRIMALOUT=*SAS-data-set*
POUT=*SAS-data-set*
specifies the output data set for the primal solution. This data set contains the primal solution information. See the section "Data Input and Output" on page 1086 for details.

Presolve Option

PRESOLVER=*option*
specifies a presolve *option* or its corresponding value *num*, as listed in Table 16.2.

Table 16.2. Values for PRESOLVER= Option

Number	Option	Description
–1	AUTOMATIC	Apply the default level of presolve processing.
0	NONE	Disable presolver.
1	BASIC	Perform minimal presolve processing.
2	MODERATE	Apply a higher level of presolve processing.
3	AGGRESSIVE	Apply the highest level of presolve processing.

The default value is AUTOMATIC.

Control Options

ABSOBJGAP=*num*
specifies a stopping criterion. When the absolute difference between the best integer objective and the objective of the best node remaining falls below the value of *num*, the procedure stops. The value of *num* can be any positive number; the default value is 1E−6.

CUTOFF=*num*
cuts off any nodes in a minimization (maximization) problem with an objective value above (below) *num*. The value of *num* can be any number; the default value is the positive (negative) number that has the largest absolute value representable in your operating environment.

EMPHASIS=*option*
specifies a search emphasis *option* or its corresponding value *num* as listed in Table 16.3. The default value is BALANCE.

Table 16.3. Values for EMPHASIS= Option

Number	Option	Description
0	BALANCE	Perform a balanced search.
1	OPTIMAL	Emphasize optimality over feasibility.
2	FEASIBLE	Emphasize feasibility over optimality.

INTTOL=*num*

specifies the amount by which an integer variable value can differ from an integer and still be considered integer feasible. The value of *num* can be any number between 0.0 and 1.0; the default value is 1E−5. PROC OPTMILP attempts to find an optimal solution with integer infeasibility less than *num*. If you assign a value smaller than 1E−10 to *num* and the best solution found by PROC OPTMILP has integer infeasibility between *num* and 1E−10, then PROC OPTMILP ends with a solution status of OPTIMAL_COND (see the section "Macro Variable _OROPTMILP_ " on page 1083).

MAXNODES=*num*

specifies the maximum number of branch-and-bound nodes to be processed. The value of *num* can be any nonnegative number; the default value is the positive integer that has the largest absolute value representable in your operating environment.

MAXSOLS=*num*

specifies a stopping criterion. If *num* solutions have been found, then the procedure stops. The value of *num* can be any positive integer; the default value is the positive integer that has the largest absolute value representable in your operating environment.

MAXTIME=*num*

specifies the maximum time allowed for PROC OPTMILP to find a solution. The type of time, either CPU time or real time, is determined by the value of the TIMETYPE= option. The value of *num* can be any positive number; the default value is the positive number that has the largest absolute value representable in your operating environment.

PRINTFREQ=*num*

specifies how often information is printed in the node log. The value of *num* can be any nonnegative number; the default value is 100. If *num* is set to 0, then the node log is disabled. If *num* is positive, then an entry will be made in the node log at the first node, at the last node, and at intervals dictated by the value of *num*. An entry will also be made each time a better integer solution is found.

PRINTLEVEL=*0* | *1*

specifies whether or not a summary of the problem and solution should be printed. If PRINTLEVEL=1, then two Output Delivery System (ODS) tables named "ProblemSummary" and "SolutionSummary" are produced and printed. If PRINTLEVEL=0, then no ODS tables are produced or printed. The default value of this option is 1. For details about the ODS tables created by PROC OPTMILP, see the section "ODS Tables" on page 1095.

PRINTLEVEL2=*option*

controls the amount of information displayed in the SAS log by the solver, from a short description of presolve information and summary to details at each node. Table 16.4 describes the valid values for this option.

Table 16.4. Values for PRINTLEVEL2= Option

Number	Option	Description
0	NONE	Turn off all solver-related messages in SAS log.
1	BASIC	Display a solver summary after stopping.
2	MODERATE	Print a solver summary and a node log by using the interval dictated by the PRINTFREQ= option.
3	AGGRESSIVE	Print a detailed solver summary and a node log by using the interval dictated by the PRINTFREQ= option.

The default value is MODERATE.

RELOBJGAP=*num*
specifies a stopping criterion based on the best integer objective (BestInteger) and the objective of the best remaining node (BestBound). The relative objective gap is equal to

$$| \text{BestInteger} - \text{BestBound} | / (1\text{E}{-}10 + | \text{BestBound} |)$$

When this value becomes smaller than the specified gap size *num*, the procedure stops. The value of *num* can be any number between 0 and 1; the default value is 1E−4.

SCALE=*option*
indicates whether or not to scale the problem matrix. SCALE= can take either of the values AUTOMATIC (–1) and NONE (0). SCALE=AUTOMATIC scales the matrix as determined by PROC OPTMILP; SCALE=NONE disables scaling. The default value is AUTOMATIC.

TARGET=*num*
specifies a stopping criterion for minimization (maximization) problems. If the best integer objective is better than or equal to *num*, the procedure stops. The value of *num* can be any number; the default value is the negative (positive) number that has the largest absolute value representable in your operating environment.

TIMETYPE=*CPU* | *REAL*
specifies the measurement of time used in a PROC OPTMILP call. Numeric values of time can be specified in the MAXTIME= option or reported in the _OROPTMILP_ macro variable. The value of the TIMETYPE= option determines whether CPU time or real time is used. The default value of this option is CPU.

Heuristics Option

HEURISTICS=*option*
enables the user to control the level of primal heuristics applied by PROC OPTMILP. This level determines how frequently primal heuristics are applied during the branch-and-bound tree search. It also affects the maximum number of iterations allowed in iterative heuristics. Some computationally expensive heuristics might be disabled by

the solver at less aggressive levels. The values of *option* and the corresponding values of *num* are listed in Table 16.5.

Table 16.5. Values for HEURISTICS= Option

Number	Option	Description
–1	AUTOMATIC	Apply default level of heuristics, similar to MODERATE.
0	NONE	Disable all primal heuristics.
1	BASIC	Apply basic primal heuristics at low frequency.
2	MODERATE	Apply most primal heuristics at moderate frequency.
3	AGGRESSIVE	Apply all primal heuristics at high frequency.

The default value is AUTOMATIC. For details about primal heuristics, see the section "Primal Heuristics" on page 1093.

Search Options

NODESEL=*option*

specifies the node selection strategy *option* or its corresponding value *num* as listed in Table 16.6.

Table 16.6. Values for NODESEL= Option

Number	Option	Description
–1	AUTOMATIC	Use automatic node selection.
0	BESTBOUND	Choose the node with the best relaxed objective (best-bound-first strategy).
1	BESTESTIMATE	Choose the node with the best estimate of the integer objective value (best-estimate-first strategy).
2	DEPTH	Choose the most recently created node (depth-first strategy).

The default value is AUTOMATIC. For details about node selection, see the section "Node Selection" on page 1090.

PRIORITY=*0* | *1*

indicates whether or not to use specified branching priorities for integer variables. PRIORITY=0 ignores variable priorities; PRIORITY=1 uses priorities when they exist. The default value is 1. See the section "Branching Priorities" on page 1091 for details.

STRONGITER=*num*
specifies the number of simplex iterations performed for each variable in the candidate list when using the strong branching variable selection strategy. The value of *num* can be any positive number; the default value is automatically calculated by PROC OPTMILP.

STRONGLEN=*num*
specifies the number of candidates used when performing the strong branching variable selection strategy. The value of *num* can be any positive integer; the default value is 10.

VARSEL=*option*
specifies the rule for selecting the branching variable. The values of *option* and the corresponding values of *num* are listed in Table 16.7.

Table 16.7. Values for VARSEL= Option

Number	Option	Description
–1	AUTOMATIC	Use automatic branching variable selection.
0	MAXINFEAS	Choose the variable with maximum infeasibility.
1	MININFEAS	Choose the variable with minimum infeasibility.
2	PSEUDO	Choose a branching variable based on pseudocost.
3	STRONG	Use strong branching variable selection strategy.

The default value is AUTOMATIC. For details about variable selection, see the section "Variable Selection" on page 1090.

Cut Options

Table 16.8 describes the *option* and *num* values for the cut options in PROC OPTMILP.

Table 16.8. Values for Individual Cut Options

Number	Option	Description
–1	AUTOMATIC	Generate cutting planes based on a strategy determined by PROC OPTMILP.
0	NONE	Disable generation of cutting planes.
1	MODERATE	Use a moderate cut strategy.
2	AGGRESSIVE	Use an aggressive cut strategy.

You can use the ALLCUTS= option to set all cut types to the same level. You can override the ALLCUTS= value by using the options corresponding to particular cut types. For example, if you want PROC OPTMILP to generate only

Gomory cuts, specify ALLCUTS=NONE and CUTGOMORY=AUTOMATIC. If you want to generate all cuts aggressively but generate no lift-and-project cuts, set ALLCUTS=AGGRESSIVE and CUTLAP=NONE.

ALLCUTS=*option*

provides a shorthand way of setting all the cuts-related options in one setting. In other words, ALLCUTS=*num* is equivalent to setting each of the individual cuts parameters to the same value *num*. Thus, ALLCUTS=–1 has the effect of setting CUTCLIQUE=–1, CUTFLOWCOVER=–1, CUTFLOWPATH=–1, ..., CUTLAP=–1, and CUTMIR=–1. Table 16.8 lists the values that can be assigned to *option* and *num*. In addition, you can override levels for individual cuts with the CUTCLIQUE=, CUTFLOWCOVER=, CUTFLOWPATH=, CUTGOMORY=, CUTGUB=, CUTIMPLIED=, CUTKNAPSACK=, CUTLAP=, and CUTMIR= options. If the ALLCUTS= option is not specified, all the cuts-related options are either at their individually specified values (if the corresponding option is specified) or at their default values (if that option is not specified).

CUTCLIQUE=*option*

specifies the level of clique cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTCLIQUE= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTFLOWCOVER=*option*

specifies the level of flow cover cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTFLOWCOVER= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTFLOWPATH=*option*

specifies the level of flow path cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTFLOWPATH= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTGOMORY=*option*

specifies the level of Gomory cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTGOMORY= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTGUB=*option*

specifies the level of generalized upper bound (GUB) cover cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTGUB= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTIMPLIED=*option*

specifies the level of implied bound cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTIMPLIED= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTKNAPSACK=*option*

specifies the level of knapsack cover cuts generated by PROC OPTMILP. Table 16.8

lists the values that can be assigned to *option* and *num*. The CUTKNAPSACK= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTLAP=*option* **(experimental)**
specifies the level of lift-and-project (LAP) cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTLAP= option overrides the ALLCUTS= option. The default value is NONE.

CUTMIR=*option*
specifies the level of mixed integer rounding (MIR) cuts generated by PROC OPTMILP. Table 16.8 lists the values that can be assigned to *option* and *num*. The CUTMIR= option overrides the ALLCUTS= option. The default value is AUTOMATIC.

CUTSFACTOR=*num*
specifies a row multiplier factor for cuts. The number of cuts added is limited to *num* times the original number of rows. The value of *num* can be any nonnegative number less than or equal to 100; the default value is 3.0.

Macro Variable _OROPTMILP_

The OPTMILP procedure defines a macro variable named _OROPTMILP_. This variable contains a character string that indicates the status of the OPTMILP procedure upon termination. The various terms of the variable are interpreted as follows.

STATUS
indicates the solver status at termination. It can take one of the following values:

OK	procedure terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data
OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing data
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
OPTIMAL_AGAP	optimal solution within absolute gap specified using ABSOBJGAP= option
OPTIMAL_RGAP	optimal solution within relative gap specified using RELOBJGAP= option

OPTIMAL_COND	solution is optimal, but some infeasibilities (primal, bound, or integer) exceed tolerances due to scaling or choice of small INTTOL= value
TARGET	solution not worse than target specified using TARGET= option
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	problem is infeasible or unbounded
SOLUTION_LIM	solver reached maximum number of solutions specified using option MAXSOLS=
NODE_LIM_SOL	solver reached maximum number of nodes specified using MAXNODES= option and found a solution
NODE_LIM_NOSOL	solver reached maximum number of nodes specified using MAXNODES= option and did not find a solution
TIME_LIM_SOL	solver reached the execution time limit specified using MAXTIME= option and found a solution
TIME_LIM_NOSOL	solver reached the execution time limit specified using MAXTIME= option and did not find a solution
ABORT_SOL	solver was stopped by user but still found a solution
ABORT_NOSOL	solver was stopped by user and did not find a solution
OUTMEM_SOL	solver ran out of memory but still found a solution
OUTMEM_NOSOL	solver ran out of memory and either did not find a solution or failed to output solution due to insufficient memory
FAIL_SOL	solver stopped due to errors but still found a solution
FAIL_NOSOL	solver stopped due to errors and did not find a solution

OBJECTIVE
indicates the objective value obtained by the solver at termination.

RELATIVE_GAP
specifies the relative gap between the best integer objective (BestInteger) and the

objective of the best remaining node (BestBound) upon termination of the MILP solver. The relative gap is equal to

$$| \text{BestInteger} - \text{BestBound} | / (1\text{E}{-}10 + | \text{BestBound} |)$$

ABSOLUTE_GAP
specifies the absolute gap between the best integer objective (BestInteger) and the objective of the best remaining node (BestBound) upon termination of the MILP solver. The absolute gap is equal to $| \text{BestInteger} - \text{BestBound} |$.

PRIMAL_INFEASIBILITY
indicates the (relative) primal infeasibility of the constraints at the solution.

BOUND_INFEASIBILITY
indicates the maximum violation by the solution of the lower and/or upper bounds.

INTEGER_INFEASIBILITY
specifies the maximum violation of the integrality of an integer variable returned by the MILP solver.

NODES
specifies the number of nodes enumerated by the MILP solver by using the branch-and-bound algorithm.

ITERATIONS
indicates the number of simplex iterations taken to solve the problem.

PRESOLVE_TIME
indicates the time (in seconds) used in preprocessing.

SOLUTION_TIME
indicates the time (in seconds) taken to solve the problem.

Note: The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time (default) or real time. The type is determined by the TIMETYPE= option.

Details: OPTMILP Procedure

This section describes in detail the various data sets pertaining to PROC OPTMILP in both regular and warm starting scenarios. The section also includes brief overviews of the following:

- data set input and output
- warm start
- the branch-and-bound algorithm
- controlling the branch-and-bound algorithm
- the types of presolve techniques available
- cutting planes
- primal heuristics

- node log

The section concludes with a description of the ODS tables generated by PROC OPTMILP.

Data Input and Output

This subsection describes the PRIMALIN= data set required to warm start PROC OPTMILP, as well as the PRIMALOUT= and DUALOUT= data sets.

Definitions of Variables in the PRIMALIN= Data Set

The PRIMALIN= data set has two required variables defined as follows:

VAR
specifies the variable (column) names of the problem. The values should match the column names in the DATA= data set for the current problem.

VALUE
specifies the solution value for each variable in the problem.

Note: If PROC OPTMILP produces a feasible solution, the primal output data set from that run can be used as the PRIMALIN= data set for a subsequent run, provided that the input solution is feasible for the subsequent run.

Definitions of Variables in the PRIMALOUT= Data Set

PROC OPTMILP stores the current best integer feasible solution of the problem in the data set specified by the PRIMALOUT= option. The variables in this data set are defined as follows:

_OBJ_ID_
specifies the identifier of the objective function.

_RHS_ID_
specifies the identifier of the right-hand side.

VAR
specifies the variable (column) names.

TYPE
specifies the variable type. _TYPE_ can take one of the following values:

C continuous variable

I general integer variable

B binary variable (0 or 1)

OBJCOEF
specifies the coefficient of the variable in the objective function.

LBOUND
specifies the lower bound on the variable.

UBOUND
specifies the upper bound on the variable.

VALUE
specifies the value of the variable in the current solution.

Definitions of the DUALOUT= Data Set Variables

The DUALOUT= data set contains the constraint activities corresponding to the primal solution in the PRIMALOUT= data set. Information about additional objective rows of the MILP problem is not included. The variables in this data set are defined as follows:

_OBJ_ID_
specifies the identifier of the objective function from the input data set.

_RHS_ID_
specifies the identifier of the right-hand side from the input data set.

ROW
specifies the constraint (row) name.

TYPE
specifies the constraint type. _TYPE_ can take one of the following values:

L "less than or equal" constraint

E equality constraint

G "greater than or equal" constraint

R ranged constraint (both "less than or equal" and "greater than or equal")

RHS
specifies the value of the right-hand side of the constraint. It takes a missing value for a ranged constraint.

_L_RHS_
specifies the lower bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

_U_RHS_
specifies the upper bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

ACTIVITY
specifies the activity of a constraint for a given primal solution. In other words, the value of _ACTIVITY_ for the ith constraint is equal to $\mathbf{a}_i^T \mathbf{x}$, where $\mathbf{a}_i$ refers to the ith row of the constraint matrix and $\mathbf{x}$ denotes the vector of the current primal solution.

Warm Start

PROC OPTMILP enables you to input an integer feasible solution by using the PRIMALIN= option. PROC OPTMILP checks that the decision variables named in _VAR_ are the same as those in the MPS-format SAS data set; the procedure also checks that the input solution is integer feasible. If either condition is not true, PROC OPTMILP issues a warning and ignores the input solution. The input solution provides an incumbent solution as well as an upper (min) or lower (max) bound for the branch-and-bound algorithm. PROC OPTMILP uses the input solution to reduce the search space and to guide the search process. When it is difficult to find a good integer feasible solution for a problem, warm start can reduce solution time significantly.

The Branch-and-Bound Algorithm

The branch-and-bound algorithm, first proposed by Land and Doig (1960), is an effective approach to solving mixed integer linear programs. The following discussion outlines the approach and explains how PROC OPTMILP enhances the basic algorithm by using several advanced techniques.

The branch-and-bound algorithm solves a mixed integer linear program by dividing the search space and generating a sequence of subproblems. The search space of a mixed integer linear program can be represented by a tree. Each node in the tree is identified with a subproblem derived from previous subproblems on the path leading to the root of the tree. The subproblem (MILP^0) associated with the root is identical to the original problem, which we will call (MILP), given in the section "Overview: OPTMILP Procedure" on page 1071.

The linear programming relaxation (LP^0) of (MILP^0) can be written as

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

The branch-and-bound algorithm generates subproblems along the nodes of the tree by using the following scheme. Consider $\bar{x}^0$, the optimal solution to (LP^0), which is usually obtained using the dual simplex algorithm. If $\bar{x}_i^0$ is integer for all $i \in \mathcal{S}$, then $\bar{x}^0$ is an optimal solution to (MILP). Suppose that for some $i \in \mathcal{S}$, $\bar{x}_i^0$ is nonintegral. In that case the algorithm defines two new subproblems (MILP^1) and (MILP^2), descendants of the parent subproblem (MILP^0). The subproblem (MILP^1) is identical to (MILP^0) except for the additional constraint

$$x_i \leq \lfloor \bar{x}_i^0 \rfloor$$

and the subproblem (MILP^2) is identical to (MILP^0) except for the additional constraint

$$x_i \geq \lceil \bar{x}_i^0 \rceil$$

The notation $\lfloor y \rfloor$ represents the largest integer less than or equal to y, and the notation $\lceil y \rceil$ represents the smallest integer greater than or equal to y. The two preceding constraints can be handled by modifying the bounds of the variable x_i rather than by explicitly adding the constraints to the constraint matrix. The two new subproblems do not have $\bar{x}^0$ as a feasible solution, but the integer solution to (MILP) must satisfy one of the preceding constraints. The two subproblems thus defined are called *active nodes* in the branch-and-bound tree, and the variable x_i is called the *branching variable*.

In the next step the branch-and-bound algorithm chooses one of the active nodes and attempts to solve the linear programming relaxation of that subproblem. The relaxation might be infeasible, in which case the subproblem is dropped (fathomed). If the subproblem can be solved and the solution is *integer feasible* (that is, x_i is an integer for all $i \in \mathcal{S}$), then its objective value provides an *upper bound* for the objective value in the minimization problem (MILP); if the solution is not integer feasible, then it defines two new subproblems. Branching continues in this manner until there are no active nodes. At this point the best integer solution found is an optimal solution for (MILP). If no integer solution has been found, then (MILP) is integer infeasible. You can specify other criteria to stop the branch-and-bound algorithm before it processes all the active nodes; see the section "Controlling the Branch-and-Bound Algorithm" on page 1090 for details.

Upper bounds from integer feasible solutions can be used to *fathom* or *cut off* active nodes. Since the objective value of an optimal solution cannot be greater than an upper bound, active nodes with lower bounds higher than an existing upper bound can be safely deleted. In particular, if z is the objective value of the current best integer solution, then any active subproblems whose relaxed objective value is greater than or equal to z can be discarded.

It is important to realize that mixed integer linear programs are NP-hard. Roughly speaking, this means that the effort required to solve a mixed integer linear program grows exponentially with the size of the problem. For example, a problem with 10 binary variables can in the worst case generate $2^{10} = 1,024$ nodes in the branch-and-bound tree. A problem with 20 binary variables can in the worst case generate $2^{20} = 1,048,576$ nodes in the branch-and-bound tree. Although it is unlikely that the branch-and-bound algorithm will have to generate every single possible node, the need to explore even a small fraction of the potential number of nodes for a large problem can be resource intensive.

A number of techniques can speed up the search progress of the branch-and-bound algorithm. Heuristics are used to find feasible solutions, which can improve the upper bounds on solutions of mixed integer linear programs. Cutting planes can reduce the search space and thus improve the lower bounds on solutions of mixed integer linear programs. When using cutting planes, the branch-and-bound algorithm is also called the *branch-and-cut algorithm*. Preprocessing can reduce problem size and improve problem solvability. PROC OPTMILP employs various heuristics, cutting planes, preprocessing, and other techniques, which you can control through corresponding options.

Controlling the Branch-and-Bound Algorithm

There are numerous strategies that can be used to control the branch-and-bound search (see Linderoth and Savelsbergh 1998, Achterberg, Koch, and Martin 2005). PROC OPTMILP implements the most widely used strategies and provides several options that enable you to direct the choice of the next active node and of the branching variable. In the discussion that follows, let (LP^k) be the linear programming relaxation of subproblem (MILP^k). Also, let

$$f_i(k) = \bar{x}_i^k - \lfloor \bar{x}_i^k \rfloor$$

where $\bar{x}^k$ is the optimal solution to the relaxation problem (LP^k) solved at node k.

Node Selection

The NODESEL= option specifies the strategy used to select the next active node. The valid keywords for this option are AUTOMATIC, BESTBOUND, BESTESTIMATE, and DEPTH. The following list describes the strategy associated with each keyword.

AUTOMATIC — allows PROC OPTMILP to choose the best node selection strategy based on problem characteristics and search progress. This is the default setting.

BESTBOUND — chooses the node with the smallest (or largest, in the case of a maximization problem) relaxed objective value. The best-bound strategy tends to reduce the number of nodes to be processed and can improve lower bounds quickly. If there is no good upper bound, however, the number of active nodes can be large. This can result in the solver running out of memory.

BESTESTIMATE — chooses the node with the smallest (or largest, in the case of a maximization problem) objective value of the estimated integer solution. Besides improving lower bounds, the best-estimate strategy also attempts to process nodes that can yield good feasible solutions.

DEPTH — chooses the node that is deepest in the search tree. Depth-first search is effective in locating feasible solutions, since such solutions are usually deep in the search tree. Compared to the costs of the best-bound and best-estimate strategies, the cost of solving LP relaxations is less in the depth-first strategy. The number of active nodes is generally small, but it is possible that the depth-first search will remain in a portion of the search tree with no good integer solutions. This occurrence is computationally expensive.

Variable Selection

The VARSEL= option specifies the strategy used to select the next branching variable. The valid keywords for this option are AUTOMATIC, MAXINFEAS, MININFEAS,

PSEUDO, and STRONG. The following list describes the action taken in each case when $\bar{x}^k$, a relaxed optimal solution of (MILP^k), is used to define two active subproblems. In the following list, "INTTOL" refers to the value assigned using the INTTOL= option. For details about the INTTOL= option, see the section "Control Options" on page 1077.

AUTOMATIC enables PROC OPTMILP to choose the best variable selection strategy based on problem characteristics and search progress. This is the default setting.

MAXINFEAS chooses as the branching variable the variable x_i such that i maximizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and } \text{INTTOL} \leq f_i(k) \leq 1 - \text{INTTOL}\}$$

MININFEAS chooses as the branching variable the variable x_i such that i minimizes

$$\{\min\{f_i(k), 1 - f_i(k)\} \mid i \in \mathcal{S} \text{ and } \text{INTTOL} \leq f_i(k) \leq 1 - \text{INTTOL}\}$$

PSEUDO chooses as the branching variable the variable x_i such that i maximizes the weighted up and down pseudocosts. Pseudocost branching attempts to branch on significant variables first, quickly improving lower bounds. Pseudocost branching estimates significance based on historical information; however, this approach might not be accurate for future search.

STRONG chooses as the branching variable the variable x_i such that i maximizes the estimated improvement in the objective value. Strong branching first generates a list of candidates, then branches on each candidate and records the improvement in the objective value. The candidate with the largest improvement is chosen as the branching variable. Strong branching can be effective for combinatorial problems, but it is usually computationally expensive.

Branching Priorities

In some cases, it is possible to speed up the branch-and-bound algorithm by branching on variables in a specific order. You can accomplish this in PROC OPTMILP by attaching branching priorities to the integer variables in your model.

There are two ways in which you can set branching priorities for use by PROC OPTMILP. You can specify the branching priorities directly in the input MPS-format data set; see the section "BRANCH Section (Optional)" on page 1016 for details. If you are constructing a model in PROC OPTMODEL, you can set branching priorities for integer variables by using the .priority suffix. More information about this suffix is available in the section "Integer Variable Suffixes." For an example in which branching priorities are used, see Example 9.3.

Presolve

PROC OPTMILP includes a variety of presolve techniques to reduce problem size, improve numerical stability, and detect infeasibility or unboundedness (Andersen and Andersen 1995; Gondzio 1997). During presolve, redundant constraints and variables are identified and removed. Presolve can further reduce the problem size by substituting variables. Variable substitution is a very effective technique, but it might occasionally increase the number of nonzero entries in the constraint matrix. Presolve might also modify the constraint coefficients to tighten the formulation of the problem.

In most cases, using presolve is very helpful in reducing solution times. You can enable presolve at different levels by specifying the PRESOLVER= option.

Cutting Planes

The feasible region of every linear program forms a *polyhedron*. Every polyhedron in n-space can be written as a finite number of half-spaces (equivalently, inequalities). In our notation, this polyhedron is defined by the set $\mathcal{Q} = \{x \in \mathbb{R}^n \mid Ax \leq b, l \leq x \leq u\}$. After we add the restriction that some variables must be integral, the set of feasible solutions, $\mathcal{F} = \{x \in \mathcal{Q} \mid x_i \in \mathbb{Z}\ \forall i \in \mathcal{S}\}$, no longer forms a polyhedron.

The *convex hull* of a set X is the minimal convex set containing X. In solving a mixed integer linear program, in order to take advantage of LP-based algorithms we want to find the convex hull, conv($\mathcal{F}$), of $\mathcal{F}$. If we can find conv($\mathcal{F}$) and describe it compactly, then we can solve a mixed integer linear program with a linear programming solver. This is generally very difficult, so we must be satisfied with finding an approximation. Typically, the better the approximation, the more efficiently the LP-based branch-and-bound algorithm can perform.

As described in the section "The Branch-and-Bound Algorithm" on page 1088, the branch-and-bound algorithm begins by solving the linear programming relaxation over the polyhedron $\mathcal{Q}$. Clearly, $\mathcal{Q}$ contains the convex hull of the feasible region of the original integer program; that is, conv($\mathcal{F}$) $\subseteq \mathcal{Q}$.

Cutting plane techniques are used to tighten the linear relaxation to better approximate conv($\mathcal{F}$). Assume we are given a solution $\bar{x}$ to some intermediate linear relaxation during the branch-and-bound algorithm. A cut, or valid inequality ($\pi x \leq \pi^0$), is some half-space with the following characteristics:

- The half-space contains conv($\mathcal{F}$); that is, every integer feasible solution is feasible for the cut ($\pi x \leq \pi^0, \forall x \in \mathcal{F}$).
- The half-space does not contain the current solution $\bar{x}$; that is, $\bar{x}$ is not feasible for the cut ($\pi \bar{x} > \pi^0$).

Cutting planes were first made popular by Dantzig, Fulkerson, and Johnson (1954) in their work on the traveling salesman problem. The two major classifications of cutting planes are *generic cuts* and *structured cuts*. The first class of cuts is based solely on algebraic arguments and can be applied to any relaxation of any integer program.

The second class of cuts is specific to certain structures that can be found in some relaxations of the mixed integer linear program. These structures are automatically discovered during the cut initialization phase of PROC OPTMILP. Table 16.9 lists the various types of cutting planes that are built into PROC OPTMILP. Included in each type are algorithms for numerous variations based on different relaxations and lifting techniques. For a survey of cutting plane techniques for mixed integer programming, see Marchand et al. (1999). For a survey of lifting techniques, see Atamturk (2004).

Table 16.9. Cutting Planes in PROC OPTMILP

Generic Cutting Planes	Structured Cutting Planes
Gomory Mixed Integer	Cliques
Lift-and-Project	Flow Cover
Mixed Integer Rounding	Flow Path
	Generalized Upper Bound Cover
	Implied Bound
	Knapsack Cover

You can set levels for individual cuts by using the CUTCLIQUE=, CUTFLOWCOVER=, CUTFLOWPATH=, CUTGOMORY=, CUTGUB=, CUTIMPLIED=, CUTKNAPSACK=, CUTLAP=, and CUTMIR= options.

The valid levels for these options are given in Table 16.8 on page 1081.

The cut level determines the internal strategy used by PROC OPTMILP for generating the cutting planes. The strategy consists of several factors, including how frequently the cut search is called, the number of cuts allowed, and the aggressiveness of the search algorithms.

Sophisticated cutting planes, such as those included in PROC OPTMILP, can take a great deal of CPU time. Typically the additional tightening of the relaxation helps to speed up the overall process as it provides better bounds for the branch-and-bound tree and helps guide the LP solver toward integer solutions. In rare cases shutting off cutting planes completely might lead to faster overall run times.

The default settings of PROC OPTMILP have been tuned to work well for most instances. However, problem-specific expertise might suggest adjusting one or more of the strategies. These options give you that flexibility.

Primal Heuristics

Primal heuristics, an important component of PROC OPTMILP, are applied during the branch-and-bound algorithm. They are used to find integer feasible solutions early in the search tree, thereby improving the upper bound for a minimization problem. Primal heuristics play a role complementary to cutting planes in reducing the gap

between the upper and lower bounds, thus reducing the size of the branch-and-bound tree.

Applying primal heuristics in the branch-and-bound algorithm assists in the following areas:

- finding a good upper bound early in the tree search; this can lead to earlier fathoming, resulting in fewer subproblems to be processed.
- locating a reasonably good feasible solution when that is sufficient; sometimes a good feasible solution is the best the solver can produce within certain time or resource limits.
- tightening bounds on integer variables by using reduced cost fixing.

The OPTMILP procedure implements several heuristic methodologies. Some algorithms, such as rounding and iterative rounding (diving) heuristics, attempt to construct an integer feasible solution by using fractional solutions to the continuous relaxation at each node of the branch-and-cut tree. Other algorithms start with an incumbent solution and attempt to find a better solution within a neighborhood of the current best solution.

The HEURISTICS= option enables you to control the level of primal heuristics applied by PROC OPTMILP. This level determines how frequently primal heuristics are applied during the tree search. Some expensive heuristics might be disabled by the solver at less aggressive levels. Setting the HEURISTICS= option to a lower level also reduces the maximum number of iterations allowed in iterative heuristics.

The valid values for this option are listed in Table 16.5 on page 1080.

Node Log

The following information about the status of the branch-and-bound algorithm is printed in the node log:

Node	indicates the sequence number of the current node in the search tree.
Active	indicates the current number of active nodes in the branch-and-bound tree.
Sols	indicates the number of feasible solutions found so far.
BestInteger	indicates the best upper bound (assuming minimization) found so far.
BestBound	indicates the best lower bound (assuming minimization) found so far.
Gap	indicates the relative gap between BestInteger and BestBound, displayed as a percentage. If the relative gap is larger than 1000, then the absolute gap is displayed. If there are no active nodes remaining, the value of Gap is 0.

Time indicates the elapsed real time.

The PRINTFREQ= and PRINTLEVEL2= options can be used to control the amount of information printed in the node log. By default a new entry is included in the log at the first node, at the last node, and at 100-node intervals. A new entry is also included each time a better integer solution is found. The PRINTFREQ= option enables you to change the interval between entries in the node log. Figure 16.4 shows a sample node log.

```
NOTE: The problem ex1data has 10 variables (0 binary, 10 integer, 0 free, 0
      fixed).
NOTE: The problem has 2 constraints (2 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 20 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 10 variables, 2 constraints, and 20 constraint
      coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active     Sols    BestInteger      BestBound      Gap    Time
             0       1        1              0     88.0955497  100.00%       0
             0       1        2     83.0000000     88.0626822    5.75%       0
             0       1        2     83.0000000     87.9665871    5.65%       0
             0       1        2     83.0000000     87.9660825    5.65%       0
             0       1        3     85.0000000     87.9331742    3.34%       0
             0       1        3     85.0000000     87.9140538    3.31%       0
NOTE: OPTMILP added 3 cuts with 30 cut coefficients at the root.
             5       2        4     86.0000000     87.6821242    1.92%       0
             8       3        5     87.0000000     87.6821242    0.78%       0
NOTE: Optimal.
NOTE: Objective = 87.
NOTE: The data set WORK.EX1SOLN has 10 observations and 8 variables.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.06 seconds
      cpu time            0.04 seconds
```

Figure 16.4. Sample Node Log

ODS Tables

PROC OPTMILP creates two Output Delivery System (ODS) tables by default unless you specify PRINTLEVEL=0. The first table, "ProblemSummary," is a summary of the input MILP problem. The second table, "SolutionSummary," is a brief summary of the attempt to solve the problem. You can refer to these tables when using ODS. An example output of PROC OPTMILP is shown in Figure 16.5 (Problem Summary) and Figure 16.6 (Solution Summary). The name, path, and label of each table are listed in Table 16.10 and Table 16.11. For more information about ODS, see *SAS Output Delivery System: User's Guide*.

The OPTMILP Procedure

Problem Summary

Problem Name	EX_MIP
Objective Sense	Minimization
Objective Function	COST
RHS	RHS
Number of Variables	3
Bounded Above	0
Bounded Below	3
Bounded Above and Below	0
Free	0
Fixed	0
Binary	0
Integer	3
Number of Constraints	3
LE (<=)	2
EQ (=)	0
GE (>=)	1
Range	0
Constraint Coefficients	8

Figure 16.5. Example PROC OPTMILP Output: Problem Summary

Solution Summary

Objective Function	COST
Solution Status	Optimal
Objective Value	-7
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Nodes	1
Iterations	2
Presolve Time	0.00
Solution Time	0.00

Figure 16.6. Example PROC OPTMILP Output: Solution Summary

Table 16.10. PROC OPTMILP ODS Table Reference: Problem Summary

Name	ProblemSummary
Label	Problem Summary
Path	Optmilp.ProblemSummary
Label Path	'The Optmilp Procedure'.'Problem Summary'

Table 16.11. PROC OPTMILP ODS Table Reference: Solution Summary

Name	SolutionSummary
Label	Solution Summary
Path	Optmilp.SolutionSummary
Label Path	'The Optmilp Procedure'.'Solution Summary'

You can create output data sets from these tables by using the ODS OUTPUT statement. The output data sets from the preceding example are displayed in Figure 16.7 and Figure 16.8, where you can also find variable names for the tables used in the ODS template of the OPTMILP procedure.

Problem Summary

Obs	Label1	cValue1	nValue1
1	Problem Name	EX_MIP	.
2	Objective Sense	Minimization	.
3	Objective Function	COST	.
4	RHS	RHS	.
5			.
6	Number of Variables	3	3.000000
7	Bounded Above	0	0
8	Bounded Below	3	3.000000
9	Bounded Above and Below	0	0
10	Free	0	0
11	Fixed	0	0
12	Binary	0	0
13	Integer	3	3.000000
14			.
15	Number of Constraints	3	3.000000
16	LE (<=)	2	2.000000
17	EQ (=)	0	0
18	GE (>=)	1	1.000000
19	Range	0	0
20			.
21	Constraint Coefficients	8	8.000000

Figure 16.7. ODS Output Data Set: Problem Summary

Solution Summary

Obs	Label1	cValue1	nValue1
1	Objective Function	COST	.
2	Solution Status	Optimal	.
3	Objective Value	−7	−7.000000
4			.
5	Relative Gap	0	0
6	Absolute Gap	0	0
7	Primal Infeasibility	0	0
8	Bound Infeasibility	0	0
9	Integer Infeasibility	0	0
10			.
11	Nodes	1	1.000000
12	Iterations	2	2.000000
13	Presolve Time	0.00	0
14	Solution Time	0.00	0

Figure 16.8. ODS Output Data Set: Solution Summary

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the OPTMILP procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

Examples: OPTMILP Procedure

This section contains examples intended to illustrate the options and syntax of PROC OPTMILP. Example 16.1 demonstrates a model contained in an MPS-format SAS data set and finds an optimal solution by using PROC OPTMILP. Example 16.2 illustrates the use of standard MPS files in PROC OPTMILP. Example 16.3 demonstrates how to warm start PROC OPTMILP. More detailed examples of mixed integer linear programs, along with example SAS code, are given in Chapter 9.

Example 16.1. Simple Integer Linear Program

This example illustrates a model in an MPS-format SAS data set. This data set is passed to PROC OPTMILP and a solution is found.

Consider a scenario where you have a container with a set of limiting attributes (volume V and weight W) and a set I of items that you want to pack. Each item type i has a certain value p_i, a volume v_i, and a weight w_i. You must choose at most four items of each type so that the total value is maximized and all the chosen items fit into the container. Let x_i be the number of items of type i to be included in the container. This model can be formulated as the following integer linear program:

$$
\begin{array}{lllll}
\max & \displaystyle\sum_{i \in I} p_i x_i & & & \\
\text{s.t.} & \displaystyle\sum_{i \in I} v_i x_i & \leq & V & (\text{volume\_con}) \\
& \displaystyle\sum_{i \in I} w_i x_i & \leq & W & (\text{weight\_con}) \\
& x_i & \leq & 4 \quad \forall i \in I & \\
& x_i \in \mathbb{Z}^+ & & \forall i \in I &
\end{array}
$$

Constraint (volume_con) enforces the volume capacity limit, while constraint (weight_con) enforces the weight capacity limit. An instance of this problem can be saved in an MPS-format SAS data set by using the following code:

```
data ex1data;
   input field1 $ field2 $ field3 $ field4 field5 $ field6;
   datalines;
NAME                    ex1data           .                     .
ROWS                                      .                     .
MAX         z                             .                     .
L           volume_con                    .                     .
L           weight_con                    .                     .
COLUMNS                                   .                     .
            .MRK0       'MARKER'          .     'INTORG'        .
            x[1]        z                 1     volume_con     10
            x[1]        weight_con       12                     .
            x[2]        z                 2     volume_con    300
```

```
             x[2]          weight_con       15                      .
             x[3]          z                 3     volume_con      250
             x[3]          weight_con       72                      .
             x[4]          z                 4     volume_con      610
             x[4]          weight_con      100                      .
             x[5]          z                 5     volume_con      500
             x[5]          weight_con      223                      .
             x[6]          z                 6     volume_con      120
             x[6]          weight_con       16                      .
             x[7]          z                 7     volume_con       45
             x[7]          weight_con       73                      .
             x[8]          z                 8     volume_con      100
             x[8]          weight_con       12                      .
             x[9]          z                 9     volume_con      200
             x[9]          weight_con      200                      .
             x[10]         z                10     volume_con       61
             x[10]         weight_con      110                      .
             .MRK1         'MARKER'          .     'INTEND'         .
RHS                                          .                      .
             .RHS.         volume_con     1000                      .
             .RHS.         weight_con      500                      .
BOUNDS                                       .                      .
UP           .BOUNDS.      x[1]              4                      .
UP           .BOUNDS.      x[2]              4                      .
UP           .BOUNDS.      x[3]              4                      .
UP           .BOUNDS.      x[4]              4                      .
UP           .BOUNDS.      x[5]              4                      .
UP           .BOUNDS.      x[6]              4                      .
UP           .BOUNDS.      x[7]              4                      .
UP           .BOUNDS.      x[8]              4                      .
UP           .BOUNDS.      x[9]              4                      .
UP           .BOUNDS.      x[10]             4                      .
ENDATA                                       .                      .
;
```

In the COLUMNS section of this data set, the name of the objective is z, and the objective coefficients p_i appear in field4. The coefficients v_i of (volume_con) appear in field6. The coefficients w_i of (weight_con) appear in field4. In the RHS section, the bounds V and W appear in field4.

This problem can be solved by using the following statement to call the OPTMILP procedure:

```
proc optmilp data=ex1data primalout=ex1soln;
run;
```

The progress of the solver is shown in Output 16.1.1.

Output 16.1.1. Simple Integer Linear Program PROC OPTMILP Log

```
NOTE: The problem ex1data has 10 variables (0 binary, 10 integer, 0 free, 0
      fixed).
NOTE: The problem has 2 constraints (2 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 20 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 10 variables, 2 constraints, and 20 constraint
      coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active     Sols    BestInteger      BestBound      Gap    Time
             0       1        1              0     88.0955497  100.00%       0
             0       1        2     83.0000000     88.0626822    5.75%       0
             0       1        2     83.0000000     87.9665871    5.65%       0
             0       1        2     83.0000000     87.9660825    5.65%       0
             0       1        3     85.0000000     87.9331742    3.34%       0
             0       1        3     85.0000000     87.9140538    3.31%       0
NOTE: OPTMILP added 3 cuts with 30 cut coefficients at the root.
             5       2        4     86.0000000     87.6821242    1.92%       0
             8       3        5     87.0000000     87.6821242    0.78%       0
NOTE: Optimal.
NOTE: Objective = 87.
NOTE: The data set WORK.EX1SOLN has 10 observations and 8 variables.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.06 seconds
      cpu time            0.06 seconds
```

The data set ex1soln is shown in Output 16.1.2.

Output 16.1.2. Simple Integer Linear Program Solution

Example 1 Solution Data

Objective Function ID	RHS ID	Variable Name	Variable Type	Objective Coefficient	Lower Bound	Upper Bound	Variable Value
z	.RHS.	x[1]	I	1	0	4	0
z	.RHS.	x[2]	I	2	0	4	0
z	.RHS.	x[3]	I	3	0	4	0
z	.RHS.	x[4]	I	4	0	4	0
z	.RHS.	x[5]	I	5	0	4	0
z	.RHS.	x[6]	I	6	0	4	3
z	.RHS.	x[7]	I	7	0	4	1
z	.RHS.	x[8]	I	8	0	4	4
z	.RHS.	x[9]	I	9	0	4	0
z	.RHS.	x[10]	I	10	0	4	3

The optimal solution is $x_6 = 3, x_7 = 1, x_8 = 4$, and $x_{10} = 3$, with a total value of 87. From this solution, you can compute the total volume used, which is 988 ($\leq V = 1000$); the total weight used is 499 ($\leq W = 500$). The problem summary and solution summary are shown in Output 16.1.3.

Output 16.1.3. Simple Integer Linear Program Summary

Problem Summary

Problem Name	ex1data
Objective Sense	Maximization
Objective Function	z
RHS	.RHS.
Number of Variables	10
Bounded Above	0
Bounded Below	0
Bounded Above and Below	10
Free	0
Fixed	0
Binary	0
Integer	10
Number of Constraints	2
LE (<=)	2
EQ (=)	0
GE (>=)	0
Range	0
Constraint Coefficients	20

Solution Summary

Objective Function	z
Solution Status	Optimal
Objective Value	87
Relative Gap	0.0077795131
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Nodes	9
Iterations	34
Presolve Time	0.00
Solution Time	0.02

Example 16.2. MIPLIB Benchmark Instance

The following example illustrates the conversion of a standard MPS-format file into an MPS-format SAS data set. The problem is re-solved several times, each time by using a different control option. For such a small example, it is necessary to disable cuts and heuristics in order to see the computational savings gained by using other options. For larger or more complex examples, the benefits of using the various control options are more pronounced.

The standard set of MILP benchmark cases is called MIPLIB (Bixby et al. 1998, Achterberg, Koch, and Martin 2003) and can be found at http://miplib.zib.de/. The following code uses the %MPS2SASD macro to convert an example from MIPLIB to a SAS data set:

```
%mps2sasd(mpsfile="p0282.mps", outdata=mpsdata);
```

The problem can then be solved using PROC OPTMILP on the data set created by the conversion:

```
proc optmilp data=mpsdata allcuts=none heuristics=none;
run;
```

The resulting log is shown in Output 16.2.1.

Output 16.2.1. MIPLIB PROC OPTMILP Log

```
NOTE: The problem P0282 has 282 variables (282 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 241 constraints (241 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1966 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 80 variables and 80 constraints.
NOTE: The OPTMILP presolver removed 682 constraint coefficients.
NOTE: The OPTMILP presolver modified 46 constraint coefficients.
NOTE: The presolved problem has 202 variables, 161 constraints, and 1284
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .         180000        .       0
            45      43       1         258993         180000   43.88%       0
           100      63       1         258993         254952    1.59%       0
           150      84       2         258723         255333    1.33%       0
           200     105       2         258723         255541    1.25%       0
           300     147       2         258723         255840    1.13%       0
           400     142       2         258723         256436    0.89%       0
           500     118       2         258723         257233    0.58%       0
           584      46       3         258411         258083    0.13%       0
           600      32       3         258411         258213    0.08%       0
           626       8       3         258411         258387    0.01%       0
NOTE: Optimal within relative gap.
NOTE: Objective = 258411.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.57 seconds
      cpu time            0.56 seconds
```

Suppose you do not have a bound for the solution. If there is an objective value that, even if it is not optimal, satisfies your requirements, then you can save time by using the TARGET= option. The following PROC OPTMILP call solves the problem with a target value of 260,000:

```
proc optmilp data=mpsdata allcuts=none heuristics=none
   target=260000;
run;
```

The relevant results from this run are displayed in Output 16.2.2. In this case, there is a decrease in CPU time, but the objective value has increased.

Output 16.2.2. MIPLIB PROC OPTMILP Log with TARGET= Option

```
NOTE: The problem P0282 has 282 variables (282 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 241 constraints (241 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1966 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 80 variables and 80 constraints.
NOTE: The OPTMILP presolver removed 682 constraint coefficients.
NOTE: The OPTMILP presolver modified 46 constraint coefficients.
NOTE: The presolved problem has 202 variables, 161 constraints, and 1284
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .         180000        .       0
            45      44       1         258993         180000   43.88%       0
NOTE: Target reached.
NOTE: Objective of the best integer solution found = 258993.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.15 seconds
      cpu time            0.15 seconds
```

When the objective value of a solution is within a certain relative gap of the optimal objective value, the procedure stops. The acceptable relative gap can be changed using the RELOBJGAP= option, as demonstrated in the following example:

```
proc optmilp data=mpsdata allcuts=none heuristics=none
   relobjgap=0.10;
run;
```

The relevant results from this run are displayed in Output 16.2.3. In this case, since the specified RELOBJGAP= value is larger than the default value, the number of nodes as well as the CPU time have decreased from their values in the original run. Note that these savings are exchanged for an increase in the objective value of the solution.

Output 16.2.3. MIPLIB PROC OPTMILP Log with RELOBJGAP= Option

```
NOTE: The problem P0282 has 282 variables (282 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 241 constraints (241 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1966 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 80 variables and 80 constraints.
NOTE: The OPTMILP presolver removed 682 constraint coefficients.
NOTE: The OPTMILP presolver modified 46 constraint coefficients.
NOTE: The presolved problem has 202 variables, 161 constraints, and 1284
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .         180000        .       0
            45      43       1         258993         180000   43.88%       0
            56      42       1         258993         241020    7.46%       0
NOTE: Optimal within relative gap.
NOTE: Objective = 258993.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.18 seconds
      cpu time            0.15 seconds
```

The MAXTIME= option enables you to accept the best solution produced by PROC OPTMILP in a specified amount of time. The following example illustrates the use of the MAXTIME= option:

```
proc optmilp data=mpsdata allcuts=none heuristics=none
   maxtime=0.3;
run;
```

The relevant results from this run are displayed in Output 16.2.4. Once again, a reduction in solution time is traded for an increase in objective value.

Output 16.2.4. MIPLIB PROC OPTMILP Log with MAXTIME= Option

```
NOTE: The problem P0282 has 282 variables (282 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 241 constraints (241 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1966 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 80 variables and 80 constraints.
NOTE: The OPTMILP presolver removed 682 constraint coefficients.
NOTE: The OPTMILP presolver modified 46 constraint coefficients.
NOTE: The presolved problem has 202 variables, 161 constraints, and 1284
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .         180000        .       0
            45      43       1         258993         180000   43.88%       0
           100      63       1         258993         254952    1.59%       0
           150      84       2         258723         255333    1.33%       0
           200     105       2         258723         255541    1.25%       0
           300     147       2         258723         255840    1.13%       0
NOTE: CPU time limit reached.
NOTE: Objective of the best integer solution found = 258723.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.35 seconds
      cpu time            0.32 seconds
```

The MAXNODES= option enables you to limit the number of nodes generated by PROC OPTMILP. The following example illustrates the use of the MAXNODES= option:

```
proc optmilp data=mpsdata allcuts=none heuristics=none
   maxnodes=500;
run;
```

The relevant results from this run are displayed in Output 16.2.5. PROC OPTMILP displays the best objective value of all the solutions produced.

Output 16.2.5. MIPLIB PROC OPTMILP Log with MAXNODES= Option

```
NOTE: The problem P0282 has 282 variables (282 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 241 constraints (241 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1966 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 80 variables and 80 constraints.
NOTE: The OPTMILP presolver removed 682 constraint coefficients.
NOTE: The OPTMILP presolver modified 46 constraint coefficients.
NOTE: The presolved problem has 202 variables, 161 constraints, and 1284
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .         180000        .       0
            45      43       1         258993         180000   43.88%       0
           100      63       1         258993         254952    1.59%       0
           150      84       2         258723         255333    1.33%       0
           200     105       2         258723         255541    1.25%       0
           300     147       2         258723         255840    1.13%       0
           400     142       2         258723         256436    0.89%       0
           500     119       2         258723         257221    0.58%       0
NOTE: Node limit reached.
NOTE: Objective of the best integer solution found = 258723.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           0.51 seconds
      cpu time            0.48 seconds
```

Example 16.3. Facility Location

This advanced example demonstrates how to warm start PROC OPTMILP by using the PRIMALIN= option. The model is constructed in PROC OPTMODEL and saved in an MPS-format SAS data set for use in PROC OPTMILP. Note that this problem can also be solved from within PROC OPTMODEL; see Chapter 9 for details.

Consider the classical facility location problem. Given a set L of customer locations and a set F of candidate facility sites, you must decide which sites to build facilities on and assign coverage of customer demand to these sites so as to minimize cost. All customer demand d_i must be satisfied, and each facility has a demand capacity limit C. The total cost is the sum of the distances c_{ij} between facility j and its assigned customer i, plus a fixed charge f_j for building a facility at site j. Let $y_j = 1$ represent choosing site j to build a facility, and 0 otherwise. Also, let $x_{ij} = 1$ represent the assignment of customer i to facility j, and 0 otherwise. This model can be formulated

as the following integer linear program:

$$
\begin{array}{lllll}
\min & \displaystyle\sum_{i\in L}\sum_{j\in F} c_{ij}x_{ij} + \sum_{j\in F} f_j y_j & & & \\
\text{s.t.} & \displaystyle\sum_{j\in F} x_{ij} = 1 & \forall i \in L & & \text{(assign\_def)} \\
& x_{ij} \le y_j & \forall i \in L, j \in F & & \text{(link)} \\
& \displaystyle\sum_{i\in L} d_i x_{ij} \le C y_j & \forall j \in F & & \text{(capacity)} \\
& x_{ij} \in \{0,1\} & \forall i \in L, j \in F & & \\
& y_j \in \{0,1\} & \forall j \in F & &
\end{array}
$$

Constraint (assign_def) ensures that each customer is assigned to exactly one site. Constraint (link) forces a facility to be built if any customer has been assigned to that facility. Finally, constraint (capacity) enforces the capacity limit at each site.

Let us also consider a variation of this same problem where there is no cost for building a facility. This problem is typically easier to solve than the original problem. For this variant, let the objective be

$$\min \quad \sum_{i\in L}\sum_{j\in F} c_{ij}x_{ij}$$

First, let us construct a random instance of this problem by using the following DATA steps:

```
%let NumCustomers  = 50;
%let NumSites      = 10;
%let SiteCapacity  = 35;
%let MaxDemand     = 10;
%let xmax          = 200;
%let ymax          = 100;
%let seed          = 1234;

/* generate random customer locations */
data cdata(drop=i);
   length name $8;
   do i = 1 to &NumCustomers;
      name = compress('C'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      demand = ranuni(&seed) * &MaxDemand;
      output;
   end;
run;

/* generate random site locations and fixed charge */
```

```
data sdata(drop=i);
   length name $8;
   do i = 1 to &NumSites;
      name = compress('SITE'||put(i,best.));
      x = ranuni(&seed) * &xmax;
      y = ranuni(&seed) * &ymax;
      fixed_charge = (abs(&xmax/2-x) + abs(&ymax/2-y)) / 2;
      output;
   end;
run;
```

In the following PROC OPTMODEL code, we generate the model and define both variants of the cost function:

```
proc optmodel;
   set <str> CUSTOMERS;
   set <str> SITES;

   /* x and y coordinates of CUSTOMERS and SITES */
   num x {CUSTOMERS union SITES};
   num y {CUSTOMERS union SITES};
   num demand {CUSTOMERS};
   num fixed_charge {SITES};

   /* distance from customer i to site j */
   num dist {i in CUSTOMERS, j in SITES}
       = sqrt((x[i] - x[j])^2 + (y[i] - y[j])^2);

   read data cdata into CUSTOMERS=[name] x y demand;
   read data sdata into SITES=[name] x y fixed_charge;

   var Assign {CUSTOMERS, SITES} binary;
   var Build {SITES} binary;

   /* each customer assigned to exactly one site */
   con assign_def {i in CUSTOMERS}:
       sum {j in SITES} Assign[i,j] = 1;

   /* if customer i assigned to site j, then facility must be */
   /* built at j                                              */
   con link {i in CUSTOMERS, j in SITES}:
       Assign[i,j] <= Build[j];

   /* each site can handle at most &SiteCapacity demand */
   con capacity {j in SITES}:
       sum {i in CUSTOMERS} demand[i] * Assign[i,j]
       <= &SiteCapacity * Build[j];

   min CostNoFixedCharge
       = sum {i in CUSTOMERS, j in SITES} dist[i,j] * Assign[i,j];

   save mps nofcdata;

   min CostFixedCharge
       = CostNoFixedCharge
         + sum {j in SITES} fixed_charge[j] * Build[j];
```

```
   save mps fcdata;

quit;
```

We first solve the problem for the fixed-charge model by using the following code. The information printed in the log by PROC OPTMILP is displayed in Output 16.3.1.

```
proc optmilp data=fcdata primalout=fcout;
run;
```

Output 16.3.1. PROC OPTMILP Log for Facility Location with Fixed Charges

```
NOTE: The problem fcdata has 510 variables (510 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 560 constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
           Node  Active    Sols    BestInteger      BestBound      Gap    Time
              0       1       0              .   1870.6874826        .       0
              0       1       0              .   1887.7583984        .       0
              0       1       2   2030.5371397   1892.7586646    7.28%       0
              0       1       5   1921.6837559   1895.6168522    1.38%       0
              0       1       5   1921.6837559   1896.3025743    1.34%       0
              0       1       5   1921.6837559   1898.2934630    1.23%       0
              0       1       5   1921.6837559   1898.3693562    1.23%       0
              0       1       5   1921.6837559   1898.4146301    1.23%       0
NOTE: OPTMILP added 19 cuts with 1970 cut coefficients at the root.
              8       9       7   1914.2169970   1898.4146301    0.83%       1
             14      15       8   1911.7679632   1898.4146301    0.70%       1
            100      71       8   1911.7679632   1902.2772813    0.50%       3
            200     147       8   1911.7679632   1902.6557849    0.48%       4
            300     203       8   1911.7679632   1903.0087266    0.46%       6
            315     202       9   1911.5023765   1903.0622409    0.44%       6
            316     202      10   1911.5023718   1903.0622409    0.44%       6
            400     261      10   1911.5023718   1903.2980984    0.43%       7
            500     330      10   1911.5023718   1903.4521820    0.42%       8
            600     370      10   1911.5023718   1904.1571741    0.39%      10
            648     382      11   1910.9293121   1904.6205473    0.33%      11
            649     383      12   1910.9293120   1904.6205473    0.33%      11
            650     383      13   1910.9293116   1904.6205473    0.33%      11
            700     397      13   1910.9293116   1904.7170348    0.33%      12
            800     412      13   1910.9293116   1905.3160494    0.29%      14
            900     413      13   1910.9293116   1905.5502045    0.28%      17
            983     355      14   1908.7057490   1905.8021302    0.15%      19
            984     356      15   1908.7057291   1905.8021302    0.15%      19
           1000     341      16   1908.7057291   1905.9104785    0.15%      20
           1100     307      16   1908.7057291   1906.2754335    0.13%      23
           1200     245      16   1908.7057291   1906.6901654    0.11%      27
           1300     203      16   1908.7057291   1907.0403752    0.09%      30
           1400     119      16   1908.7057291   1907.7107465    0.05%      35
           1500      25      16   1908.7057291   1908.3671027    0.02%      39
           1517      10      16   1908.7057291   1908.5296514    0.01%      40
NOTE: Optimal within relative gap.
NOTE: Objective = 1908.7057291.
NOTE: The data set WORK.FCOUT has 510 observations and 8 variables.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           48.79 seconds
      cpu time            44.18 seconds
```

Next, we solve the problem for the model with no fixed charge by using the following code. The first PROC SQL call populates the macro variables varcostNo. This macro variable is used to display the objective value when the results are plotted. The second PROC SQL call generates a data set which is used to plot the results. The information printed in the log by PROC OPTMILP is displayed in Output 16.3.2.

```
proc optmilp data=nofcdata primalout=nofcout;
run;

proc sql noprint;
   select put(sum(_objcoef_ * _value_),6.1) into :varcostNo
   from nofcout;
quit;

proc sql;
   create table CostNoFixedCharge_Data as
   select
      scan(p._var_,2,'[],') as customer,
      scan(p._var_,3,'[],') as site,
      c.x as xi, c.y as yi, s.x as xj, s.y as yj
   from
      cdata as c,
      sdata as s,
      nofcout(where=(substr(_var_,1,6)='Assign' and
                     round(_value_) = 1)) as p
   where calculated customer = c.name and calculated site = s.name;
quit;
```

Output 16.3.2. PROC OPTMILP Log for Facility Location with No Fixed Charges

```
NOTE: The problem nofcdata has 510 variables (510 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 560 constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 10 variables and 500 constraints.
NOTE: The OPTMILP presolver removed 1010 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 500 variables, 60 constraints, and 1000
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       0              .   1520.1872420        .       0
             0       1       0              .   1539.2006720        .       0
             0       1       0              .   1544.8618998        .       0
             0       1       2   1568.4968729   1545.8084732    1.47%       0
             0       1       2   1568.4968729   1547.5231740    1.36%       0
             0       1       2   1568.4968729   1548.8648329    1.27%       0
             0       1       3   1563.3984995   1549.2087898    0.92%       0
             0       1       3   1563.3984995   1549.2087915    0.92%       0
NOTE: OPTMILP added 14 cuts with 1015 cut coefficients at the root.
            12      10       4   1563.3719799   1549.2087915    0.91%       0
            14       9       5   1554.1656132   1551.0382988    0.20%       0
            19      11       6   1554.0107314   1551.0382988    0.19%       0
            30       1       7   1553.8285559   1553.1324191    0.04%       1
NOTE: Optimal.
NOTE: Objective = 1553.8285559.
NOTE: The data set WORK.NOFCOUT has 510 observations and 8 variables.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           1.81 seconds
      cpu time            1.68 seconds
```

Finally, we re-solve the fixed-charge model by using the following code. Note that the solution to the model with no fixed charge is feasible for the fixed-charge model and

should provide a good starting point for PROC OPTMILP. We use the PRIMALIN= option to provide an incumbent solution ("warm start"). The two PROC SQL calls perform the same functions as in the case with no fixed charges. The results from this approach are shown in Output 16.3.3. As the output shows, warm starting PROC OPTMILP can reduce the number of nodes processed and the amount of time spent to find the optimal solution.

```
proc optmilp data=fcdata primalin=nofcout;
run;

proc sql noprint;
   select put(sum(_objcoef_ * _value_), 6.1) into :varcost
   from fcout(where=(substr(_var_,1,6)='Assign'));
   select put(sum(_objcoef_ * _value_), 5.1) into :fixcost
   from fcout(where=(substr(_var_,1,5)='Build'));
   select put(sum(_objcoef_ * _value_), 6.1) into :totalcost
   from fcout;
quit;

proc sql;
   create table CostFixedCharge_Data as
   select
      scan(p._var_,2,'[],') as customer,
      scan(p._var_,3,'[],') as site,
      c.x as xi, c.y as yi, s.x as xj, s.y as yj
   from
      cdata as c,
      sdata as s,
      fcout(where=(substr(_var_,1,6)='Assign' and
                   round(_value_) = 1)) as p
   where calculated customer = c.name and calculated site = s.name;
quit;
```

Output 16.3.3. PROC OPTMILP Log for Facility Location with Fixed Charges, Using Warm Start

```
NOTE: The problem fcdata has 510 variables (510 binary, 0 integer, 0 free, 0
      fixed).
NOTE: The problem has 560 constraints (510 LE, 50 EQ, 0 GE, 0 range).
NOTE: The problem has 2010 constraint coefficients.
NOTE: The OPTMILP presolver value AUTOMATIC is applied.
NOTE: The OPTMILP presolver removed 0 variables and 0 constraints.
NOTE: The OPTMILP presolver removed 0 constraint coefficients.
NOTE: The OPTMILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 510 variables, 560 constraints, and 2010
      constraint coefficients.
NOTE: The MIXED INTEGER LINEAR solver is called.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       1   1967.9351072   1870.6874826    5.20%       0
             0       1       1   1967.9351072   1887.7584203    4.25%       0
             0       1       1   1967.9351072   1892.7586866    3.97%       0
             0       1       4   1921.6837648   1895.0700047    1.40%       0
             0       1       4   1921.6837648   1895.9244295    1.36%       0
             0       1       5   1917.3437318   1898.8816348    0.97%       0
             0       1       5   1917.3437318   1899.0762206    0.96%       0
             0       1       5   1917.3437318   1899.1900845    0.96%       0
             0       1       5   1917.3437318   1899.1953627    0.96%       0
NOTE: OPTMILP added 14 cuts with 1095 cut coefficients at the root.
            13      14       6   1914.2727588   1899.1953627    0.79%       1
            19      19       7   1913.5664405   1899.1953627    0.76%       1
           100      77       7   1913.5664405   1903.1016611    0.55%       3
           133      95       8   1912.3862595   1903.5805067    0.46%       3
           141     103       9   1912.3862151   1903.5805067    0.46%       4
           156     105      10   1911.7679486   1904.4133622    0.39%       4
           158     107      11   1911.7679302   1904.4133622    0.39%       4
           200     110      11   1911.7679302   1905.5157753    0.33%       5
           241      50      12   1909.0542475   1906.1034550    0.15%       7
           243      51      13   1909.0542475   1906.1034550    0.15%       7
           251      43      14   1908.7057172   1906.4339809    0.12%       8
           300      30      14   1908.7057172   1907.4082707    0.07%      10
           339       7      14   1908.7057172   1908.5557179    0.01%      12
NOTE: Optimal within relative gap.
NOTE: Objective = 1908.7057172.
NOTE: The data set WORK.FCOUT has 510 observations and 8 variables.
NOTE: PROCEDURE OPTMILP used (Total process time):
      real time           15.18 seconds
      cpu time            13.45 seconds
```

The following two SAS programs produce a plot of the solutions for both variants of the model, using data sets produced by PROC SQL from the PRIMALOUT= data sets produced by PROC OPTMILP.

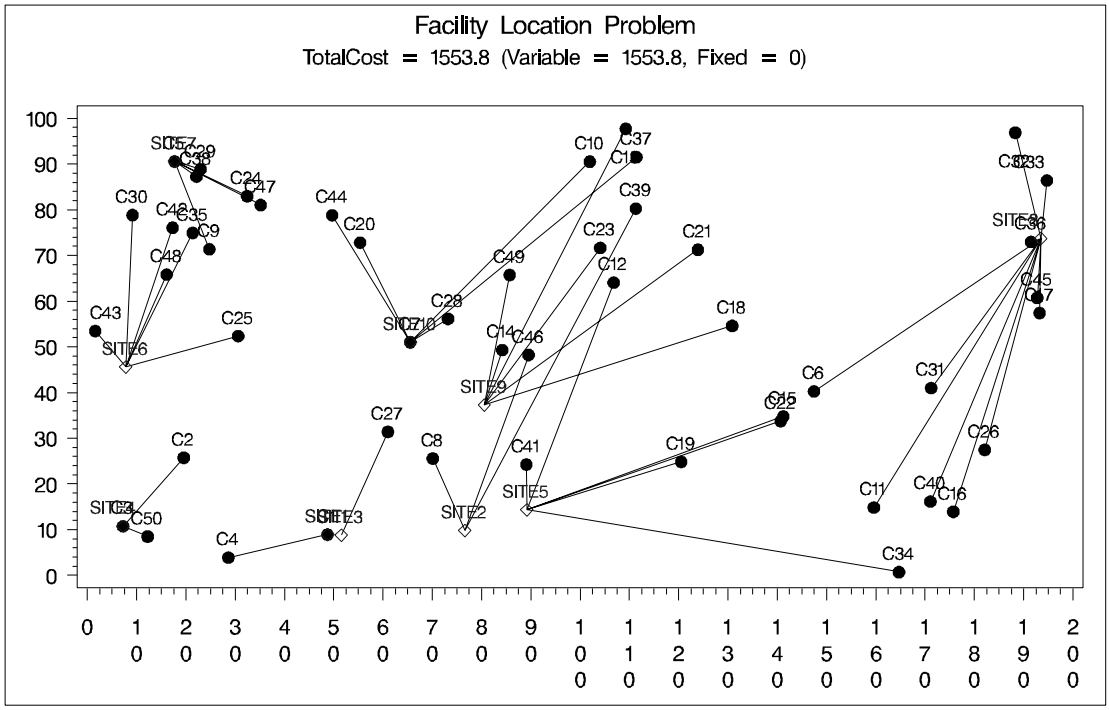
Note: Execution of this code requires SAS/GRAPH software.

```
title1 "Facility Location Problem";
title2 "TotalCost = &varcostNo (Variable = &varcostNo, Fixed = 0)";
data csdata;
   set cdata(rename=(y=cy)) sdata(rename=(y=sy));
run;
/* create Annotate data set to draw line between customer and */
/* assigned site                                              */
%annomac;
data anno(drop=xi yi xj yj);
   %SYSTEM(2, 2, 2);
```

```
set CostNoFixedCharge_Data(keep=xi yi xj yj);
   %LINE(xi, yi, xj, yj, *, 1, 1);
run;
proc gplot data=csdata anno=anno;
   axis1 label=none order=(0 to &xmax by 10);
   axis2 label=none order=(0 to &ymax by 10);
   symbol1 value=dot interpol=none
      pointlabel=("#name" height=0.7) cv=black;
      symbol2 value=diamond interpol=none
      pointlabel=("#name" color=blue height=0.7) cv=blue;
   plot cy*x sy*x / overlay haxis=axis1 vaxis=axis2;
run;
quit;
```

The output from the first program appears in Output 16.3.4.

Output 16.3.4. Solution Plot for Facility Location with No Fixed Charges

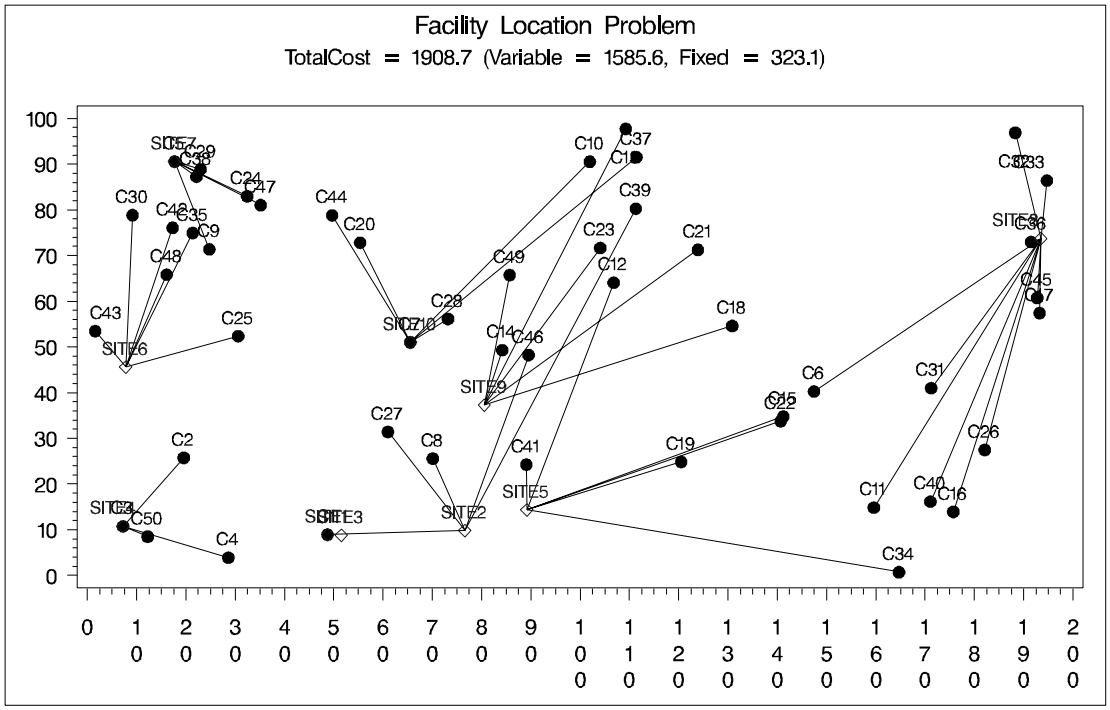



```
title1 "Facility Location Problem";
title2 "TotalCost = &totalcost (Variable = &varcost, Fixed = &fixcost)";
/* create Annotate data set to draw line between customer and */
/* assigned site                                              */
data anno(drop=xi yi xj yj);
   %SYSTEM(2, 2, 2);
   set CostFixedCharge_Data(keep=xi yi xj yj);
   %LINE(xi, yi, xj, yj, *, 1, 1);
run;
proc gplot data=csdata anno=anno;
   axis1 label=none order=(0 to &xmax by 10);
```

```
    axis2 label=none order=(0 to &ymax by 10);
    symbol1 value=dot interpol=none
        pointlabel=("#name" height=0.7) cv=black;
        symbol2 value=diamond interpol=none
        pointlabel=("#name" color=blue height=0.7) cv=blue;
    plot cy*x sy*x / overlay haxis=axis1 vaxis=axis2;
run;
quit;
```

The output from the second program appears in Output 16.3.5.

Output 16.3.5. Solution Plot for Facility Location with Fixed Charges



The economic tradeoff for the fixed-charge model forces us to build fewer sites and push more demand to each site.

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Chapter 17
The OPTQP Procedure

Chapter Contents

Chapter 17
The OPTQP Procedure

Overview: OPTQP Procedure

The OPTQP procedure solves quadratic programs—problems with quadratic objective function and a collection of linear constraints, including lower and/or upper bounds on the decision variables.

Mathematically, a quadratic programming (QP) problem can be stated as follows:

$$\begin{aligned} \min \quad & \tfrac{1}{2}\,\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

where

$\mathbf{Q} \in \mathbb{R}^{n \times n}$	is the quadratic (also known as Hessian) matrix
$\mathbf{A} \in \mathbb{R}^{m \times n}$	is the constraints matrix
$\mathbf{x} \in \mathbb{R}^{n}$	is the vector of decision variables
$\mathbf{c} \in \mathbb{R}^{n}$	is the vector of linear objective function coefficients
$\mathbf{b} \in \mathbb{R}^{m}$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l} \in \mathbb{R}^{n}$	is the vector of lower bounds on the decision variables
$\mathbf{u} \in \mathbb{R}^{n}$	is the vector of upper bounds on the decision variables

The quadratic matrix $\mathbf{Q}$ is assumed to be symmetric; i.e.,

$$q_{ij} = q_{ji}, \quad \forall i, j = 1, \ldots, n$$

Indeed, it is easy to show that even if $\mathbf{Q} \neq \mathbf{Q}^{\mathrm{T}}$, the simple modification

$$\tilde{\mathbf{Q}} = \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^{\mathrm{T}})$$

produces an equivalent formulation $\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \mathbf{x}^{\mathrm{T}}\tilde{\mathbf{Q}}\mathbf{x}$; hence symmetry is assumed. When specifying a quadratic matrix it suffices to list only lower triangular coefficients.

In addition to being symmetric, $\mathbf{Q}$ is also required to be positive semidefinite:

$$\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^{n}$$

for minimization type of models; it is required to be negative semidefinite for the maximization type of models. Convexity can come as a result of a matrix-matrix multiplication

$$\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$$

or as a consequence of physical laws, etc. See Figure 17.1 for examples of convex, concave, and nonconvex objective functions.

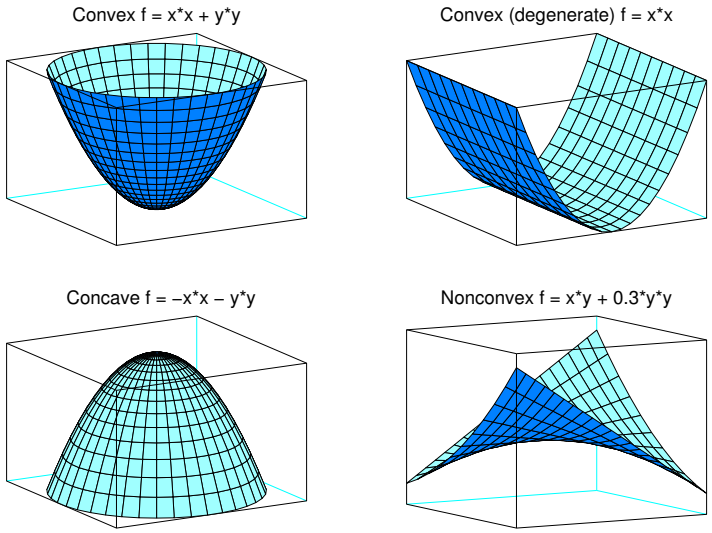

Figure 17.1. Examples of Convex, Concave, and Nonconvex Objective Functions

The order of constraints is insignificant. Some or all components of $\mathbf{l}$ or $\mathbf{u}$ (lower/upper bounds) can be omitted.

Getting Started: OPTQP Procedure

Consider a small illustrative example. Suppose you want to minimize a two-variable quadratic function $f(x_1, x_2)$ on the nonnegative quadrant, subject to two constraints:

$$\begin{array}{rrcrcrcrcl} \min & 2x_1 & + & 3x_2 & + & x_1^2 & + & 10x_2^2 & + & 2.5x_1x_2 \\ \text{subject to} & x_1 & - & x_2 & \leq & 1 & & & & \\ & x_1 & + & 2x_2 & \geq & 100 & & & & \\ & x_1 & & & \geq & 0 & & & & \\ & & & x_2 & \geq & 0 & & & & \end{array}$$

The linear objective function coefficients, vector of right-hand sides, and lower and upper bounds are identified immediately as

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 100 \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}$$

Let us carefully construct the quadratic matrix $\mathbf{Q}$. Observe that you can use symmetry to separate the main-diagonal and off-diagonal elements:

$$\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \frac{1}{2}\sum_{i,j=1}^{n} x_i\, q_{ij}\, x_j = \frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2 + \sum_{i>j} x_i\, q_{ij}\, x_j$$

The first expression

$$\frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2$$

sums the main-diagonal elements. Thus in this case you have

$$q_{11} = 2, \quad q_{22} = 20$$

Notice that the main-diagonal values are doubled in order to accommodate the 1/2 factor. Now the second term

$$\sum_{i>j} x_i\, q_{ij}\, x_j$$

sums the off-diagonal elements in the strict lower triangular part of the matrix. The only off-diagonal ($x_i\, x_j,\ i \neq j$) term in the objective function is $2.5\, x_1\, x_2$, so you have

$$q_{21} = 2.5$$

Notice that you do not need to specify the upper triangular part of the quadratic matrix.

Finally, the matrix of constraints is as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

The QPS-format SAS input data set for the preceding problem can be expressed in the following manner:

```
data gsdata;
   input field1 $ field2 $ field3$ field4 field5 $ field6 @;
datalines;
NAME    .       EXAMPLE    .              .          .
ROWS    .       .          .              .          .
N       OBJ     .          .              .          .
L       R1      .          .              .          .
G       R2      .          .              .          .
COLUMNS .       .          .              .          .
.       X1      R1          1.0           R2         1.0
.       X1      OBJ         2.0           .          .
.       X2      R1         -1.0           R2         2.0
.       X2      OBJ         3.0           .          .
RHS     .       .          .              .          .
.       RHS     R1         1.0            .          .
.       RHS     R2         100            .          .
RANGES  .       .          .              .          .
BOUNDS  .       .          .              .          .
QUADOBJ .       .          .              .          .
.       X1      X1         2.0            .          .
.       X1      X2         2.5            .          .
.       X2      X2         20             .          .
ENDATA  .       .          .              .          .
;
```

For more details about the QPS-format data set, see Chapter 14, "The MPS-Format SAS Data Set."

Alternatively, if you have a QPS-format flat file named gs.qps, then the following call to the SAS macro %MPS2SASD translates that file into a SAS data set, named gsdata:

```
%mps2sasd(mpsfile =gs.qps, outdata = gsdata);
```

Note: The SAS macro %MPS2SASD is provided in SAS/OR software. See the section "Converting an MPS/QPS-Format File: %MPS2SASD" on page 1019 for details.

You can use the following call to PROC OPTQP:

```
proc optqp data=gsdata
  primalout = gspout
  dualout   = gsdout;
run;
```

The procedure output is displayed in Figure 17.2.

The OPTQP Procedure

Problem Summary

Problem Name	EXAMPLE
Objective Sense	Minimization
Objective Function	OBJ
RHS	RHS
Number of Variables	2
Bounded Above	0
Bounded Below	2
Bounded Above and Below	0
Free	0
Fixed	0
Number of Constraints	2
LE (<=)	1
EQ (=)	0
GE (>=)	1
Range	0
Constraint Coefficients	4
Hessian Diagonal Elements	2
Hessian Elements Above the Diagonal	1

The OPTQP Procedure

Solution Summary

Objective Function	OBJ
Solution Status	Optimal
Objective Value	15018.000761
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	9.9575984E-8
Complementarity	0.0014952645
Iterations	10
Presolve Time	0.00
Solution Time	0.00

Figure 17.2. Procedure Output

The optimal primal solution is displayed in Figure 17.3.

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient
1	OBJ	RHS	X1	N	2
2	OBJ	RHS	X2	N	3

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status
1	0	1.7977E308	34.0000	O
2	0	1.7977E308	33.0000	O

Figure 17.3. Optimal Solution

The SAS log shown in Figure 17.4 provides information about the problem, convergence information after each iteration, and the optimal objective value.

```
NOTE: The problem EXAMPLE has 2 variables (0 free, 0 fixed).
NOTE: The problem has 2 constraints (1 LE, 0 EQ, 1 GE, 0 range).
NOTE: The problem has 4 constraint coefficients.
NOTE: The objective function has 2 Hessian diagonal elements and 1 Hessian
      elements above the diagonal.
NOTE: The OPTQP presolver value AUTOMATIC is applied.
NOTE: The OPTQP presolver removed 0 variables and 0 constraints.
NOTE: The OPTQP presolver removed 0 constraint coefficients.
NOTE: The QUADRATIC ITERATIVE solver is called.
NOTE:                                      Primal        Bound         Dual
      Iter    Complement  Duality Gap      Infeas       Infeas       Infeas
         0      63669937     1.985060           0            0            0
         1      12781299     1.952937           0            0            0
         2       2189517     1.877587           0            0            0
         3        363689     1.669055           0            0            0
         4         57278     1.058687           0            0            0
         5   5621.204805     0.239307           0            0            0
         6    914.397217     0.015841           0            0            0
         7     14.822037     0.000796           0            0            0
         8      0.605195  0.000039820           0            0            0
         9      0.029919  0.000001991           0            0            0
        10      0.001495 9.9575984E-8           0            0            0
NOTE: Optimal.
NOTE: Objective = 15018.000761.
```

Figure 17.4. Iteration Log

See the section "Interior Point Algorithm: Overview" on page 1133 and the section "Iteration Log for the OPTQP Procedure" on page 1135 for more details about convergence information given by the iteration log.

Syntax: OPTQP Procedure

The following statement is used in PROC OPTQP:

PROC OPTQP < *options* > **;**

Functional Summary

Table 17.1 outlines the options available for the OPTQP procedure classified by function.

Table 17.1. Options in the OPTQP Procedure

Description	Option
Data Set Options:	
QPS-format input SAS data set	DATA=
dual solution output SAS data set	DUALOUT=
objective sense (maximization or minimization)	OBJSENSE=
primal solution output SAS data set	PRIMALOUT=
save output data sets only if optimal	SAVE_ONLY_IF_OPTIMAL
Control Options:	
maximum number of iterations	MAXITER=
maximum real time	MAXTIME=
type of presolve	PRESOLVER=
enable/disable iteration log	PRINTFREQ=
enable/disable printing summary	PRINTLEVEL=
stopping criterion based on duality gap	STOP_DG=
stopping criterion based on dual infeasibility	STOP_DI=
stopping criterion based on primal infeasibility	STOP_PI=

PROC OPTQP Statement

The following options can be specified in the PROC OPTQP statement.

DATA=*SAS-data-set*
specifies the input SAS data set. This data set can also be created from a QPS-format flat file by using the SAS macro %MPS2SASD. If the DATA= option is not specified, PROC OPTQP will use the most recently created SAS data set. See Chapter 14, "The MPS-Format SAS Data Set," for more details.

DUALOUT=*SAS-data-set*
DOUT=*SAS-data-set*
specifies the output data set containing the dual solution. See the section "Output Data Sets" on page 1131 for details.

MAXITER=*k*
specifies the maximum number of predictor-corrector iterations performed by the interior point algorithm (see the section "Interior Point Algorithm: Overview" on page 1133). The value k is an integer greater than or equal to 1. If you do not specify this option, the procedure does not stop based on the number of iterations performed.

MAXTIME=*k*
specifies an upper limit of k seconds of real time for the optimization process. If you do not specify this option, the procedure does not stop based on the amount of time elapsed.

OBJSENSE=*option*
specifies whether the QP model is a minimization or a maximization problem. You specify OBJSENSE=MIN for a minimization problem and OBJSENSE=MAX for a maximization problem. Alternatively, you can specify the objective sense in the input data set; see the section "ROWS Section" on page 1011 for details. If for some reason the objective sense is specified differently in these two places, this option supersedes the objective sense specified in the input data set. If the objective sense is not specified anywhere, then PROC OPTQP interprets and solves the quadratic program as a minimization problem.

PRESOLVER=*option*
PRESOL=*option*
specifies one of the following presolve options:

Option	Description
NONE (0)	Disable presolver.
AUTOMATIC (-1)	Apply presolver by using default setting.
BASIC (1)	Apply basic presolver.
MODERATE (2)	Apply moderate presolver.
AGGRESSIVE (3)	Apply aggressive presolver.

You can also specify the option by integers from -1 to 3. The integer value for each option is indicated in parentheses. The default option is AUTOMATIC.

PRIMALOUT=*SAS-data-set*
POUT=*SAS-data-set*
specifies the output data set containing the primal solution. See the section "Output Data Sets" on page 1131 for details.

PRINTFREQ=k

specifies that the printing of the solution progress to the iteration log should occur after every k iterations. The print frequency, k, is an integer greater than or equal to zero. The value $k = 0$ disables the printing of the progress of the solution. The default value of this option is 1.

PRINTLEVEL=*0* | *1*

specifies whether a summary of the problem and solution should be printed. If PRINTLEVEL=1, then two ODS (Output Delivery System) tables named "ProblemSummary" and "SolutionSummary" are produced and printed. If PRINTLEVEL=0, then no ODS tables are produced or printed. The default value of this option is 1.

For details about the ODS tables created by PROC OPTQP, see the section "ODS Tables" on page 1136.

SAVE_ONLY_IF_OPTIMAL

specifies that the PRIMALOUT= and DUALOUT= data sets be saved only if the final solution obtained by the solver at termination is optimal. If the PRIMALOUT= and DUALOUT= options are specified, and this option is not specified, then PROC OPTQP always saves the solutions obtained at termination, regardless of the final status.

STOP_DG=δ

specifies the desired relative duality gap, $\delta \in$[1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

STOP_DI=β

specifies the maximum allowed relative dual constraints violation, $\beta \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

STOP_PI=α

specifies the maximum allowed relative bound and primal constraints violation, $\alpha \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

PROC OPTQP Macro Variable

The OPTQP procedure defines a macro variable named _OROPTQP_. This variable contains a character string that indicates the status of the procedure. The various terms of the variable are interpreted as follows.

STATUS

indicates the solver status at termination. It can take one of the following values:

OK	procedure terminated normally
SYNTAX_ERROR	incorrect use of syntax
DATA_ERROR	inconsistent input data

OUT_OF_MEMORY	insufficient memory allocated to the procedure
IO_ERROR	problem in reading or writing of data
ERROR	status that cannot be classified into any of the preceding categories

SOLUTION_STATUS
indicates the solution status at termination. It can take one of the following values:

OPTIMAL	solution is optimal
CONDITIONAL_OPTIMAL	optimality of the solution cannot be proven
INFEASIBLE	solution is infeasible
UNBOUNDED	problem is unbounded
INFEASIBLE_OR_UNBOUNDED	solution is infeasible or problem is unbounded
ITERATION_LIMIT_REACHED	maximum allowable iterations reached
TIME_LIMIT_REACHED	maximum time limit reached
FAILED	solver failed to converge, possibly due to numerical issues
NONCONVEX	quadratic matrix is nonconvex (minimization)
NONCONCAVE	quadratic matrix is nonconcave (maximization)

OBJECTIVE
indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY
indicates the (relative) infeasibility of the primal constraints at the optimal solution. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

DUAL_INFEASIBILITY
indicates the (relative) infeasibility of the dual constraints at the optimal solution. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

BOUND_INFEASIBILITY
indicates the (relative) violation of the optimal solution over the lower and upper bounds. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

DUALITY_GAP
indicates the (relative) duality gap. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

COMPLEMENTARITY
indicates the (absolute) complementarity at the optimal solution. See the section "Interior Point Algorithm: Overview" on page 1133 for details.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time for preprocessing (seconds).

SOLUTION_TIME
indicates the time taken by the interior point algorithm to perform iterations for solving the problem (seconds).

Details: OPTQP Procedure

Output Data Sets

This subsection describes the PRIMALOUT= and DUALOUT= output data sets.

Definitions of Variables in the PRIMALOUT= Data Set

The PRIMALOUT= data set contains the primal solution to the QP model. The variables in the data set have the following names and meanings.

_OBJ_ID_
specifies the name of the objective function. This is particularly useful when there are multiple objective functions, in which case each objective function has a unique name. See "ROWS Section" on page 1011 for details.

Note: PROC OPTQP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_
specifies the name of the variable containing the right-hand-side value of each constraint. See "ROWS Section" on page 1011 for details.

VAR
specifies the name of the decision variable.

TYPE
specifies the type of the decision variable. _TYPE_ can take one of the following values:

N nonnegative variable

D bounded variable with either lower or upper bound

F free variable

X fixed variable

O other

OBJCOEF
specifies the coefficient of the decision variable in the linear component of the objective function.

LBOUND
specifies the lower bound on the decision variable.

UBOUND
specifies the upper bound on the decision variable.

VALUE
specifies the value of the decision variable.

STATUS
specifies the status of the decision variable. _STATUS_ can indicate one of the following two cases:

O QP problem is optimal.

I QP problem could be infeasible or unbounded, or PROC OPTQP was not able to solve the problem.

Definitions of Variables in the DUALOUT= Data Set

The DUALOUT= data set contains the dual solution to the QP model. Information about the objective rows of the QP problems is not included. The variables in the data set have the following names and meanings.

_OBJ_ID_
specifies the name of the objective function. This is particularly useful when there are multiple objective functions, in which case each objective function has a unique name. See "ROWS Section" on page 1011 for details.

Note: PROC OPTQP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_
specifies the name of the variable containing the right-hand-side value of each constraint. See "ROWS Section" on page 1011 for details.

ROW
specifies the name of the constraint. See "ROWS Section" on page 1011 for details.

TYPE
specifies the type of the constraint. _TYPE_ can take one of the following values:

L "less than or equals" constraint

E equality constraint

G "greater than or equals" constraint

R ranged constraint (both "less than or equals" and "greater than or equals")

See "ROWS Section" on page 1011 and "RANGES Section (Optional)" on page 1014 for details.

RHS
specifies the value of the right-hand side of the constraints. It takes a missing value for a ranged constraint.

_L_RHS_
specifies the lower bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

_U_RHS_
specifies the upper bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

VALUE
specifies the value of the dual variable associated with the constraint.

STATUS
specifies the status of the constraint. _STATUS_ can indicate one of the following two cases:

O QP problem is optimal.

I QP problem could be infeasible or unbounded, or PROC OPTQP was not able to solve the problem.

ACTIVITY
specifies the value of a constraint. In other words, the value of _ACTIVITY_ for the ith constraint would be equal to $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$, where $\mathbf{a}_i$ refers to the ith row of the constraints matrix and $\mathbf{x}$ denotes the vector of current decision variable values.

Interior Point Algorithm: Overview

The interior point solver in PROC OPTQP implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following QP formulation (the primal):

$$\begin{aligned} \min \quad & \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

The corresponding dual is as follows:

$$\begin{aligned} \max \quad & -\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{y} \\ \text{subject to} \quad & -\mathbf{Q}\mathbf{x} + \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of slack variables in the dual problem.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker) conditions, which can be stated as follows:

$$\begin{aligned}
\mathbf{Ax} - \mathbf{s} &= \mathbf{b} && (\text{Primal Feasibility}) \\
-\mathbf{Qx} + \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} &= \mathbf{c} && (\text{Dual Feasibility}) \\
\mathbf{WXe} &= \mathbf{0} && (\text{Complementarity}) \\
\mathbf{SYe} &= \mathbf{0} && (\text{Complementarity}) \\
\mathbf{x},\ \mathbf{y},\ \mathbf{w},\ \mathbf{s} &\geq \mathbf{0}
\end{aligned}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal slack variables.

Note: Slack variables (the s vector) are automatically introduced by the solver when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the solver to handle slack variables much more efficiently.

The letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{S}$ denote matrices with corresponding x, y, w, and s on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations representing the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original QP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{Q} - \mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively; Θ and Ξ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. PROC OPTQP uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point solver is that it takes full advantage of the sparsity in the constraint and quadratic matrices, thereby enabling it to efficiently solve large-scale quadratic programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore it is of interest to observe the following four measures:

- Relative primal infeasibility measure α:

$$\alpha = \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- Relative dual infeasibility measure β:

$$\beta = \frac{\|\mathbf{Q}\mathbf{x} + \mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- Relative duality gap δ:

$$\delta = \frac{|\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- Absolute complementarity γ:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

where $\|v\|_2$ is the Euclidean norm of the vector v. These measures are displayed in the iteration log.

Iteration Log for the OPTQP Procedure

The interior point solver in PROC OPTQP implements an infeasible primal-dual predictor-corrector interior point algorithm. The following information is displayed in the iteration log:

Iter	indicates the iteration number
Complement	indicates the (absolute) complementarity
Duality Gap	indicates the (relative) duality gap
Primal Infeas	indicates the (relative) primal infeasibility measure
Bound Infeas	indicates the (relative) bound infeasibility measure
Dual Infeas	indicates the (relative) dual infeasibility measure

If the sequence of solutions converges to an optimal solution of the problem, you should see all columns in the iteration log converge to zero or very close to zero. If they do not, it can be the result of insufficient iterations being performed to reach optimality. In this case, you might need to increase the value specified in the option MAXITER= or MAXTIME=. If the complementarity and/or the duality gap do not converge, the problem might be infeasible or unbounded. If the infeasibility columns do not converge, the problem might be infeasible.

ODS Tables

PROC OPTQP creates two ODS (Output Delivery System) tables by default unless the option PRINTLEVEL=0 is specified. One table is a summary of the input QP problem. The other is a brief summary of the solution status. PROC OPTQP assigns a name to each table it creates. You can use these names to reference the table when using the ODS to select tables and create output data sets. These names are listed in Table 17.2. For more information about ODS, see *SAS Output Delivery System: User's Guide*.

Table 17.2. ODS Tables Produced by PROC OPTQP

ODS Table Name	Description	Statement	Option
ProblemSummary	Summary of the input QP problem	OPTQP	default
SolutionSummary	Summary of the solution status	OPTQP	default

A typical output of PROC OPTQP is shown in Figure 17.5.

The OPTQP Procedure

Problem Summary

Problem Name	BANDM
Objective Sense	Minimization
Objective Function	1
RHS	ZZZZ0001
Number of Variables	472
Bounded Above	0
Bounded Below	472
Bounded Above and Below	0
Free	0
Fixed	0
Number of Constraints	305
LE (<=)	0
EQ (=)	305
GE (>=)	0
Range	0
Constraint Coefficients	2494
Hessian Diagonal Elements	25
Hessian Elements Above the Diagonal	16

The OPTQP Procedure

Solution Summary

Objective Function	1
Solution Status	Optimal
Objective Value	16352.342414
Primal Infeasibility	1.693635E-11
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	1.533061E-7
Complementarity	0.0040753441
Iterations	26
Presolve Time	0.00
Solution Time	0.17

Figure 17.5. Typical OPTQP Output

You can create output data sets from these tables by using the ODS OUTPUT statement. This can be useful, for example, when you want to create a report to summarize multiple PROC OPTQP runs. The output data sets corresponding to the preceding output are shown in Figure 17.6, where you can also find (in the row following the heading of each data set in the display) the variable names that are used in the table definition (template) of each table.

Problem Summary

Obs	Label1	cValue1	nValue1
1	Problem Name	BANDM	.
2	Objective Sense	Minimization	.
3	Objective Function	1	.
4	RHS	ZZZZ0001	.
5			.
6	Number of Variables	472	472.000000
7	Bounded Above	0	0
8	Bounded Below	472	472.000000
9	Bounded Above and Below	0	0
10	Free	0	0
11	Fixed	0	0
12			.
13	Number of Constraints	305	305.000000
14	LE (<=)	0	0
15	EQ (=)	305	305.000000
16	GE (>=)	0	0
17	Range	0	0
18			.
19	Constraint Coefficients	2494	2494.000000
20			.
21	Hessian Diagonal Elements	25	25.000000
22	Hessian Elements Above the Diagonal	16	16.000000

Solution Summary

Obs	Label1	cValue1	nValue1
1	Objective Function	1	.
2	Solution Status	Optimal	.
3	Objective Value	16352.342414	16352
4			.
5	Primal Infeasibility	1.693635E-11	1.693635E-11
6	Dual Infeasibility	0	0
7	Bound Infeasibility	0	0
8	Duality Gap	1.533061E-7	0.000000153
9	Complementarity	0.0040753441	0.004075
10			.
11	Iterations	26	26.000000
12	Presolve Time	0.00	0
13	Solution Time	0.17	0.172000

Figure 17.6. ODS Output Data Sets

Memory Limit

The system option MEMSIZE sets a limit on the amount of memory used by the SAS System. If you do not specify a value for this option, then the SAS System sets a default memory limit. Your operating environment determines the actual size of the default memory limit, which is sufficient for many applications. However, to solve most realistic optimization problems, the OPTQP procedure might require more memory. Increasing the memory limit can reduce the chance of an out-of-memory condition.

Note: The MEMSIZE system option is not available in some operating environments. See the documentation for your operating environment for more information.

You can specify -MEMSIZE 0 to indicate all available memory should be used, but this setting should be used with caution. In most operating environments, it is better to specify an adequate amount of memory than to specify -MEMSIZE 0. For example, if you are running PROC OPTLP to solve LP problems with only a few hundred thousand variables and constraints, -MEMSIZE 500M might be sufficient to enable the procedure to run without an out-of-memory condition. When problems have millions of variables, -MEMSIZE 1000M or higher might be needed. These are "rules of thumb"—problems with atypical structure, density, or other characteristics can increase the optimizer's memory requirements.

The MEMSIZE option can be specified at system invocation, on the SAS command line, or in a configuration file. The syntax is described in the SAS Companion for your operating environment.

To report a procedure's memory consumption, you can use the FULLSTIMER option. The syntax is described in the SAS Companion for your operating environment.

Examples: OPTQP Procedure

This section contains examples that illustrate the use of the OPTQP procedure. Example 17.1 illustrates how to model a linear least-squares problem and solve it by using PROC OPTQP. Example 17.2 and Example 17.3 explain in detail how to model the portfolio optimization/selection problem.

Example 17.1. Linear Least-Squares Problem

The linear least-squares problem arises in the context of determining a solution to an over-determined set of linear equations. In practice, these could arise in data fitting and estimation problems. An 1 system of linear equations can be defined as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $m > n$. Since this system usually does not have a solution, we need to be satisfied with some sort of approximate solution. The most widely used approximation is the least-squares solution, which minimizes $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.

This problem is called a least-squares problem for the following reason. Let $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ be defined as previously. Let $k_i(x)$ be the kth component of the vector $\mathbf{A}\mathbf{x} - \mathbf{b}$:

$$k_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i, \ \ i = 1, 2, \ldots, m$$

By definition of the Euclidean norm, the objective function can be expressed as follows:

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \sum_{i=1}^{m} k_i(x)^2$$

Therefore, the function we minimize is the sum of squares of m terms $k_i(x)$; hence the term least-squares. The following example is an illustration of the *linear* least-squares problem; i.e., each of the terms k_i is a linear function of x.

Consider the following least-squares problem defined by

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This translates to the following set of linear equations:

$$4x_1 = 1, \quad -x_1 + x_2 = 0, \quad 3x_1 + 2x_2 = 1$$

The corresponding least-squares problem is

$$\text{minimize} \quad (4x_1 - 1)^2 + (-x_1 + x_2)^2 + (3x_1 + 2x_2 - 1)^2$$

The preceding objective function can be expanded to

$$\text{minimize} \quad 26x_1^2 + 5x_2^2 + 10x_1x_2 - 14x_1 - 4x_2 + 2$$

In addition, we impose the following constraint so that the equation $3x_1 + 2x_2 = 1$ is satisfied within a tolerance of 0.1:

$$0.9 \leq 3x_1 + 2x_2 \leq 1.1$$

You can create the QPS-format input data set by using the following SAS code:

```
data lsdata;
   input field1 $ field2 $ field3$ field4 field5 $ field6 @;
datalines;
NAME     .       LEASTSQ      .             .          .
ROWS     .       .            .             .          .
N        OBJ     .            .             .          .
G        EQ3     .            .             .          .
COLUMNS  .       .            .             .          .
.        X1      OBJ        -14             EQ3        3
.        X2      OBJ         -4             EQ3        2
RHS      .       .            .             .          .
.        RHS     OBJ         -2             EQ3      0.9
RANGES   .       .            .             .          .
.        RNG     EQ3        0.2             .          .
BOUNDS   .       .            .             .          .
FR     BND1      X1           .             .          .
FR     BND1      X2           .             .          .
QUADOBJ  .       .            .             .          .
.        X1      X1          52             .          .
.        X1      X2          10             .          .
.        X2      X2          10             .          .
ENDATA   .       .            .             .          .
;
```

The decision variables x_1 and x_2 are free, so they have bound type FR in the BOUNDS section of the QPS-format data set.

You can use the following SAS code to solve the least-squares problem:

```
proc optqp data=lsdata
  primalout = lspout;
run;
```

The optimal solution is displayed in Output 17.1.1.

Output 17.1.1. Solution to the Least-Squares Problem

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient
1	OBJ	RHS	X1	F	-14
2	OBJ	RHS	X2	F	-4

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status
1	-1.7977E308	1.7977E308	0.23809	O
2	-1.7977E308	1.7977E308	0.16190	O

The iteration log is shown in Output 17.1.2.

Output 17.1.2. Iteration Log

```
NOTE: The problem LEASTSQ has 2 variables (2 free, 0 fixed).
NOTE: The problem has 1 constraints (0 LE, 0 EQ, 0 GE, 1 range).
NOTE: The problem has 2 constraint coefficients.
NOTE: The objective function has 2 Hessian diagonal elements and 1 Hessian
      elements above the diagonal.
NOTE: The OPTQP presolver value AUTOMATIC is applied.
NOTE: The OPTQP presolver removed 0 variables and 0 constraints.
NOTE: The OPTQP presolver removed 0 constraint coefficients.
NOTE: The QUADRATIC ITERATIVE solver is called.
NOTE:                                          Primal        Bound         Dual
      Iter   Complement  Duality Gap       Infeas       Infeas       Infeas
         0     10001805     2.001029  1052.607895   416.588001            0
         1       501893     2.094827    52.630395    20.750266            0
         2        26887    34.114879     2.631520     0.958349            0
         3  2997.321110   600.993085     0.131576            0            0
         4   734.867262   111.975745     0.006579            0            0
         5    26.413788     5.701869     0.000329            0            0
         6     0.877292     0.287945  0.000016608            0            0
         7     0.051222     0.017041  0.000000983            0            0
         8     0.007624     0.001934  0.000000103            0            0
         9     0.001232     0.000171 5.7776383E-9            0            0
        10  0.000047950  0.000008773 2.898554E-10            0            0
        11  0.000001381  0.000000438 1.449279E-11            0            0
NOTE: Optimal.
NOTE: Objective = 0.0095238096.
```

Example 17.2. Portfolio Optimization

Consider a portfolio optimization example. The two competing goals of investment are (1) long-term growth of capital and (2) low risk. A good portfolio grows steadily without wild fluctuations in value. The Markowitz model is an optimization model for balancing the return and risk of a portfolio. The decision variables are the amounts invested in each asset. The objective is to minimize the variance of the portfolio's total return, subject to the constraints that (1) the expected growth of the portfolio reaches at least some target level and (2) you do not invest more capital than you have.

Let $x_1, \ldots, x_n$ be the amount invested in each asset, $\mathcal{B}$ be the amount of capital you have, $\mathbf{R}$ be the random vector of asset returns over some period, and $\mathbf{r}$ be the expected value of $\mathbf{R}$. Let G be the minimum growth you hope to obtain, and $\mathcal{C}$ be the covariance matrix of $\mathbf{R}$. The objective function is $\mathrm{Var}\left(\sum_{i=1}^{n} x_i R_i\right)$, which can be equivalently denoted as $\mathbf{x}^{\mathrm{T}}\mathcal{C}\mathbf{x}$.

Assume, for example, $n = 4$. Let $\mathcal{B} = 10{,}000$, $G = 1000$, $\mathbf{r} = [0.05, -0.2, 0.15, 0.30]$,

and

$$\mathcal{C} = \begin{bmatrix} 0.08 & -0.05 & -0.05 & -0.05 \\ -0.05 & 0.16 & -0.02 & -0.02 \\ -0.05 & -0.02 & 0.35 & 0.06 \\ -0.05 & -0.02 & 0.06 & 0.35 \end{bmatrix}$$

The QP formulation can be written as follows:

$$\begin{array}{rlllllllll} \min & 0.08x_1^2 & - & 0.1x_1x_2 & - & 0.1x_1x_3 & - & 0.1x_1x_4 & + & \\ & 0.16x_2^2 & - & 0.04x_2x_3 & - & 0.04x_2x_4 & + & 0.35x_3^2 & + & \\ & 0.12x_3x_4 & + & 0.35x_4^2 & & & & & & \\ \text{subject to} & & & & & & & & & \\ (\text{budget}) & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 10000 \\ (\text{growth}) & 0.05x_1 & - & 0.2x_2 & + & 0.15x_3 & + & 0.30x_4 & \geq & 1000 \\ & & & & & & & x_1,\, x_2,\, x_3,\, x_4 & \geq & 0 \end{array}$$

The corresponding QPS-format input data set is as follows:

```
data portdata;
   input field1 $ field2 $ field3$ field4 field5 $ field6 @;
datalines;
NAME     .          PORT      .          .          .
ROWS     .          .         .          .          .
N        OBJ.FUNC   .         .          .          .
L        BUDGET     .         .          .          .
G        GROWTH     .         .          .          .
COLUMNS  .          .         .          .          .
.        X1         BUDGET    1.0        GROWTH     0.05
.        X2         BUDGET    1.0        GROWTH     -.20
.        X3         BUDGET    1.0        GROWTH     0.15
.        X4         BUDGET    1.0        GROWTH     0.30
RHS      .          .         .          .          .
.        RHS        BUDGET    10000      .          .
.        RHS        GROWTH    1000       .          .
RANGES   .          .         .          .          .
BOUNDS   .          .         .          .          .
QUADOBJ  .          .         .          .          .
.        X1         X1        0.16       .          .
.        X1         X2        -.10       .          .
.        X1         X3        -.10       .          .
.        X1         X4        -.10       .          .
.        X2         X2        0.32       .          .
.        X2         X3        -.04       .          .
.        X2         X4        -.04       .          .
.        X3         X3        0.70       .          .
.        X3         X4        0.12       .          .
.        X4         X4        0.70       .          .
ENDATA   .          .         .          .          .
;
```

Use the following SAS code to solve the problem:

```
proc optqp data=portdata
  primalout = portpout
  dualout   = portdout;
run;
```

The optimal solution is shown in Output 17.2.1.

Output 17.2.1. Portfolio Optimization

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient
1	OBJ.FUNC	RHS	X1	N	0
2	OBJ.FUNC	RHS	X2	N	0
3	OBJ.FUNC	RHS	X3	N	0
4	OBJ.FUNC	RHS	X4	N	0

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status
1	0	1.7977E308	3452.86	O
2	0	1.7977E308	0.00	O
3	0	1.7977E308	1068.81	O
4	0	1.7977E308	2223.45	O

Thus, the minimum variance portfolio that earns an expected return of at least 10% is $x_1 = 3452.86$, $x_2 = 0$, $x_3 = 1068.81$, $x_4 = 2223.45$. Asset 2 gets nothing, because its expected return is -20% and its covariance with the other assets is not sufficiently negative for it to bring any diversification benefits. What if we drop the nonnegativity assumption? You need to update the BOUNDS section in the existing QPS-format data set to indicate that the decision variables are free.

```
...
RANGES   .          .          .          .          .
BOUNDS   .          .          .          .          .
FR       BND1       X1         .          .          .
FR       BND1       X2         .          .          .
FR       BND1       X3         .          .          .
FR       BND1       X4         .          .          .
QUADOBJ  .          .          .          .          .
...
```

Financially, that means you are allowed to short-sell—i.e., sell low-mean-return assets and use the proceeds to invest in high-mean-return assets. In other words, you

put a negative portfolio weight in low-mean assets and "more than 100%" in high-mean assets. You can see in the optimal solution displayed in Output 17.2.2 that the decision variable x_2, denoting Asset 2, is equal to -1563.61, which means short sale of that asset.

Output 17.2.2. Portfolio Optimization with Short-Sale Option

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient
1	OBJ.FUNC	RHS	X1	F	0
2	OBJ.FUNC	RHS	X2	F	0
3	OBJ.FUNC	RHS	X3	F	0
4	OBJ.FUNC	RHS	X4	F	0

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status
1	-1.7977E308	1.7977E308	1684.35	O
2	-1.7977E308	1.7977E308	-1563.61	O
3	-1.7977E308	1.7977E308	682.51	O
4	-1.7977E308	1.7977E308	1668.95	O

Example 17.3. Portfolio Selection with Transactions

Consider a portfolio selection problem with a slight modification. You are now required to take into account the current position and transaction costs associated with buying and selling assets. The objective is to find the minimum variance portfolio. In order to understand the scenario better, consider the following data.

You are given three assets. The current holding of the three assets is denoted by the vector **c** = [200, 300, 500], the amount of asset bought and sold is denoted by b_i and s_i, respectively, and the net investment in each asset is denoted by x_i and is defined by the following relation:

$$x_i - b_i + s_i = c_i, \quad i = 1, 2, 3$$

Let us say that you pay a transaction fee of 0.01 every time you buy or sell. Let the covariance matrix $\mathcal{C}$ be defined as

$$\mathcal{C} = \begin{bmatrix} 0.027489 & -0.00874 & -0.00015 \\ -0.00874 & 0.109449 & -0.00012 \\ -0.00015 & -0.00012 & 0.000766 \end{bmatrix}$$

Assume that you hope to obtain at least 12% growth. Let **r** = [1.109048, 1.169048, 1.074286] be the vector of expected return on the three assets, and let $\mathcal{B}$=1000 be

the available funds. Mathematically, this problem can be written in the following manner:

$$\begin{aligned}\min \quad & 0.027489x_1^2 - 0.01748x_1x_2 - 0.0003x_1x_3 + 0.109449x_2^2 \\ & -0.00024x_2x_3 + 0.000766x_3^2\end{aligned}$$

$$\begin{array}{lrcll}
\text{subject to} & & & & \\
\text{(return)} & \sum_{i=1}^{3} r_i x_i & \geq & 1.12\mathcal{B} & \\
\text{(budget)} & \sum_{i=1}^{3} x_i + \sum_{i=1}^{3} 0.01(b_i + s_i) & = & \mathcal{B} & \\
\text{(balance)} & x_i - b_i + s_i & = & c_i, & i = 1, 2, 3 \\
 & x_i,\ b_i,\ s_i & \geq & 0, & i = 1, 2, 3
\end{array}$$

The QPS-format input data set is as follows:

```
data potrdata;
   input field1 $ field2 $ field3$ field4 field5 $ field6 @;
datalines;
NAME   .         POTRAN    .              .          .
ROWS   .         .         .              .          .
N      OBJ.FUNC  .         .              .          .
G      RETURN    .         .              .          .
E      BUDGET    .         .              .          .
E      BALANC1   .         .              .          .
E      BALANC2   .         .              .          .
E      BALANC3   .         .              .          .
COLUMNS .        .         .              .          .
.      X1        RETURN    1.109048       BUDGET     1.0
.      X1        BALANC1   1.0            .          .
.      X2        RETURN    1.169048       BUDGET     1.0
.      X2        BALANC2   1.0            .          .
.      X3        RETURN    1.074286       BUDGET     1.0
.      X3        BALANC3   1.0            .          .
.      B1        BUDGET    .01            BALANC1   -1.0
.      B2        BUDGET    .01            BALANC2   -1.0
.      B3        BUDGET    .01            BALANC3   -1.0
.      S1        BUDGET    .01            BALANC1    1.0
.      S2        BUDGET    .01            BALANC2    1.0
.      S3        BUDGET    .01            BALANC3    1.0
RHS    .         .         .              .          .
.      RHS       RETURN    1120           .          .
.      RHS       BUDGET    1000           .          .
.      RHS       BALANC1   200            .          .
.      RHS       BALANC2   300            .          .
.      RHS       BALANC3   500            .          .
RANGES .         .         .              .          .
BOUNDS .         .         .              .          .
QUADOBJ .        .         .              .          .
.      X1        X1        0.054978       .          .
.      X1        X2        -.01748        .          .
.      X1        X3        -.0003         .          .
.      X2        X2        0.218898       .          .
```

```
.    X2        X3        -.00024        .          .
.    X3        X3        0.001532       .          .
ENDATA .       .         .              .          .
;
```

Use the following SAS code to solve the problem:

```
proc optqp data=potrdata
  primalout = potrpout
  dualout   = potrdout;
run;
```

The optimal solution is displayed in Output 17.3.1.

Output 17.3.1. Portfolio Selection with Transactions

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient
1	OBJ.FUNC	RHS	X1	N	0
2	OBJ.FUNC	RHS	X2	N	0
3	OBJ.FUNC	RHS	X3	N	0
4	OBJ.FUNC	RHS	B1	N	0
5	OBJ.FUNC	RHS	B2	N	0
6	OBJ.FUNC	RHS	B3	N	0
7	OBJ.FUNC	RHS	S1	N	0
8	OBJ.FUNC	RHS	S2	N	0
9	OBJ.FUNC	RHS	S3	N	0

Obs	Lower Bound	Upper Bound	Variable Value	Variable Status
1	0	1.7977E308	397.584	O
2	0	1.7977E308	406.115	O
3	0	1.7977E308	190.165	O
4	0	1.7977E308	197.584	O
5	0	1.7977E308	106.115	O
6	0	1.7977E308	0.000	O
7	0	1.7977E308	0.000	O
8	0	1.7977E308	0.000	O
9	0	1.7977E308	309.835	O

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